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Elliptic Boundary Value Problems in Domains with Point Singularities

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ABSTRACT. The book contains a systematic treatment of linear elliptic boundary value problems in domains with either smooth boundaries or conical or cuspidal boundary points. The authors concentrate on the following fundamental results: estimates for solutions in usual and weighted Sobolev spaces of arbitrary integer order, solvability of the bundary value problem, regularity assertions and asymptotic formulas for the solutions near singular points. The book could be of interest to researchers and graduate students working in the field of partial differential equations.

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Contents

Introduction

The goal of this book is a systematic and self-contained exposition of a theory of linear elliptic boundary value problems in domains with isolated singularities on the boundary.

Roots of the theory. Elliptic boundary value problems play an important role in mathematical physics. Starting with the 18th century an innumerable number of works was dedicated to special boundary value problems. The theory of *general* elliptic boundary value problems in smooth domains was developed in the second half of the 20th century by I. G. Petrovskiĭ [**195**], M. I. Vishik [**246**], Ya. B. Lopatinskiĭ [**127**], L. Hörmander [**86, 87**], S. Agmon [**3, 6**], S. Agmon, A. Douglis, L. Nirenberg [**7, 8**], F. E. Browder [**37, 38**], M. Schechter [**213, 217**], J. Peetre [**192**], A. I. Koshelev [**102**], Yu. M. Berezanskiĭ [**26, 27**], V. A. Solonnikov [**236**], Ya. A. Roĭtberg [**201**] - [**205**], Ya. A. Roĭtberg, Z. G. Sheftel' [**207, 208**], J. Nečas [**184**], J.-L. Lions, E. Magenes [**126**], B.-W. Schulze, G. Wildenhain [**229**], I. V. Gel'man, V. G. Maz'ya [**74**], and others.

Fundamental results in this theory are:

- a priori estimates for the solutions in different function spaces
- the Fredholm property of the operator corresponding to the boundary value problem
- regularity assertions for the solutions

The construction of *parametrices* (approximately inverse operators) to the operators of elliptic boundary value problems in domains with smooth boundaries resulted in the development of the theory of pseudodifferential boundary value problems. This theory has its origin in papers of A. P. Calderón, A. Zygmund [**42**], M. I. Vishik, G. I. Eskin [**247, 248, 249**], G. I. Eskin [**67, 68**], and L. Boutet de Monvel [**34, 35**]. One of the most important results in the theory of pseudodifferential boundary value problems is the calculation of the index for elliptic boundary value problems in topological terms. A formula for the index of boundary value problems in two-dimensional domains was found by A. I. Vol'pert [**253**]. M. F. Atiyah and I. M. Singer [**18**] calculated the index of elliptic operators on compact manifolds without boundary, while an index formula for elliptic boundary value problems was derived by M. F. Atiyah and R. Bott [**17**]. We refer further to the papers of M. S. Agranovich [**10**], A. P. Calderón [**41**], and L. Boutet de Monvel [**35**]). A more recent treatment of the theory of pseudodifferential boundary value problems is given, e.g., in the monographs of S. Rempel, B.-W. Schulze [**200**] and G. Grubb [**81**].

Parallel with the theory of general elliptic boundary value problems in smooth domains, such problems in domains with singularities on the boundary were investigated. The question on the behaviour of solutions of elliptic boundary value problems near boundary singularities is of great importance for many applications, e.g., in aerodynamics, hydrodynamics, fracture machanics. The treatment of elliptic boundary value problems in domains with singularities required a new theory. On one hand, the methods which were developed for domains with smooth boundaries cannot be directly applied to domains with singularities. On the other hand, many results of the theory for smooth domains are not true if the boundary of the domain contains singularities.

The pioneering work in the development of a general theory for elliptic boundary value problems in domains with angular and conical points was done by G. I. Eskin [**65, 66**], Ya. B. Lopatinskiĭ [**128**], and V. A. Kondrat'ev [**98, 99**]. The first two authors investigated boundary value problems in plane domains with angular points applying the Mellin transformation to an integral equation on the boundary. V. A. Kondrat'ev considered elliptic boundary value problems in domains of arbitrary dimension with conical points. He applied the Mellin transformation to a model problem connected with the boundary value problem and proved the Fredholm property of the operator of the boundary value problem in weighted and usual L_2 Sobolev spaces of positive integer order. Furthermore, he described the asymptotics of the solutions near conical points. Analogous asymptotic representations for solutions of elliptic boundary value problems in an infinite cylinder were found in 1963 by S. Agmon and L. Nirenberg [**9**].

The gap between V. A. Kondrat'ev's theory and applications was narrowed in the works of V. G. Maz'ya and B. A. Plamenevskiĭ [**142, 143, 144, 147, 149**]. These two authors extended the results of V. A. Kondrat'ev to other function spaces (L_p Sobolev spaces, Hölder classes, spaces with inhomogeneous norms), calculated the coefficients in the asymptotics and described the singularities of the Green functions. In the monograph of M. Dauge [**53**] L_2 Sobolev spaces of fractional order were admitted. Several papers of V. G. Maz'ya, B. A. Plamenevskiĭ [**150, 151, 152**], V. A. Kozlov, V. G. Maz'ya [**111, 112, 113**], V. A. Kozlov (e.g., [**105, 106, 107, 108**]), V. A. Kozlov, J. Roßmann [**115, 116**]), and M. Costabel, M. Dauge [**51**] contain a detailed analysis of the singularities of the solutions to elliptic boundary value problems near conical points. Other results in this field are estimates of the L_p-means (V. A. Kozlov, V. G. Maz'ya [**109, 110**]), the Miranda-Agmon maximum principle (V. G. Maz'ya, B. A. Plamenevskiĭ [**150**], V. G. Maz'ya, J. Roßmann [**155**]), and the construction of stable asymptotics (V. G. Maz'ya, J. Roßmann [**156**], M. Costabel, M. Dauge [**52**]).

We also mention the books of P. Grisvard [**79, 80**], A. Kufner, A.-M. Sändig [**121**], V. G. Maz'ya, S. A. Nazarov, B. A. Plamenevskiĭ [**138**], S. A. Nazarov, B. A. Plamenevskiĭ [**182**] and S. Nicaise [**186**], where different aspects of the theory of elliptic boundary value problems in domains with angular and conical points are considered.

A theory of elliptic boundary value problems in domains with cusps or in quasi-cylindrical domains was established in papers of V. I. Feĭgin [**69**], L. A. Bagirov and V. I. Feĭgin [**23**], V. G. Maz'ya and B. A. Plamenevskiĭ [**139, 147**], A. B. Movchan and S. A. Nazarov [**170, 171, 173**], J.-L. Steux [**239, 240**], and M. Dauge [**56**]. Elliptic equations on manifolds with cusps were studied further by B. W. Schulze, B. Sternin, and V. Shatalov [**227, 228**].

Moreover, there are many works concerning other boundary singularities, such as edges, polyhedral vertices, and domains of class $C^{0,1}$, which are not studied here.

The class of boundary value problems. A principal new feature of this book in comparison with other monographs and papers on elliptic boundary value problems in domains with conical points is the consideration of solutions in Sobolev spaces of both positive and negative order.

In this book we consider boundary value problems for differential equations. Avoiding the use of pseudodifferential operators ensures a more elementary character of the book. Moreover, in most of applications of the theory of elliptic boundary value problems only differential operators occur.

Pseudo-differential operators on manifolds with conical points were studied by R. Melrose and G. Mendoza [**164**], B. A. Plamenevskiĭ [**199**], and B. W. Schulze [**222**]–[**225**]. A. O. Derviz [**59**], E. Schrohe and B.-W. Schulze [**218, 219**] extended the results to pseudo-differential boundary value problems on manifolds with conical points. They constructed algebras of pseudo-differential boundary value problems and parametrices for elliptic elements. Studying the structure of the parametrices, they obtained regularity assertions and the asymptotics of the solutions near the conical point. We refer further to the work [**163**] of R. Melrose which is dedicated to index theorems of Atiyah-Patodi-Singer type for pseudo-differential operators on manifolds with conical points.

A boundary value problem in the classical form consists of a differential equation (or a system of differential equations)

$$Lu = f \tag{1}$$

for the unknown function (vector-function) u in a domain $\Omega \subset \mathbb{R}^n$ and some conditions

$$Bu = g \tag{2}$$

which have to be satisfied on the boundary $\partial\Omega$. Here B is a vector (or matrix) differential operator. The equations (2) are called *boundary conditions.*

In contrast to other monographs, we consider boundary conditions, where additionally to the unknown functions in the domain Ω also an unknown vector-function $\underline{u}$ on the boundary $\partial\Omega$ appears, i.e., boundary conditions of the form

$$Bu + C\underline{u} = g \quad \text{on } \partial\Omega. \tag{3}$$

Here B is a vector (or a matrix) differential operator on Ω and C is a matrix differential operator on $\partial\Omega$. Naturally, boundary value problems of the form (1), (2) are contained in the class of the problems (1), (3).

The reason for considering the boundary value problems of the form (1), (3), which appeared first in the works of B. Lawruk [**124**], is that the adjoint problem belongs to the same class of problems. This is not true if we restrict ourselves to classical boundary value problems (1), (2). Let us consider, e.g., the Laplace equation in a plane domain Ω with the boundary condition

$$b_1 \frac{\partial u}{\partial \nu} + b_2 \frac{\partial u}{\partial \tau} = g \quad \text{on } \partial\Omega,$$

where $\partial/\partial\nu$ denotes the derivative in the direction of the exterior normal, $\partial/\partial\tau$ denotes the derivative in the tangential direction to $\partial\Omega$, and b_1, b_2 are smooth real-valued functions satisfying the condition $|b_1|+|b_2| \neq 0$ on $\partial\Omega$. Then an adjoint problem is

$$\Delta v = f \quad \text{in } \Omega,$$
$$v - b_1\, v_1 = g_1, \quad \frac{\partial v}{\partial \nu} - \frac{\partial}{\partial \tau}\,(b_2\, v_1) = g_2 \quad \text{on } \partial\Omega.$$

Clearly, if $b_1 \neq 0$ on $\partial\Omega$, then the unknown v_1 on $\partial\Omega$ can be eliminated. However, if $b_1 = 0$ on a nonempty subset of $\partial\Omega$, then the adjoint problem can not be written in the form (1), (2).

Note that the operators

$$\mathcal{A} = \begin{pmatrix} L & 0 \\ B & C \end{pmatrix} \tag{4}$$

of the boundary value problem (1), (3) form a subalgebra of Boutet de Monvel's algebra for pseudodifferential boundary value problems which consists of matrices of the form

$$\begin{pmatrix} L+G & K \\ B & C \end{pmatrix}, \tag{5}$$

where L is a pseudodifferential operator, G is a singular Green operator, B is a trace operator acting from Ω to $\partial\Omega$, C is a pseudodifferential operator on the boundary $\partial\Omega$, and K is a Poisson operator acting from $\partial\Omega$ to Ω.

The inclusion of adjoint boundary value problems has several advantages. For example, it is not necessary to construct a regularizer in order to prove the Fredholm property of the operator (4). It suffices to prove a priori estimates and regularity assertions for the solutions of the boundary value problem (1), (3). The cokernel of the operator (4) can be described by the solutions of the homogeneous formally adjoint problem. Furthermore, with the help of the adjoint problem we are able to construct an extension of the operator (4) to Sobolev spaces of an arbitrary order. For elliptic boundary value problems of the form (1), (2) this extension was constructed in papers of Ya. A. Roĭtberg [**201**] - [**205**] and Ya. A. Roĭtberg, Z. G. Sheftel' [**207**], [**208**].

The structure of the book. The book consists of three parts. In the first part (Chapters 1-4) we consider the boundary value problem (1), (3) in a domain with smooth boundary. We give a detailed proof for the equivalence of the ellipticity, the Fredholm property of the operator (4) and the validity of a priori estimates for the solutions in corresponding Sobolev spaces. In these assertions the operator (4) is considered in Sobolev spaces of both positive and negative order.

The main step in the proof of the Fredholm property is the derivation of necessary and sufficient conditions for the unique solvability of boundary value problems with constant coefficients in the half-space $x_n > 0$ in Chapter 2. Here we use Sobolev spaces of functions which are periodic in the variables $x_1, \dots, x_{n-1}$. The theorem on the unique solvability of boundary value problems with constant coefficients in the half-space implies, in particular, regularity assertions and a priori estimates for the solutions. These are extended to elliptic problems with variable coefficients in the half-space.

Chapter 3 deals with elliptic boundary value problems in smooth bounded domains. Based on the results of Chapter 2, we obtain the Fredholm property and regularity assertions for the solution. Furthermore, we prove the existence of Green's functions for arbitrary elliptic problems of the form (1), (3) and get a representation of the solutions by means of these functions. Finally, elliptic boundary value problems with a complex parameter are considered in Chapter 3. Such problems arise, in particular, from boundary value problems in a cone if one applies the Mellin transformation $|x| = r \to \lambda$ to the principal parts of the operators L, B, and C. Spectral properties of the so obtained operator pencils are used later in the study of boundary value problems in domains with conical points.

For the sake of simplicity, Chapters 1-3 deal only with boundary value problems for differential equations of order $2m$, where the order of the derivatives in the operator B is less than $2m$. In Chapter 4 we generalize the results to arbitrary elliptic boundary value problems for systems of differential equations. Furthermore, a special section in Chapter 4 is dedicated to boundary value problems in the variational form.

The second part of the book (Chapters 5-8) is concerned with elliptic boundary value problems in cylinders, cones and bounded domains with conical points. Chapter 5 deals with boundary value problems in an infinite cylinder $\mathcal{C} = \{(x,t) : x \in \Omega,\ -\infty < t < +\infty\}$, where Ω is a smooth bounded domain in $\mathbb{R}^n$. First the coefficients of the differential operators are assumed to be independent from the variable t. We obtain necessary and sufficient conditions for the unique solvability of such boundary value problems in weighted Sobolev spaces, where also Sobolev spaces of small positive and nonpositive orders are involved. Furthermore, we obtain the asymptotics of the solutions at infinity and derive formulas for the coefficients in the asymptotics. These formulas contain special solutions of the homogeneous adjoint problem. Here it turns out to be an advantage that we have considered boundary value problems of the form (1), (3) with unknowns both in the domain and on the boundary. Thus, we do not have to restrict ourselves to boundary value problems for which the classical Green formula is valid. In the case of t-dependent coefficients satisfying the so-called stabilization condition at infinity we obtain the Fredholm property of the operator to the boundary value problem, regularity assertions, and a priori estimates for the solutions.

The results of Chapter 5 are applied in Chapter 6 to obtain analogous results for elliptic boundary value problems in infinite cones and bounded domains with angular or conical points. The main results concerning problems in bounded domains are the Fredholm property of the operator (4) in weighted Sobolev spaces of arbitrary integer order, regularity assertions and a priori estimates for the solutions, asymptotic decomposition of the solutions near the conical points, and formulas for the coefficients in the asymptotics. Moreover, we study the Green functions of the boundary value problem. Again for the sake of simplicity, we restrict ourselves in Chapters 5 and 6 to boundary value problems for a $2m$ order differential equation with boundary operators B_k of order less than $2m$. Chapter 7 is dedicated to the generalization of the results to elliptic boundary value problems for systems of differential equations without such restriction on the boundary conditions. Furthermore, elliptic boundary value problems in variational form are considered here.

The class of weighted Sobolev spaces used in Chapters 6 and 7 does not contain the usual Sobolev spaces without weight. Therefore, in Chapter 8 we consider the boundary value problem also in another class of weighted Sobolev spaces with so-called inhomogeneous norms which contains the usual Sobolev spaces. We investigate the solvability of the boundary value problem in such spaces and find the asymptotics of the solutions near the conical points.

The third part of the book consisting of Chapters 9 and 10 concerns boundary value problems in domains with other isolated singularities. In particular, boundary value problems in domains with point singularities of interior and exterior of a cusp type are considered. Here the solvability of the boundary value problem in special weighted Sobolev spaces depending on the geometry of the domain near the point singularities is studied.

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Part 1

Elliptic boundary value problems in domains with smooth boundary

CHAPTER 1

Boundary value problems for ordinary differential equations on the half-axis

This chapter deals with boundary value problems for linear ordinary differential equations of even order $2m$ with constant coefficients on the interval $(0, +\infty)$. It prepares the treatment of boundary value problems for partial differential operators in the half-space and in bounded domains of $\mathbb{R}^n$. We introduce the notion of regularity and show that it is necessary and sufficient for the unique solvability of the boundary value problem in Sobolev spaces of arbitrary integer order. Furthermore, we study the connections between the formally adjoint boundary value problem and the adjoint (in the functional analytic sense) problem.

1.1. The boundary value problem and its formally adjoint

In the beginning of the first section we describe the class of boundary value problems on $\mathbb{R}_+ = (0, +\infty)$ which are considered in Chapter 1. While in classical boundary value problems only an unknown function u on the semi-axis has to be found, the boundary value problems considered here contain also an additional unknown vector $\underline{u} \in \mathbb{C}^J$. We present a Green formula for these problems. This formula allows to introduce a formally adjoint problem which has the same form as the starting problem.

1.1.1. Formulation of the problem. Let

$$L(D_t) = \sum_{j=0}^{2m} a_j\, D_t^j \tag{1.1.1}$$

be a linear differential operator of order $2m$ with constant coefficients a_j, where $a_{2m} \neq 0$. Here D_t denotes the derivative $D_t = -i\partial_t = -i\, d/dt$. Furthermore, let

$$B_k(D_t) = \sum_{j=0}^{\mu_k} b_{k,j}\, D_t^j$$

$(k = 1, \dots, m + J)$ be linear differential operators of order μ_k, and let

$$C = \Big(c_{k,j}\Big)_{1 \le k \le m+J,\ 1 \le j \le J}$$

be a constant $(m + J) \times J$-matrix. Here μ_k are integer numbers. We allow μ_k to be negative. In this case the operator B_k is assumed to be identically equal to zero.

We consider the problem

$$L(D_t)\, u(t) = f(t)\ , \quad t > 0, \tag{1.1.2}$$

$$B(D_t)\, u(t)|_{t=0} + C\, \underline{u} = \underline{g}, \tag{1.1.3}$$

where $B(D_t)$ denotes the vector of the operators $B_1(D_t), \dots, B_{m+J}(D_t)$, f is a given function on $\mathbb{R}_+$, and $\underline{g}$ is a given vector from $\mathbb{C}^{m+J}$. We seek a function u on $\mathbb{R}_+$ and a vector $\underline{u} = (u_1, \dots, u_J)$ such that u is a solution of the differential equation (1.1.2), and the pair $(u, \underline{u})$ satisfies the boundary conditions (1.1.3) which can be written in the coordinate form as

$$B_k(D_t)u(t)|_{t=0} + \sum_{j=1}^{J} c_{k,j}\, u_j = g_k\,, \quad k = 1, \dots, m+J.$$

REMARK 1.1.1. Here and in the following we will not make a distinction between column and row vectors. In (1.1.3) $\underline{u}$ and $\underline{g}$ are considered as column vectors.

1.1.2. The Green formula and the formally adjoint problem. In order to define the formally adjoint problem to (1.1.2), (1.1.3), we use a modification of the classical Green formula.

First we consider the case $\mu_k < 2m$. Let

$$L^+(D_t) = \sum_{j=0}^{2m} \overline{a}_j\, D_t^j$$

be the formally adjoint operator to L. Furthermore, let $\mathcal{D}$ denote the vector

$$\mathcal{D} = (1, D_t, \dots, D_t^{2m-1}). \tag{1.1.4}$$

Then the operator $B(D_t)$ can be written in the form

$$B(D_t) = Q \cdot \mathcal{D} \tag{1.1.5}$$

(here $\mathcal{D}$ is considered as a column vector), where the elements of the $(m+J) \times 2m$ matrix

$$Q = \Big(q_{k,j}\Big)_{1 \le k \le m+J,\ 1 \le j \le 2m}$$

are defined by the coefficients of the operators B_k as follows:

$$q_{k,j} = \begin{cases} b_{k,j-1} & \text{for} \quad j = 1, \dots, \mu_k + 1\,, \\ 0 & \text{for} \qquad j > \mu_k + 1\,. \end{cases}$$

THEOREM 1.1.1. *The following Green formula is satisfied for all infinitely differentiable functions u, v on $\overline{\mathbb{R}}_+$ with compact support and all vectors $\underline{u} \in \mathbb{C}^J$, $\underline{v} \in \mathbb{C}^{m+J}$:*

$$\int_0^\infty Lu \cdot \overline{v}\, dt + \big(B(D_t)u|_{t=0} + C\,\underline{u}\,,\, \underline{v}\big)_{\mathbb{C}^{m+J}} \tag{1.1.6}$$
$$= \int_0^\infty u \cdot \overline{L^+ v}\, dt + \big((\mathcal{D}u)(0)\,,\, P(D_t)v|_{t=0} + Q^*\,\underline{v}\big)_{\mathbb{C}^{2m}} + (\underline{u}, C^*\underline{v})_{\mathbb{C}^J}\,.$$

Here $P(D_t)$ denotes the vector with the components

$$P_j(D_t) = -i \sum_{s=0}^{2m-j} \overline{a}_{j+s}\, D_t^s \quad j = 1, \dots, 2m, \tag{1.1.7}$$

and Q^*, C^* are the adjoint matrices to Q and C, respectively.

Proof: Let L_j be the following differential operators:

$$L_j = \sum_{s=0}^{j-1} a_s\, D_t^s \quad \text{for } j = 1, \ldots , 2m, \qquad L_0 = 0. \tag{1.1.8}$$

We prove by induction that

$$\int_0^\infty u\, \overline{L^+ v}\, dt = \int_0^\infty (L_j u \cdot \overline{v} - i\, D_t^j u \cdot \overline{P_j v})\, dt - \sum_{s=1}^{j} (D_t^{s-1} u)(0) \cdot \overline{(P_s v)(0)} \tag{1.1.9}$$

for smooth functions u, v with compact support. Obviously, (1.1.9) is satisfied for $j = 0$. Suppose that (1.1.9) is valid for a given nonnegative integer $j = j_0 < 2m$. Using the equations

$$\begin{aligned} L_j\, u &= -a_j\, D_t^j\, u + L_{j+1}\, u\,, \\ P_j\, v &= -i\, \overline{a}_j\, v + D_t\, P_{j+1}\, v \end{aligned}$$

and integrating by parts, we get

$$\begin{aligned} &\int_0^\infty (L_j u \cdot \overline{v} - i\, D_t^j u \cdot \overline{P_j v})\, dt \\ &= \int_0^\infty (L_{j+1} u \cdot \overline{v} - i\, D_t^{j+1} u \cdot \overline{P_{j+1} v})\, dt - (D_t^j u)(0) \cdot \overline{P_{j+1} v)(0)}\,. \end{aligned} \tag{1.1.10}$$

Consequently, (1.1.9) is satisfied for $j = j_0 + 1$ and therefore for each nonnegative integer $j \le 2m$. In particular, for $j = 2m$ we have

$$\int_0^\infty u \cdot \overline{L^+ v}\, dt = \int_0^\infty L u \cdot \overline{v}\, dt - \sum_{s=1}^{2m} (D_t^{s-1} u)(0) \cdot \overline{(P_s v)(0)}\,. \tag{1.1.11}$$

Furthermore, we have

$$\big(B(D_t)u|_{t=0}\,,\, \underline{v}\big)_{\mathbb{C}^{m+J}} = \big(Q \cdot (\mathcal{D}u)(0)\,, \underline{v}\big)_{\mathbb{C}^{m+J}} = \big((\mathcal{D}u)(0)\,,\, Q^* \underline{v}\big)_{\mathbb{C}^{2m}} \tag{1.1.12}$$

and

$$\big(C\underline{u}, \underline{v}\big)_{\mathbb{C}^{m+J}} = \big(\underline{u}, C^* \underline{v}\big)_{\mathbb{C}^J}\,. \tag{1.1.13}$$

The equalities (1.1.11)–(1.1.13) yield (1.1.6). ■

Let $P(D_t)$ be the operator given in the Green formula (1.1.6). By (1.1.7), there is the representation

$$P = T \cdot \mathcal{D}$$

with the triangular matrix

$$T = -i \begin{pmatrix} \overline{a}_1 & \cdots & \overline{a}_{2m-1} & \overline{a}_{2m} \\ \overline{a}_2 & \cdots & \overline{a}_{2m} & 0 \\ & \vdots & & \\ \overline{a}_{2m} & \cdots & 0 & 0 \end{pmatrix}, \tag{1.1.14}$$

It is natural to define the formally adjoint problem to (1.1.2)–(1.1.3) by the operators on the right-hand side of (1.1.6).

DEFINITION 1.1.1. Assume that the Green formula (1.1.6) is valid. Then the problem

$$L^+(D_t)\,v(t) = f(t) \quad \text{for } t > 0, \tag{1.1.15}$$

$$P(D_t)\,v(t)|_{t=0} + Q^*\,\underline{v} = \underline{g}\;, \qquad C^*\,\underline{v} = \underline{h} \tag{1.1.16}$$

is said to be *formally adjoint* to (1.1.2), (1.1.3).

By the representation of the elements $q_{k,j}$ of the matrix Q, the boundary conditions (1.1.16) of the formally adjoint problem have the following form

$$P_j(D_t)\,v(t)|_{t=0} + \sum_{\substack{k=1\\ \mu_k \geq j-1}}^{m+J} \bar{b}_{k,j-1}\,v_k = g_j\;, \qquad j = 1, \ldots, 2m\,,$$

$$\sum_{k=1}^{m+J} \bar{c}_{k,j} v_k = h_j\;, \qquad j = 1, \ldots, J\,.$$

The formally adjoint problem has the same structure as the starting problem. However, the number of the boundary conditions and of the unknowns is greater than in (1.1.2), (1.1.3).

1.1.3. Boundary operators of higher order. Now we consider the boundary value problem (1.1.2), (1.1.3) without the restriction $\mu_k < 2m$ on the orders of the differential operators B_k. Let κ be an integer number such that

$$\kappa \geq 2m, \qquad \kappa > \max \mu_k \text{ for } k = 1, \ldots, m+J,$$

and let $\mathcal{D}^{(\kappa)}$ be the column vector with the components $1, D_t, \ldots, D_t^{\kappa-1}$. Then the vector $B(D_t)$ can be written in the form

$$B(D_t) = Q^{(\kappa)} \cdot \mathcal{D}^{(\kappa)}\,, \tag{1.1.17}$$

where $Q^{(\kappa)}$ is a $(m+J) \times \kappa$ matrix of complex numbers. Furthermore, according to (1.1.11), we have

$$\int_0^\infty Lu \cdot \bar{v}\,dx = \int_0^\infty u \cdot \overline{L^+ v}\,dx + \left((\mathcal{D}^{(\kappa)}u)(0)\,,\,(P^{(\kappa)}v)(0)\right)_{\mathbb{C}^\kappa}\,, \tag{1.1.18}$$

where $P^{(\kappa)}$ is the vector with the components $P_1(D_t), \ldots, P_{2m}(D_t), 0, \ldots, 0$. We introduce the $(\kappa - 2m) \times \kappa$ matrix

$$R^{(\kappa)} = \begin{pmatrix} a_0 & a_1 & & \cdots & & a_{2m} & 0 & \cdots & 0 \\ 0 & a_0 & & \cdots & & a_{2m-1} & a_{2m} & \cdots & 0 \\ & & \ddots & & & & & \ddots & \\ 0 & 0 & \cdots & a_0 & a_1 & & \cdots & & a_{2m} \end{pmatrix}$$

Obviously, the vector $\mathcal{D}^{(\kappa-2m)}\,L(D_t)$ has the representation

$$\mathcal{D}^{(\kappa-2m)}\,L(D_t) = R^{(\kappa)} \cdot \mathcal{D}^{(\kappa)} \tag{1.1.19}$$

Therefore, we obtain the following Green formula which is valid for all infinitely differentiable functions u, v on $\overline{\mathbb{R}}$, $\underline{u} \in \mathbb{C}^J$, $\underline{v} \in \mathbb{C}^{m+J}$, $\underline{w} \in \mathbb{C}^{\kappa-2m}$:

$$\int_0^\infty Lu \cdot \overline{v}\, dt + \big((\mathcal{D}^{(\kappa-2m)}Lu)(0)\,,\, \underline{w}\big)_{\mathbb{C}^{\kappa-2m}} + \big((Bu)(0) + C\,\underline{u}\,,\, \underline{v}\big)_{\mathbb{C}^{m+J}} \tag{1.1.20}$$

$$= \int_0^\infty u \cdot \overline{L^+ v}\, dt + \Big((\mathcal{D}^{(\kappa)}u)(0)\,,\, (P^{(\kappa)}v)(0) + (R^{(\kappa)})^*\, \underline{w} + (Q^{(\kappa)})^*\, \underline{v}\Big)_{\mathbb{C}^\kappa}$$

$$+ \,(\underline{u}, C^* \underline{v})_{\mathbb{C}^J}\,.$$

The boundary value problem

$$L^+(D_t)v(t) = f(t) \quad \text{for } t > 0, \tag{1.1.21}$$

$$P^{(\kappa)}(D_t)v(t)\big|_{t=0} + (R^{(\kappa)})^* \underline{w} + (Q^{(\kappa)})^* \underline{v} = \underline{g}, \qquad C^* \underline{v} = \underline{h} \tag{1.1.22}$$

is said to be *formally adjoint* to problem (1.1.2), (1.1.3) with respect to the Green formula (1.1.20). In the case $\kappa = 2m$ this problem coincides with problem (1.1.15), (1.1.16).

Note that the boundary conditions (1.1.22) of the formally adjoint problem contain only derivatives up to order $2m - 1$.

1.2. Solvability of the boundary value problem on the half-axis

The goal of this section is to prove that the regularity of the boundary value problem (1.1.2), (1.1.3) is necessary and sufficient for the unique solvability of this problem in the Cartesian product of the Sobolev space $W_2^l(\mathbb{R}_+)$ with the set $\mathbb{C}^J$. We give several equivalent definitions for the regularity and show that any boundary value problem and its formally adjoint problem are simultaneously regular.

1.2.1. Sobolev spaces on the half-axis. Let $C_0^\infty(\mathbb{R}_+)$, $C_0^\infty(\overline{\mathbb{R}}_+)$ be the sets of infinitely differentiable functions on $\overline{\mathbb{R}}_+ = [0, +\infty)$ with compact support in $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$, respectively. We define the Sobolev spaces $\overset{\circ}{W}{}_2^l(\mathbb{R}_+)$ and $W_2^l(\mathbb{R}_+)$ for nonnegative integer l as the closure of $C_0^\infty(\mathbb{R}_+)$, $C_0^\infty(\overline{\mathbb{R}}_+)$ with respect to the norm

$$\|u\|_{W_2^l(\mathbb{R}_+)} = \Big(\int_0^\infty \sum_{j=0}^{l} |D_t^j\, u(t)|^2\, dt \Big)^{1/2}.$$

By Sobolev's lemma, the space $W_2^l(\mathbb{R}_+)$ is continuously imbedded into $C^{l-1}(\overline{\mathbb{R}}_+)$ [1]. Consequently, the derivatives $(D_t^j u)(0)$ $(j = 0, 1, \dots, l-1)$ at the point $t = 0$ exist for functions from $W_2^l(\mathbb{R}_+)$. The subspace $\overset{\circ}{W}{}_2^l(\mathbb{R}_+)$ can be characterized as the set of all functions $u \in W_2^l(\mathbb{R}_+)$ such that $(D_t^j u)(0) = 0$ for $j = 0, \dots, l-1$.

Analogously to $W_2^l(\mathbb{R}_+)$, the space $W_2^l(\mathbb{R})$ can be defined. Note that every function $u \in W_2^l(\mathbb{R}_+)$ can be continuously extended to a function $v \in W_2^l(\mathbb{R})$. For example, the function

$$v(t) = \begin{cases} u(t) & \text{for} \quad t > 0, \\ \chi(t) \sum_{j=0}^{l-1} \frac{1}{j!} (D_t^j u)(0)\, (it)^j & \text{for} \quad t \le 0, \end{cases} \tag{1.2.1}$$

[1] $C^{l-1}(\overline{\mathbb{R}}_+)$ denotes the space of all functions on $\overline{\mathbb{R}}_+$ which have continuous derivatives up to the order $l - 1$.

where χ is an arbitrary smooth function with compact support equal to one in the interval $(-1,+1)$, is an extension of u. This extension satisfies the inequality

$$\|v\|_{W_2^l(\mathbb{R})} \le c\,\|u\|_{W_2^l(\mathbb{R}_+)}$$

with a constant c independent of u. An equivalent norm in $W_2^l(\mathbb{R})$ for arbitrary integer l is

$$\|u\|_{H^l(\mathbb{R})} = \Big(\int_{-\infty}^{+\infty} (1+\tau^2)^l\, |(\mathcal{F}_{t\to\tau}u)(\tau)|^2\, d\tau \Big)^{1/2} .$$

Here $\mathcal{F}_{t\to\tau}$ denotes the *Fourier transformation*

$$(\mathcal{F}_{t\to\tau}u)(\tau) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{-it\tau}\, u(t)\, dt \,. \tag{1.2.2}$$

1.2.2. Regularity of the boundary value problems on the half-axis. We want to investigate the solvability of the boundary value problem (1.1.2), (1.1.3) in the space $W_2^l(\mathbb{R}_+)\times\mathbb{C}^J$. Here the notion of regularity of boundary value problems plays an important role. To introduce this notion, we denote by $\mathcal{M}^+$ the set of the so-called *stable solutions* of the homogeneous differential equation $L(D_t)\,u(t) = 0$ which tend to zero as $t \to +\infty$. Obviously, $\mathcal{M}^+$ is the linear span of the functions

$$t^s\, e^{i\tau_j t}\,, \qquad s = 0, \dots, r_j - 1\,, \tag{1.2.3}$$

where $\tau_1, \dots, \tau_\mu$ are the zeros of the polynomial

$$L(\tau) = \sum_{j=0}^{2m} a_j\, \tau^j$$

in the upper half-plane $\operatorname{Im}\tau > 0$ and r_j denotes the multiplicity of τ_j. The space $\mathcal{M}^+$ can be also characterized as the set of all solutions of the equation $L(D_t)\,u(t) = 0$ which belong to a Sobolev space $W_2^l(\mathbb{R}_+)$.

DEFINITION 1.2.1. The boundary value problem (1.1.2), (1.1.3) is said to be *regular*, if

(i) The polynomial $L(\tau)$ has no real zeros and exactly m zeros (counting multiplicity) of $L(\tau)$ lie in the upper half-plane $\operatorname{Im}\tau > 0$.

(ii) The system of the homogeneous boundary conditions (1.1.3)

$$B(D_t)\,u(t)|_{t=0} + C\,\underline{u} = 0$$

has only the trivial solution $(u, \underline{u}) = 0$ in $\mathcal{M}^+ \times \mathbb{C}^J$.

REMARK 1.2.1. In particular, from condition (ii) it follows that the equation $C\underline{u} = 0$ has only the trivial solution or, equivalently, the rank of the matrix C is equal to J.

1.2.3. Equivalent definitions of regularity.

LEMMA 1.2.1. *Let condition (i) be satisfied. We denote by* $\tau_1, \dots, \tau_\mu$ *the zeros of* $L(\tau)$ *lying in the upper half-plane* $\operatorname{Im}\tau > 0$ *and by* $r_1, \dots, r_\mu$ *their multiplicities. Then the following assertions are equivalent:*

1) *The boundary value problem (1.1.2), (1.1.3) is regular.*

2) *For every* $\underline{g} \in \mathbb{C}^{m+J}$ *there exist exactly one function* $u \in \mathcal{M}^+$ *and one vector* $\underline{u} \in \mathbb{C}^J$ *such that the boundary conditions (1.1.3) are satisfied.*
3) *The vector-polynomials* $\tau \to \big(B_k(\tau),\, c_{k,1}\,,\,\dots\,,\, c_{k,J}\big)$, $k = 1, \dots, m+J$, *are linearly independent modulo the vector-polynomial* $\tau \to \big(L_+(\tau), 0, \dots, 0\big)$, *where*
$$L_+(\tau) = (\tau - \tau_1)^{r_1} \cdots (\tau - \tau_\mu)^{r_\mu}\,.$$
This means, the equation
$$\sum_{k=1}^{m+J} \beta_k \left(B_k(\tau), c_{k,1}, \dots, c_{k,J}\right) = P(\tau)\left(L_+(\tau), 0, \dots, 0\right)$$
with an arbitrary polynomial P *implies* $\beta_1 = \ldots = \beta_{m+J} = 0$.

Proof: From (i) it follows that $r_1 + \cdots + r_\mu = m$. The set $\mathcal{M}^+$ consists of all functions of the form
$$u(t) = \sum_{j=1}^{\mu} \sum_{s=0}^{r_j-1} \alpha_{j,s}\, \partial_\tau^s\, e^{it\tau}\Big|_{\tau=\tau_j}$$
Inserting this function and an arbitrary vector $\underline{u}$ into the boundary condition (1.1.3), we get the algebraic system

$$\sum_{j=1}^{\mu} \sum_{s=0}^{r_j-1} \alpha_{j,s}\, \partial_\tau^s\, B(\tau)\Big|_{\tau=\tau_j} + C\,\underline{u} = \underline{g}\,. \tag{1.2.4}$$

with the unknowns $\alpha_{j,s}$ $(j = 1, \dots, \mu,\ s = 0, \dots, r_j - 1)$ and $\underline{u}$. Here we have used the identity $B(D_t)\partial_\tau^s e^{it\tau}|_{t=0} = \partial_\tau^s B(\tau)$. The algebraic system (1.2.4) is uniquely solvable for each vector $\underline{g}$ if and only if the corresponding homogeneous equation
$$\sum_{j=1}^{\mu} \sum_{s=0}^{r_j-1} \alpha_{j,s}\, \partial_\tau^s\, B(\tau)\Big|_{\tau=\tau_j} + C\,\underline{u} = 0\,.$$
has only the trivial solution $\alpha_{j,s} = 0$ $(j = 1, \dots, \mu,\ s = 0, \dots, r_j - 1)$, $\underline{u} = 0$. Consequently, the assertions 1) and 2) are equivalent.

Furthermore, the algebraic system (1.2.4) is uniquely solvable if and only if the homogeneous algebraic system with the transposed coefficients matrix

$$\sum_{k=1}^{m+J} \beta_k\, \partial_\tau^s\, B_k(\tau)\Big|_{\tau=\tau_j} = 0\,, \qquad j = 1, \dots, \mu;\ s = 0, \dots, r_j - 1, \tag{1.2.5}$$

$$\sum_{k=1}^{m+J} \beta_k\, c_{k,j} = 0 \qquad j = 1, \dots, J. \tag{1.2.6}$$

has only the trivial solution $\beta_1 = \ldots = \beta_{m+J} = 0$. Here the equations (1.2.5) are satisfied if and only if the numbers τ_j, $j = 1, \dots, \mu$, are zeros of the polynomial $\sum_{k=1}^{m+J} \beta_k\, B_k(\tau)$ with the multiplicities r_j or, with other words, if
$$\sum_{k=1}^{m+J} \beta_k\, B_k(\tau) = P(\tau)\, L_+(\tau).$$
This proves the equivalence of the assertions 2) and 3). ∎

Note that the coefficients of the polynomial L_+ analytically depend on the coefficients of L.

We give two examples for regular boundary value problems on the half-axis.

Example 1. The boundary value problem

$$-u''(t) + \eta^2 u(t) = f(t) \quad \text{for } t > 0\,,$$
$$\underline{a}\, u'(0) + \underline{b}\, u(0) + C\, \underline{u} = \underline{g}$$

with $\eta > 0$, $\underline{a} = (a_1, \dots, a_{J+1})$, $\underline{b} = (b_1, \dots, b_{J+1}) \in \mathbb{C}^{J+1}$ is regular if and only if the matrix

$$(\underline{a}\eta - \underline{b}\,, C) = \begin{pmatrix} a_1\eta - b_1 & c_{1,1} & \cdots & c_{1,J} \\ a_2\eta - b_2 & c_{2,1} & \cdots & c_{2,J} \\ \vdots & \vdots & & \vdots \\ a_{J+1}\eta - b_{J+1} & c_{J+1,1} & \cdots & c_{J+1,J} \end{pmatrix}$$

is regular.

Example 2. Let $L(D_t)$ be a differential operator of order $2m$ satisfying condition *(i)* and let $B_k(\tau)$, $k = 1, \dots, m$, be polynomials of the form

$$B_k(\tau) = p(\tau)\, p_k(q(\tau))\,, \tag{1.2.7}$$

where p, p_k and q are polynomials and the degree of p_k is equal to $k-1$. We suppose that

$$p(\tau_j) \neq 0 \quad \text{for } j = 1, \dots, \mu\,, \tag{1.2.8}$$

where $\tau_1, \dots, \tau_\mu$ denote the zeros of $L(\tau)$ in the upper half-plane and that

$$\begin{aligned} q(\tau_j) \neq q(\tau_k) \quad & \text{if} \quad \tau_j \neq \tau_k\,,\ j,k = 1, \dots, \mu, \\ q'(\tau_j) \neq 0 \quad & \text{if} \quad \tau_j \text{ is a zero of } L(\tau) \text{ of multiplicity } r_j > 1,\ j = 1, \dots, \mu. \end{aligned}$$

Then the boundary value problem

$$L(D_t)u(t) = f(t) \quad \text{for } t > 0\,,$$
$$B_k(D_t)u(t)|_{t=0} = g_k \quad (k = 1, \dots, m)$$

is regular.

Proof: Suppose that there exists a polynomial $P(\tau)$ such that

$$\sum_{k=1}^m \beta_k\, B_k(\tau) = p(\tau) \sum_{k=1}^m \beta_k\, p_k(q(\tau)) = P(\tau)\, L_+(\tau)$$

Then by (1.2.8), the polynomial

$$\sum_{k=1}^m \beta_k\, p_k(q(\tau)) \tag{1.2.9}$$

is divisible by $L_+(\tau)$. Since the expression (1.2.9) is a polynomial of degree $\leq m-1$ relative to $q(\tau)$, we can write this expression as a product of at most $m-1$ factors of the form $q(\tau) - c$. Hence the polynomial (1.2.9) is either equal to zero or there exists a constant c such that $q(\tau) - c$ is divisible by a product $(\tau - \tau_j)(\tau - \tau_k)$. The second possibility contradicts our assumption on q. Therefore, we get $\beta_1 = \ldots = \beta_m = 0$.

This proves the regularity of the boundary value problem. ■

In the special case $p(\tau) = 1$, $q(\tau) = \tau$, $p_k(\tau) = \tau^{k-1}$, we obtain the regularity of the *Dirichlet boundary conditions* $D_t^{k-1} u(t)|_{t=0} = g_k$, $k = 1, \dots, m$, for the operator $L(D_t)$.

1.2.4. Solvability of regular boundary value problems on the half-axis in a Sobolev space. Let l be an arbitrary integer number, $l \geq 2m$, $l \geq \max \mu_k + 1$. Obviously, the operator $(u, \underline{u}) \to (f, \underline{g})$ of the boundary value problem (1.1.2), (1.1.3) realizes a linear and continuous mapping

$$\mathcal{A} : W_2^l(\mathbb{R}_+) \times \mathbb{C}^J \to W_2^{l-2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J} \ . \tag{1.2.10}$$

If the boundary value problem (1.1.2), (1.1.3) is regular, then the operator (1.2.10) is injective. We prove that the operator $\mathcal{A}$ of a regular boundary value problem is an isomorphism.

THEOREM 1.2.1. *The following assertions are equivalent:*

1) *The boundary value problem (1.1.2), (1.1.3) is regular.*
2) *For every $f \in W_2^{l-2m}(\mathbb{R}_+)$, $\underline{g} \in \mathbb{C}^{m+J}$, $l \geq \max(2m, \mu_1 + 1, \dots, \mu_{k+J} + 1,)$ there exists exactly one solution $(u, \underline{u}) \in W_2^l(\mathbb{R}_+) \times \mathbb{C}^J$ of the problem (1.1.2), (1.1.3). In other words, the operator (1.2.10) is an isomorphism.*

Proof: First we show the implication 1) $\Longrightarrow$ 2). Let f be an arbitrary function from $W_2^{l-2m}(\mathbb{R}_+)$ and let $f_1 \in W_2^{l-2m}(\mathbb{R})$ be an extension of f to the whole t-axis. Then

$$v_1 = \mathcal{F}^{-1}_{\tau \to t}\Big(L(\tau)^{-1} \mathcal{F}_{t \to \tau} f_1\Big)$$

is a solution of the equation $L(D_t)\, v_1 = f_1$ in $\mathbb{R}$. Since $L(\tau) \neq 0$ for real τ, there exists the supremum

$$c_0 = \sup_\tau |(1 + \tau^2)^m \, L(\tau)^{-1}| \ .$$

Consequently, $v_1 \in W_2^l(\mathbb{R})$ and the restriction v of v_1 to the half-axis $\mathbb{R}_+$ belongs to $W_2^l(\mathbb{R}_+)$. By Lemma 1.2.2, there exists a solution $(w, \underline{u}) \in W_2^l(\mathbb{R}_+) \times \mathbb{C}^J$ of the problem

$$\begin{aligned} &L(D_t)\, w(t) = 0 \ , \qquad\qquad t > 0 \ , \\ &B(D_t) w(t) + C\, \underline{u} = \underline{g} - B(D_t) v(t)|_{t=0} \, . \end{aligned}$$

Hence $(u, \underline{u})$ with $u = v + w$ is a solution of the boundary value problem (1.1.2), (1.1.3). The uniqueness of this solution follows from Defintion 1.1.1.

Now we prove that assertion 2) implies 1). Let us start with the condition (i) of the regularity. From 2) it follows, in particular, that

$$\|u\|_{W_2^l(\mathbb{R}_+)} \leq c\, \|Lu\|_{W_2^{l-2m}(\mathbb{R}_+)} \tag{1.2.11}$$

for every $u \in W_2^l(\mathbb{R}_+)$ vanishing near $t = 0$ with a constant c independent of u. Assume that the polynomial $L(\tau)$ has a real zero τ_0. We set

$$u(t) = u_\varepsilon(t) \stackrel{def}{=} e^{i\tau_0 t}\, \zeta_\varepsilon(t),$$

where $\zeta_\varepsilon(t) = \zeta(\varepsilon t)$, ζ is a given function from $C_0^\infty(\mathbb{R}_+)$, and ε is a positive real number less than one. We prove that

$$c_1\,\varepsilon^{-1} \le \|u_\varepsilon\|^2_{W_2^l(\mathbb{R}_+)} \le c_2\,\varepsilon^{-1} \tag{1.2.12}$$

with positive constants c_1, c_2 depending only on l, τ_0, and ζ. Obviously,

$$\|u_\varepsilon\|^2_{W_2^l(\mathbb{R}_+)} \ge \|u_\varepsilon\|^2_{L_2(\mathbb{R}_+)} = \|\zeta_\varepsilon\|^2_{L_2(\mathbb{R}_+)} = \varepsilon^{-1}\,\|\zeta\|^2_{L_2(\mathbb{R}_+)}\,.$$

Furthermore,

$$\begin{aligned}\|u_\varepsilon\|^2_{W_2^l(\mathbb{R}_+)} \;&\le\; \sum_{j=0}^{l}\int_0^\infty \big|D_t^j\big(\zeta(\varepsilon t)\,e^{i\tau_0 t}\big)\big|^2\,dt \le \frac{1}{\varepsilon}\sum_{j=0}^{l}\int_0^\infty \big|\varepsilon^j\,D_t^j\big(\zeta(t)\,e^{i\tau_0 t/\varepsilon}\big)\big|^2\,dt\\ &\le\; \frac{1}{\varepsilon}\,c_2\,(1+\tau_0^{2l})\,\|\zeta\|^2_{W_2^l(\mathbb{R}_+)}\end{aligned}$$

This proves (1.2.12).

Since $L(D_t)\,e^{i\tau_0 t} = 0$, the function Lu_ε has the form

$$Lu_\varepsilon = e^{i\tau_0 t}\sum_{j=1}^{2m} c_j\,D_t^j\zeta_\varepsilon = e^{i\tau_0 t}\sum_{j=1}^{2m} c_j\,(-i\varepsilon)^j\,\zeta^{(j)}(\varepsilon t) \tag{1.2.13}$$

with coefficients c_j depending only on τ_0 and on the coefficients of the differential operator L. From this representation it follows that the norm of Lu_ε in $W_2^{l-2m}(\mathbb{R}_+)$ has an upper bound independent of ε. This contradicts the inequalities (1.2.11) and (1.2.12). Hence the polynomial $L(\tau)$ has no real zeros.

Furthermore, it follows from assertion 2) that the problem

$$\begin{aligned}&Lu = 0 \quad \text{in } \mathbb{R}_+,\\ &Bu|_{t=0} + C\underline{u} = \underline{g}\end{aligned}$$

has a unique solution in $W_2^l(\mathbb{R}_+)\times\mathbb{C}^J$ for every given $\underline{g}\in\mathbb{C}^{m+J}$. Since the set of the solutions of the equation $Lu = 0$ in $W_2^l(\mathbb{R}_+)$ coincides with $\mathcal{M}_+$, we have

$$u = \sum_{j=1}^{\mu}\sum_{s=0}^{r_j-1} \alpha_{j,s}\,\partial_\tau^s e^{i\tau t}\Big|_{\tau=\tau_j}\,,$$

where $\tau_1,\dots,\tau_\mu$ are the different zeros of the polynomial $L(\tau)$ in the upper half-plane $\operatorname{Im}\tau > 0$ and $r_1,\dots,r_\mu$ are their multiplicities. Inserting this function into the boundary condition (1.1.3), we get the linear algebraic system (1.2.4). This can be uniquely solvable only if the number of the unknowns coincides with the number of the equations, i.e., if $r_1+\dots+r_\mu = m$. Thus, condition (i) in Definition 1.2.1 is satisfied. Condition (ii) is obviously satisfied. ■

LEMMA 1.2.2. *Assume that the boundary value problem (1.1.2), (1.1.3) is regular. Then for every* $u\in W_2^l(\mathbb{R}_+)$, $l\ge 2m$, $l\ge \max\mu_k+1$, $\underline{u}\in\mathbb{C}^J$ *the following inequality is valid with a constant* c *independent of* u *and* $\underline{u}$:

$$\|u\|_{W_2^l(\mathbb{R}_+)} + |\underline{u}|_{\mathbb{C}^J} \le c\left(\|L(D_t)\,u\|_{W_2^{l-2m}(\mathbb{R}_+)} + \Big|B(D_t)u|_{t=0} + C\underline{u}\Big|_{\mathbb{C}^{m+J}}\right) \tag{1.2.14}$$

If c *is the best constant in (1.2.14), then* $1/c$ *is a Lipschitz-continuous function of the coefficients of the operators* L, B *and* C.

Proof: The estimate (1.2.14) immediately follows from Theorem 1.2.1 and from the closed graph theorem. Assume that the differences between the coefficients of the operator $\mathcal{A}$ and the corresponding coefficients of another operator $\mathcal{A}'$ are less than ε. Then the inequality

$$\|(\mathcal{A}-\mathcal{A}')\,(u,\underline{u})\|_{W_2^{l-2m}(\mathbb{R}_+)\times\mathbb{C}^{m+J}} \le c_1\,\varepsilon\left(\|u\|_{W_2^l(\mathbb{R}_+)} + |\underline{u}|_{\mathbb{C}^J}\right) \tag{1.2.15}$$

holds for every $u \in W_2^l(\mathbb{R}_+)$, $\underline{u} \in \mathbb{C}^J$. Here the constant c_1 depends only on l, m, μ_k, and J. Let c be the best constant in (1.2.14) for the operator $\mathcal{A}$ and c' the best constant for the operator $\mathcal{A}'$. By means of (1.2.15), we get

$$\begin{aligned}
&\|u\|_{W_2^l(\mathbb{R}_+)} + |\underline{u}|_{\mathbb{C}^J} \le c\,\|\mathcal{A}(u,\underline{u})\|_{W_2^{l-2m}(\mathbb{R}_+)\times\mathbb{C}^{m+J}}\\
&\le c\left(\|\mathcal{A}'(u,\underline{u})\|_{W_2^{l-2m}(\mathbb{R}_+)\times\mathbb{C}^{m+J}} + \|(\mathcal{A}-\mathcal{A}')(u,\underline{u})\|_{W_2^{l-2m}(\mathbb{R}_+)\times\mathbb{C}^{m+J}}\right)\\
&\le c\,(1+c_1c'\varepsilon)\,\|\mathcal{A}'(u,\underline{u})\|_{W_2^{l-2m}(\mathbb{R}_+)\times\mathbb{C}^{m+J}}\,.
\end{aligned}$$

Hence $c' \le c(1+c_1c'\varepsilon)$ and, analogously, $c \le c'(1+c_1c\varepsilon)$. Therefore, c and c' satisfy the estimate

$$\left|\frac{1}{c}-\frac{1}{c'}\right| \le c_1\,\varepsilon\,.$$

This proves the lemma. ■

1.2.5. Solvability of the formally adjoint problem. As we have shown, the regularity of the boundary value problem (1.1.2), (1.1.3) is necessary and sufficient for the unique solvability of this problem in $W_2^l(\mathbb{R}_+)\times\mathbb{C}^J$. We prove that the regularity is also necessary and sufficient for the unique solvability of the formally adjoint problem.

THEOREM 1.2.2. *The boundary value problem (1.1.2), (1.1.3) is regular if and only if the formally adjoint problem (1.1.21), (1.1.22) with respect to the Green formula (1.1.20) is regular. In particular, in the case* $\max\mu_k < 2m$ *the boundary value problems (1.1.2), (1.1.3) and (1.1.15), (1.1.16) are simultaneously regular.*

Proof: Since $L^+(\tau) = \overline{L(\overline{\tau})}$, condition (i) in Definition 1.2.1 is satisfied simultaneously for the operators L and L^+. We show that the validity of condition (ii) for the boundary value problem (1.1.2), (1.1.3) implies the validity of this condition for the formally adjoint problem (1.1.21), (1.1.22). Let $(v,\underline{w},\underline{v}) \in \mathcal{M}^+\times\mathbb{C}^{\kappa-2m}\times\mathbb{C}^{m+J}$ be a solution of the homogeneous problem (1.1.21), (1.1.22). Then the Green formula (1.1.20) yields

$$\int_0^\infty L(D_t)u\cdot\overline{v}\,dt + \big((\mathcal{D}^{(\kappa-2m)}Lu)(0),\underline{w}\big)_{\mathbb{C}^{\kappa-2m}} + \big((B(D_t)u|_{t=0}+C\underline{u}),\underline{v}\big)_{\mathbb{C}^{m+J}} = 0 \tag{1.2.16}$$

for arbitrary $u \in W_2^\kappa(\mathbb{R}_+)$, $\underline{u}\in\mathbb{C}^J$. By Lemma 1.2.1, there exist a function $u\in\mathcal{M}^+$ and a vector $\underline{u}\in\mathbb{C}^J$ such that $B(D_t)u|_{t=0}+C\underline{u}=\underline{v}$. Hence (1.2.16) implies $\underline{v}=0$. Using the fact that the last $\kappa-2m$ components of the vector $P^{(\kappa)}$ are zero and the last $\kappa-2m$ rows of the matrix $(R^{(\kappa)})^*$ form a triangular matrix with the numbers $\overline{a}_{2m}\neq 0$ in the diagonal, we can conclude from the equation

$$P^{(\kappa)}v|_{t=0} + (R^{(\kappa)})^*\,\underline{w} + (Q^{(\kappa)})^*\,\underline{v} = 0$$

that $\underline{w} = 0$. Furthermore, by Theorem 1.2.1, we can choose the function u and the vector $\underline{u}$ in (1.2.16) such that $L(D_t)u(t) = v$ for $t > 0$ and $B(D_t)u|_{t=0} + C\underline{u} = 0$. Then we get $v = 0$.

Consequently, the homogeneous problem (1.1.21), (1.1.22) has only the trivial solution. Analogously, the validity of the condition *(ii)* for the formally adjoint problem implies the validity of this condition for the problem (1.1.2), (1.1.3). ∎

COROLLARY 1.2.1. *Assume that the boundary value problem (1.1.2), (1.1.3) is regular. Then the formally adjoint problem (1.1.21), (1.1.22) is uniquely solvable in* $W_2^l(\mathbb{R}_+) \times \mathbb{C}^{m+J} \times \mathbb{C}^{\kappa-2m}$ *for any given* $f \in W_2^{l-2m}(\mathbb{R}_+)$, $\underline{g} \in \mathbb{C}^\kappa$, $\underline{h} \in \mathbb{C}^J$, $l \geq 2m$, *and the solution* $(v, \underline{w}, \underline{v})$ *satisfies the estimate*

$$\|v\|_{W_2^l(\mathbb{R}_+)} + |\underline{w}|_{\mathbb{C}^{\kappa-2m}} + |\underline{v}|_{\mathbb{C}^{m+J}} \leq c\left(\|f\|_{W_2^{l-2m}(\mathbb{R}_+)} + |\underline{g}|_{\mathbb{C}^\kappa} + |\underline{h}|_{\mathbb{C}^J}\right)$$

with a constant c independent of f, $\underline{g}$ and $\underline{h}$.

1.3. Solvability of regular problems on the half-axis in Sobolev spaces of negative order

In the sequel, we restrict ourselves to boundary value problems, where the order of differentiation in the boundary conditions is less than $2m$. Then the Green formula (1.1.6) is valid.

It was proved in the foregoing section that the operator $\mathcal{A}$ of a regular boundary value problem realizes an isomorphism $W_2^l(\mathbb{R}_+) \times \mathbb{C}^J \to W_2^{l-2m}(\mathbb{R}_+) \times \mathbb{C}^{m+j}$ for arbitrary integer $l \geq 2m$. This assertion can not be immediately extended to the case $l < 2m$, since the values of the functions $u, D_t u, \dots, D_t^{2m-1} u$ and, consequently, also the values of $B_k u$, at the point $t = 0$ do not exist for functions u from Sobolev spaces of lower order. To avoid this difficulty, we introduce spaces of pairs $(u, \underline{\phi})$, where u belongs to a Sobolev space on the interval $(0, +\infty)$ and $\underline{\phi}$ is a vector from $\mathbb{C}^{2m}$. Here the components of the vector $\underline{\phi}$ substitute the values of the derivatives of u at the point $t = 0$. We construct an extension of the operator $\mathcal{A}$ to the Cartesian product (or a subspace of the Cartesian product) of a Sobolev space of arbitrary integer order and the set $\mathbb{C}^{2m} \times \mathbb{C}^J$ and prove that this extension is an isomorphism if and only if the given boundary value problem is regular.

1.3.1. Sobolev spaces of negative order. Let l be an arbitrary nonnegative integer. Then we define the space $W_2^{-l}(\mathbb{R}_+)$ as the dual space of $\overset{\circ}{W}{}_2^l(\mathbb{R}_+)$ provided with the norm

$$\|u\|_{W_2^{-l}(\mathbb{R}_+)} = \sup\left\{|(u,v)_{\mathbb{R}_+}| \,:\, v \in \overset{\circ}{W}{}_2^l(\mathbb{R}_+),\ \|v\|_{W_2^l(\mathbb{R}_+)} = 1\right\}.$$

Here $(\cdot,\cdot)_{\mathbb{R}_+}$ denotes the extension of the scalar product in $L_2(\mathbb{R}_+)$ to pairs (u, v) from the product space $W_2^{-l}(\mathbb{R}_+) \times \overset{\circ}{W}{}_2^l(\mathbb{R}_+)$. Analogously, the space $W_2^l(\mathbb{R}_+)^*$ will be defined as the dual space of $W_2^l(\mathbb{R}_+)$.

Furthermore, we define the space $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ for arbitrary integer l and arbitrary noninteger k as follows.

If $l \geq 0$, then $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ is the set of all

$$(u, \underline{\phi}) = (u, \phi_1, \dots, \phi_k),$$

where $u \in W_2^l(\mathbb{R}_+)$ and $\underline{\phi} = (\phi_1, \dots, \phi_k)$ is a vector in $\mathbb{C}^k$ satisfying the condition

$$\phi_j = (D_t^{j-1} u)(0) \qquad \text{for } j \leq \min(k, l).$$

The norm in $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ is defined as

$$\|(u,\underline{\phi})\|_{\tilde{W}_2^{l,k}(\mathbb{R}_+)} = \|u\|_{W_2^l(\mathbb{R}_+)} + |\underline{\phi}|_{\mathbb{C}^k}\,. \tag{1.3.1}$$

Since only the components ϕ_j with index $j > l$ can be chosen independently of u, we can identify the space $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ with $W_2^l(\mathbb{R}_+)$ if $l \geq k$ and with $W_2^l(\mathbb{R}_+) \times \mathbb{C}^{k-l}$ if $0 \leq l < k$.

In the case $l < 0$ we set $\tilde{W}_2^{l,k}(\mathbb{R}_+) = W_2^{-l}(\mathbb{R}_+)^* \times \mathbb{C}^k$ and

$$\|(u,\underline{\phi})\|_{\tilde{W}_2^{l,k}(\mathbb{R}_+)} = \|u\|_{W_2^{-l}(\mathbb{R}_+)^*} + |\underline{\phi}|_{\mathbb{C}^k}\,. \tag{1.3.2}$$

In particular, with this notation, we have

$$\tilde{W}_2^{l,0}(\mathbb{R}_+) = \begin{cases} W_2^l(\mathbb{R}_+) & \text{for } l \geq 0, \\ W_2^{-l}(\mathbb{R}_+)^* & \text{for } l < 0. \end{cases}$$

Note that the space $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ is complete for arbitrary integer k, l, $k \geq 0$. It can be easily seen that the space $\tilde{W}_2^{l_1,k}(\mathbb{R}_+)$ is continuously imbedded into $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ if $l_1 \geq l$. Moreover, the following assertion holds.

LEMMA 1.3.1. *The space $\tilde{W}_2^{l_1,k}(\mathbb{R}_+)$ is dense in $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ if $l_1 \geq l$.*

Proof: It suffices to prove the lemma for $l_1 = l+1$. For $l < 0$ and $l \geq k$ the assertion of the lemma is trivial. Therefore, we restrict ourselves to the case $0 \leq l < k$ in the proof. Let

$$(u,\underline{\phi}) = (u,\, u(0),\, \dots\,, (D_t^{l-1}u)(0), \phi_{l+1}\,, \dots\,, \phi_k)$$

be an arbitrary element from $\tilde{W}_2^{l,k}(\mathbb{R}_+)$. Since $W_2^{l+1}(\mathbb{R}_+)$ is dense in $W_2^l(\mathbb{R}_+)$, there exists a sequence $\{u_j\}_{j=1}^\infty \subset W_2^{l+1}(\mathbb{R}_+)$ which converges to u in $W_2^l(\mathbb{R}_+)$. We set $c_j = (D_t^l u_j)(0)$ and define the functions $u_{j,\varepsilon}$ by the equality

$$u_{j,\varepsilon}(t) = u_j(t) - \frac{1}{l!}(c_j - \phi_{l+1})\,(it)^l\,\chi(\frac{t}{\varepsilon})\,,$$

where χ is an arbitrary smooth function with compact support equal to one in the interval $0 \leq t \leq 1$, and ε is a positive integer. It can be easily verified that $(D_t^l u_{j,\varepsilon})(0) = \phi_{l+1}$ and $u_{j,\varepsilon} \to u_j$ in $W_2^l(\mathbb{R}_+)$ as $\varepsilon \to 0$. Consequently,

$$\{(u_{j,\varepsilon}\, u_{j,\varepsilon}(0), \dots\,, (D_t^l u_{j,\varepsilon})(0), \phi_{l+2}\,, \dots\,, \phi_k)\}_{j=1}^\infty$$

converges to $(u,\underline{\phi})$ in $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ as $\varepsilon \to 0$. This proves the lemma. ∎

COROLLARY 1.3.1. *The set*

$$\big\{\big(u, u(0), (D_t u)(0), \dots\,, (D_t^{k-1}u)(0)\big) \;:\; u \in C_0^\infty(\overline{\mathbb{R}}_+)\big\} \tag{1.3.3}$$

is dense in $\tilde{W}_2^{l,k}(\mathbb{R}_+)$.

Proof: For $l \geq 2m$ this assertion follows immediately from the density of $C_0^\infty(\overline{\mathbb{R}}_+)$ in $W_2^l(\mathbb{R}_+)$. If $l < 2m$, then by Lemma 1.3.1, the space $\tilde{W}_2^{2m,k}(\mathbb{R}_+)$ is dense in $\tilde{W}_2^{l,k}(\mathbb{R}_+)$, while the set (1.3.3) is dense in $\tilde{W}_2^{2m,k}(\mathbb{R}_+)$. This implies the density of the set (1.3.3) in $\tilde{W}_2^{l,k}(\mathbb{R}_+)$. ∎

1.3.2. Extension of the operator of the boundary value problem to Sobolev spaces of arbitrary order. In the following, we assume that $\operatorname{ord} B_k < 2m$ for $k = 1, \ldots, m+J$. Since the space $\tilde{W}_2^{l,2m}(\mathbb{R}_+)$ can be identified with $W_2^l(\mathbb{R}_+)$ for $l \geq 2m$ (by means of the bijection $\big(u, (\mathcal{D}u)(0)\big) \leftrightarrow u$), the operator (1.2.10) can be considered as a linear and continuous mapping

$$\mathcal{A} : \tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J \to W_2^{l-2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J}, \quad l \geq 2m. \tag{1.3.4}$$

Now we construct an extension of this operator to the space $\tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ with $l < 2m$. We start with the operator

$$\tilde{W}_2^{l,2m}(\mathbb{R}_+) \ni \big(u, (\mathcal{D}u)(0)\big) \to Lu \in W_2^l(\mathbb{R}_+), \quad l \geq 2m, \tag{1.3.5}$$

which will be also denoted by L in what follows.

LEMMA 1.3.2. *The operator (1.3.5) can be uniquely extended to a continuous operator*

$$L : \tilde{W}_2^{l,2m}(\mathbb{R}_+) \to W_2^{2m-l}(\mathbb{R}_+)^* \tag{1.3.6}$$

with arbitrary integer $l < 2m$. For $(u, \underline{\phi}) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+)$, $l \leq 0$, the functional $L(u, \underline{\phi}) = f$ is given by the equality

$$(f, v)_{\mathbb{R}_+} = (u, L^+ v)_{\mathbb{R}_+} + \big(\underline{\phi}, (Pv)(0)\big)_{\mathbb{C}^{2m}}, \qquad v \in W_2^{2m-l}(\mathbb{R}_+), \tag{1.3.7}$$

while in the case $0 < l < 2m$ we have

$$(f, v)_{\mathbb{R}_+} = \int_0^\infty \big(L_l u \cdot \overline{v} - i\, D_t^l u \cdot \overline{P_l v}\big)\, dt + \sum_{j=l+1}^{2m} \phi_j \overline{(P_j v)(0)} \tag{1.3.8}$$

for all $v \in W_2^{2m-l}(\mathbb{R}_+)$. (Here P_l, L_l are the differential operators defined by (1.1.7) and (1.1.8), respectively.)

Proof: Due to (1.1.9) and (1.1.11), the operator L defined by (1.3.7) and (1.3.8) is a continuous extension of the operator (1.3.5). The uniqueness follows from the density of the space $\tilde{W}_2^{2m,2m}(\mathbb{R}_+)$ in $\tilde{W}_2^{l,2m}(\mathbb{R}_+)$ (see Lemma 1.3.1). ∎

Using the equality (1.1.5), we can extend the boundary condition (1.1.3) to pairs $(u, \underline{\phi}) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+)$ with $l < 2m$. This extension is given by

$$Q\underline{\phi} + C\underline{u} = \underline{g}.$$

Thus, the following theorem holds.

THEOREM 1.3.1. *If $\mu_k < 2m$ for $k = 1, \ldots, m+J$, then the operator (1.3.4) can be uniquely extended to a continuous operator from the space $\tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J$ with arbitrary integer $l < 2m$ into $W_2^{2m-l}(\mathbb{R}_+)^* \times \mathbb{C}^{m+J}$ This extension has the form*

$$\big(u, \underline{\phi}, \underline{u}\big) \to \big(L(u, \underline{\phi}), Q\underline{\phi} + C\underline{u}\big), \tag{1.3.9}$$

where the functional $f = L(u, \underline{\phi})$ is defined by (1.3.7) for $l \leq 0$ and by (1.3.8) for $0 < l < 2m$.

By Theorem 1.3.1, the operator of the boundary value problem (1.1.2), (1.1.3) realizes a continuous mapping

$$\mathcal{A} : \tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J \to \tilde{W}_2^{l-2m,0}(\mathbb{R}_+) \times \mathbb{C}^{m+J} \tag{1.3.10}$$

for arbitrary integer l.

REMARK 1.3.1. For $(u, \underline{\phi}, \underline{u}) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J$, $l \le 0$, the element $(f, \underline{g}) = \mathcal{A}(u, \underline{\phi}, \underline{u})$ is defined by the equality

(1.3.11)
$$(f, v)_{\mathbb{R}_+} + \big(\underline{g}, \underline{v}\big)_{\mathbb{C}^{m+J}} = (u, L^+ v)_{\mathbb{R}_+} + \big(\underline{\phi},\ (Pv)(0) + Q^* \underline{v}\big)_{\mathbb{C}^{2m}} + \big(\underline{u}, C^* \underline{v}\big)_{\mathbb{C}^J},$$

where v, $\underline{v}$ are arbitrary elements of $W_2^{2m-l}(\mathbb{R}_+)$ and $\mathbb{C}^{m+J}$, respectively. This means the operator $\mathcal{A} : (u, \underline{\phi}, \underline{u}) \to (f, \underline{g})$ is adjoint to the operator

$$\mathcal{A}^+ \ :\ W_2^{2m-l}(\mathbb{R}_+) \times \mathbb{C}^{m+J} \to W_2^{-l}(\mathbb{R}_+) \times \mathbb{C}^{2m} \times \mathbb{C}^J \tag{1.3.12}$$

of the formally adjoint problem (1.1.15), (1.1.16) for $l \le 0$.

REMARK 1.3.2. The assertions of Lemma 1.3.2 and Theorem 1.3.1 can be extended to real l, $l \ne \frac{1}{2}, \frac{3}{2}, \dots, 2m - \frac{1}{2}$. To this end, we define the space $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ for arbitrary real $l \ge 0$ as the closure of the set (1.3.3) with respect to the norm (1.3.1) and for real $l < 0$ as the closure of the set (1.3.3) with respect to the norm (1.3.2). Obviously, the mapping $(u, \underline{\phi}) \to Q\underline{\phi}$ is continuous from $\tilde{W}_2^{l,2m}(\mathbb{R}_+)$ into $\mathbb{C}^{m+J}$ for arbitrary real l and coincides with the mapping $(u, \underline{\phi}) \to (Bu)(0)$ for $l \ge 2m$.

Since the spaces $W_2^l(\mathbb{R}_+)$ and $\overset{\circ}{W}{}_2^l(\mathbb{R}_+)$ coincide for $0 \le l \le 1/2$, we have $\tilde{W}_2^{l,0}(\mathbb{R}_+) = W_2^l(\mathbb{R}_+)$ for $l \ge -1/2$. Furthermore, every function from $W_2^l(\mathbb{R}_+)$, $l < 1/2$, can be extended by zero to a function from $W_2^l(\mathbb{R})$ (see, e.g., [**242**]). From this it follows that the operators L, L^+ continuously map $W_2^l(\mathbb{R}_+)$ into $\tilde{W}_2^{l-2m,0}(\mathbb{R}_+)$ if $l > 2m - 1/2$. Analogously, the operators L_j defined by (1.1.8) continuously map $W_2^l(\mathbb{R}_+)$ into $\tilde{W}_2^{l-j+1,0}(\mathbb{R}_+)$ if $l > j - 3/2$. Therefore, the operator $\big(u, (\mathcal{D}u)(0)\big) \to Lu$ is continuous from $\tilde{W}_2^{l,2m}(\mathbb{R}_+)$ into $\tilde{W}_2^{l-2m,0}(\mathbb{R}_+)$ for $l > 2m - 1/2$. This operator can be uniquely extended to a continuous mapping $\tilde{W}_2^{l,2m}(\mathbb{R}_+) \ni (u, \underline{\phi}) \to f \in W_2^{2m-l}(\mathbb{R}_+)^*$ with $l < 2m - 1/2$, $l \ne \frac{1}{2}, \frac{3}{2}, \dots, 2m - \frac{3}{2}$. Here the functional f is defined by (1.3.7) if $l < 1/2$ and by the equality

$$(f, v)_{\mathbb{R}_+} = (L_j u, v)_{\mathbb{R}_+} - i(D_t^j u, P_j v)_{\mathbb{R}_+} + \sum_{s=j+1}^{2m} \phi_s \overline{(P_s v)(0)}, \quad v \in W_2^{2m-l}(\mathbb{R}_+),$$

if $j - 1/2 < l < j + 1/2$, $j = 1, 2, \dots, 2m - 1$.

1.3.3. Bijectivity of the operator of the boundary value problem. For the proof of the bijectivity of the operator $\mathcal{A} : \tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J \to \tilde{W}_2^{l-2m,0}(\mathbb{R}_+) \times \mathbb{C}^{m+J}$, $l < 2m$, we need the following regularity assertions.

LEMMA 1.3.3. *Let L be the differential operator (1.1.1) and let l and q be arbitrary integer numbers. We suppose that $L(\tau)$ has no real zeros. If $u \in W_2^l(\mathbb{R}_+)$ is a weak solution of the differential equation $Lu = f$ in $\mathbb{R}_+$, where $f \in W_2^{q-2m}(\mathbb{R}_+)$, then $u \in W_2^q(\mathbb{R}_+)$.*

Proof: Let $f_1 \in W_2^{q-\mu}(\mathbb{R})$ be an extension of f to the whole real t-axis. If $q - 2m > 0$, then this extension can be obtained analogously to (1.2.1), and in the case $q - 2m \le 0$ the distribution f can be extended to the following distribution $f_1 \in W_2^{q-\mu}(\mathbb{R})$:

$$(f_1, \varphi)_{\mathbb{R}} = \Big(f\,,\ \phi|_{\mathbb{R}_+} - \chi \sum_{j=0}^{2m-q-1} \frac{1}{j!} (D_t^j \varphi)(0) (it)^j\Big), \qquad \varphi \in W_2^{2m-q}(\mathbb{R}),$$

where χ is an arbitrary infinitely differentiable function on $\overline{\mathbb{R}}_+$ with compact support equal to one in a neighbourhood of zero.

Since $L(\tau) \neq 0$ for real τ, the function $v = \mathcal{F}^{-1}_{\tau\to t}\big(L(\tau)^{-1}\,\mathcal{F}_{t\to\tau} f_1\big)\big|_{t>0}$ is a solution of the equation $Lu = f$ and belongs to the space $W_2^q(\mathbb{R}_+)$. Hence $u - v$ is a solution of the homogeneous equation $L(D_t)\,(u-v) = 0$ on $\mathbb{R}_+$ and can be written as a linear combination of the functions (1.2.3). (Note that functions of the form (1.2.3) with $\operatorname{Im}\tau_j < 0$ do not belong to a Sobolev space.) Consequently, $u - v$ lies in a Sobolev space of arbitrary order. This proves the lemma. ∎

Applying the assertion of Lemma 1.3.3 to the operator (1.3.6), we obtain the following result.

LEMMA 1.3.4. *Suppose that the polynomial $L(\tau)$ has no real zeros. If $(u,\underline{\phi}) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+)$ is a solution of the equation $L(u,\underline{\phi}) = f$, where $f \in \tilde{W}_2^{q-2m,0}(\mathbb{R}_+)$, then $(u,\underline{\phi}) \in \tilde{W}_2^{q,2m}(\mathbb{R}_+)$.*

Proof: It suffices to prove the assertion for $q = l+1$. For $q > l+1$ the assertion follows by induction. Let $(u,\phi) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+)$ be a solution of the equation $L(u,\underline{\phi}) = f$. Then by (1.3.7), (1.3.8), and (1.1.9) we have

$$(f,v)_{\mathbb{R}_+} = (u, L^+ v)_{\mathbb{R}_+}$$

for every function $v \in C_0^\infty(\mathbb{R}_+)$. This means, u is a weak solution of the equation $Lu = f$ in $\mathbb{R}_+$. Therefore, from Lemma 1.3.3 it follows that $u \in W_2^{l+1}(\mathbb{R}_+)$. If $l < 0$ or $l \geq 2m$, this automatically implies $(u,\phi) \in \tilde{W}_2^{l+1,2m}(\mathbb{R}_+)$. Now let l be a nonnegative integer less than $2m$. Due to the definition of the space $\tilde{W}_2^{l,2m}(\mathbb{R}_+)$, we have $\phi_j = (D_t^{j-1}u)(0)$ for $j = 1,\dots,l$ and it remains to prove that $\phi_{l+1} = (D_t^l u)(0)$. By (1.1.10), we get

$$\begin{aligned}(1.3.13)\quad (f,v)_{\mathbb{R}_+} - \int_0^\infty \Big(L_{l+1}(D_t)u\cdot\overline{v} - i\,D_t^{l+1}u\cdot\overline{P_{l+1}(D_t)v}\Big)\,dt - \sum_{j=l+2}^{2m}\phi_j\,\overline{(P_j v)(0)} \\ = \big(\phi_{l+1} - (D_t^l u)(0)\big)\cdot\overline{(P_{l+1}v)(0)}\end{aligned}$$

for each $v \in W_2^{2m-l}(\mathbb{R}_+)$. Under the assumption of the lemma, the left-hand side of (1.3.13) defines a linear and continuous functional on $W_2^{2m-l-1}(\mathbb{R}_+)$, since the order of P_{l+1} is equal to $2m-l-1$ and the orders of the operators $P_{l+2}, \dots P_{2m}$ are less than $2m-l-1$. Hence the right-hand side of (1.3.13) is also a linear and continuous functional on $W_2^{2m-l-1}(\mathbb{R}_+)$. However, this is only possible if $\phi_{l+1} = (D_t^l u)(0)$. The proof is complete. ∎

Now it is easy to prove the following theorem.

THEOREM 1.3.2. *Let $\mu_k < 2m$ for $k = 1,\dots,m+J$.*

1) *If the boundary value problem (1.1.2), (1.1.3) is regular, then the operator (1.3.10) is an isomorphism for arbitrary integer l. Furthermore, for every triple $(u,\underline{\phi},\underline{u}) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J$ the following estimate is valid with a constant c independent of $(u,\underline{\phi},\underline{u})$:*

$$(1.3.14)\qquad \|(u,\underline{\phi})\|_{\tilde{W}_2^{l,2m}(\mathbb{R}_+)} + |\underline{u}|_{\mathbb{C}^J} \leq c\,\|\mathcal{A}(u,\underline{\phi},\underline{u})\|_{\tilde{W}_2^{l-2m,0}(\mathbb{R}_+)\times\mathbb{C}^{m+J}}$$

If c is the best constant in (1.3.14), then $1/c$ is a Lipschitz-continuous function relative to the coefficients of the operator L, B, and C.

2) *Suppose estimate (1.3.14) is satisfied for all $(u,\underline{\phi}) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+)$, $\underline{u} \in \mathbb{C}^J$, where l is a given integer number. Then the polynomial $\overline{L}(\tau)$ has no real zeros and condition (ii) in Definion 1.2.1 is satisfied. If, moreover, the operator (1.3.10) is surjective, then the boundary value problem (1.1.2), (1.1.3) is regular.*

Proof: 1) For $l \le 0$ the operator (1.3.10) is adjoint to the operator (1.3.12) of the formally adjoint problem (1.1.15), (1.1.16). By Theorem 1.2.1 and Theorem 1.2.2, the operator (1.3.12) is an isomorphism if the problem (1.1.2), (1.1.3) is regular. Hence the equation $\mathcal{A}(u,\underline{\phi},\underline{u}) = (f,\underline{g})$ is uniquely solvable in $\tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J$ for every given $f \in W_2^{2m-l}(\mathbb{R}_+)^*$, $\underline{g} \in \mathbb{C}^{m+J}$ if $l \le 0$. From this and from Lemma 1.3.4 it follows that this equation is uniquely solvable in $\tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^J$ for $0 \le l \le 2m$. Since the space $\tilde{W}_2^{l,k}(\mathbb{R}_+)$ is complete, we get the estimate (1.3.14). The assertion on the constant c in (1.3.14) can be obtained analogously to Lemma 1.2.2.

2) Suppose that (1.3.14) is satisfied for all $(u,\underline{\phi}) \in \tilde{W}_2^{l,2m}(\mathbb{R}_+)$ and $\underline{u} \in \mathbb{C}^J$. Then there exists a constant c independent of u such that

$$\|u\|_{\tilde{W}_2^{l,0}(\mathbb{R}_+)} \le c\, \|Lu\|_{\tilde{W}_2^{l-2m,0}(\mathbb{R}_+)} \tag{1.3.15}$$

for all $u \in C_0^\infty(\mathbb{R}_+)$. In the proof of Theorem 1.2.1 it was shown for the case $l \ge 2m$ that the last inequality implies the nonexistence of real zeros of the polynomial $L(\tau)$. We show that this is also true in the case $l < 2m$. Let τ_0 be a real zero of the polynomial $L(\tau)$. As in the proof of Theorem 1.2.1, we set

$$u(t) = u_\varepsilon(t) \stackrel{def}{=} e^{i\tau_0 t}\, \zeta_\varepsilon(t),$$

where $\zeta_\varepsilon(t) = \zeta(\varepsilon t)$, ζ is a function from $C_0^\infty(\mathbb{R}_+)$, and ε is a positive real number less than one. According to (1.2.12), there exist positive constants c_1, c_2, independent of ε, such that

$$c_1\, \varepsilon^{-1/2} \le \|u_\varepsilon\|_{\tilde{W}_2^{l,0}(\mathbb{R}_+)} \le c_2\, \varepsilon^{-1/2} \tag{1.3.16}$$

for $l \ge 0$. Using these inequalities, we get in the case $l < 0$

$$\begin{aligned} \|u_\varepsilon\|_{\tilde{W}_2^{l,0}(\mathbb{R}_+)} &= \|u_\varepsilon\|_{W_2^{-l}(\mathbb{R}_+)^*} = \sup_{v \in W_2^{-l}(\mathbb{R}_+),\ v \neq 0} \frac{|(u_\varepsilon, v)_{\mathbb{R}_+}|}{\|v\|_{W_2^{-l}(\mathbb{R}_+)}} \\ &\ge \frac{\|u_\varepsilon\|^2_{L_2(\mathbb{R}_+)}}{\|u_\varepsilon\|_{W_2^{-l}(\mathbb{R}_+)}} \ge c\,\varepsilon^{-1/2}, \end{aligned}$$

where c is a positive constant independent of ε. Furthermore, for $l < 0$ we have

$$\|u_\varepsilon\|_{\tilde{W}_2^{l,0}(\mathbb{R}_+)} \le \|u_\varepsilon\|_{L_2(\mathbb{R}_+)} = \|\zeta_\varepsilon\|_{L_2(\mathbb{R}_+)} = \varepsilon^{-1/2}\, \|\zeta\|_{L_2(\mathbb{R}_+)}\,.$$

Hence (1.3.16) is valid for arbitrary integer l.

By (1.2.13), the function Lu_ε is a sum of functions of the form $c_j\, \varepsilon^j\, \zeta^{(j)}(t)\, e^{i\tau_0 t}$, $j = 1, \dots, 2m$. Therefore, it follows from (1.3.16) that the norm of Lu_ε in the space $\tilde{W}_2^{l-2m,0}(\mathbb{R}_+)$ is bounded by a constant independent of ε. This contradicts (1.3.15) and (1.3.16). Consequently, the polynomial $L(\tau)$ has no real zeros.

Moreover, from (1.3.14) we conclude that the kernel of the operator (1.3.10) is trivial and, therefore, the homogeneous boundary value problem (1.1.2), (1.1.3) has only the trivial solution in $\mathcal{M}^+ \times \mathbb{C}^J$. Thus, condition (ii) of Definition 1.2.1 is satisfied.

Suppose that the operator (1.3.10) is surjective and estimate (1.3.14) is valid. Then the operator (1.3.10) is an isomorphism for the given integer $l = l_0$, and from Lemma 1.3.4 it follows that the operator (1.3.10) is an isomorphism for every $l \geq l_0$. In particular, for $l \geq 2m$ this means that the boundary value problem (1.1.2), (1.1.3) is uniquely solvable in $W_2^l(\mathbb{R}_+) \times \mathbb{C}^J$ for every $f \in W_2^{l-2m}(\mathbb{R}_+)$, $\underline{g} \in \mathbb{C}^{m+J}$. According to Theorem 1.2.1, this implies the regularity of the problem (1.1.2), (1.1.3). The proof of the theorem is complete. ■

REMARK 1.3.3. The assertions of Theorems 1.3.1, 1.3.2 are true if $\mu_k < 2m$ for $k = 1, \dots, m + J$. If this condition is not satisfied, then $\mathcal{A}$ can be continuously extended to an operator

$$\mathcal{A} : \tilde{W}_2^{l,\kappa}(\mathbb{R}_+) \times \mathbb{C}^J \to \tilde{W}_2^{l-2m,\kappa-2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J}, \tag{1.3.17}$$

where l, κ are arbitrary integer numbers, $\kappa \geq 2m$, $\kappa > \max \mu_k$. A precise description of this operator is given (for problems in n-dimensional bounded domains) in Chapter 4. Analogously to Theorem 1.3.2, it can be shown that the operator (1.3.17) realizes an isomorphism if and only if problem (1.1.2), (1.1.3) is regular.

1.4. Properties of the operator adjoint to the operator of the boundary value problem

In this section we study the operator $\mathcal{A}^*$ which is adjoint to the operator $\mathcal{A}$ in the functional analytic sense. As before, we restrict ourselves to boundary value problems with differential operators of order less than $2m$ in the boundary conditions. We show that there are close relations between the operator $\mathcal{A}^*$ and the operator $\mathcal{A}^+$ of the formally adjoint boundary value problem (1.1.15), (1.1.16) (see Theorem 1.4.1).

1.4.1. Relations between the adjoint and the formally adjoint operator. Applying Theorem 1.3.1 to the formally adjoint problem, we can extend the operator (1.3.12) with $l \leq 0$ to a continuous operator

$$\mathcal{A}^+ : \tilde{W}_2^{2m-l,2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J} \to W_2^l(\mathbb{R}_+)^* \times \mathbb{C}^{2m} \times \mathbb{C}^J \tag{1.4.1}$$

with $l \geq 0$. This extension is given by

$$(v, \underline{\psi}, \underline{v}) \to \left(L^+(v, \underline{\psi}),\ T\underline{\psi} + Q^*\underline{v},\ C^*\underline{v}\right),$$

where T is the matrix (1.1.14). In the case $l \geq 2m$ the functional $L^+(v, \underline{\psi}) = f \in W_2^l(\mathbb{R}_+)^*$ is defined as

$$(f, u)_{\mathbb{R}_+} = (v, Lu)_{\mathbb{R}_+} - \left(\underline{\psi},\ T^*(\mathcal{D}u)(0)\right)_{\mathbb{C}^{2m}}, \qquad u \in W_2^l(\mathbb{R}_+) \tag{1.4.2}$$

(cf. Lemma 1.3.2).

Now we consider the adjoint operator

$$\mathcal{A}^* : W_2^{l-2m}(\mathbb{R}_+)^* \times \mathbb{C}^{m+J} \ni (v, \underline{v}) \to (F, \underline{h}) \in W_2^l(\mathbb{R}_+)^* \times \mathbb{C}^J, \tag{1.4.3}$$

$l \geq 2m$, to the operator (1.2.10) of the boundary value problem (1.1.2), (1.1.3). The mapping (1.4.3) is defined by the equality

$$(u, F)_{\mathbb{R}_+} + (\underline{u}, \underline{h})_{\mathbb{C}^J} = (Lu, v)_{\mathbb{R}_+} + \left((Bu|_{t=0} + C\underline{u}), \underline{v}\right)_{\mathbb{C}^{m+J}}, \tag{1.4.4}$$

where u is an arbitrary function from $W_2^l(\mathbb{R}_+)$ and $\underline{u} \in \mathbb{C}^J$ is and arbitrary vector. The following theorem describes a connection between the operators $\mathcal{A}^*$ and $\mathcal{A}^+$.

THEOREM 1.4.1. *Let $\mu_k < 2m$ for $k = 1, \dots, m+J$, $f \in W_2^l(\mathbb{R}_+)^*$, $l \geq 2m$, $\underline{g} \in \mathbb{C}^{2m}$, and $\underline{h} \in \mathbb{C}^J$. Furthermore, let the functional $F \in W_2^l(\mathbb{R}_+)^*$ be defined by the equality*

$$(u, F)_{\mathbb{R}_+} = (u, f)_{\mathbb{R}_+} + \big((\mathcal{D}u)(0), \underline{g}\big)_{\mathbb{C}^{2m}}, \quad u \in W_2^l(\mathbb{R}_+). \tag{1.4.5}$$

Then $(v, \underline{\psi}, \underline{v}) \in \tilde{W}_2^{2m-l,2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ is a solution of the equation

$$\mathcal{A}^+ (v, \underline{\psi}, \underline{v}) = (f, \underline{g}, \underline{h})$$

if and only if $(v, \underline{v})$ is a solution of the equation

$$\mathcal{A}^* (v, \underline{v}) = (F, \underline{h})$$

and $\underline{\psi} = T^{-1} (\underline{g} - Q^* \underline{v})$.

Proof: 1) Let $(v, \underline{\psi}, \underline{v}) \in \tilde{W}_2^{2m-l,2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ be a solution of the equation $\mathcal{A}^+ (v, \underline{\psi}, \underline{v}) = (f, \underline{g}, \underline{h})$, i.e.,

$$(u, f)_{\mathbb{R}_+} = (Lu, v)_{\mathbb{R}_+} - \big((\mathcal{D}u)(0), T\underline{\psi}\big)_{\mathbb{C}^{2m}}, \qquad u \in W_2^l(\mathbb{R}_+), \tag{1.4.6}$$

$$T\underline{\psi} + Q^*\underline{v} = \underline{g}, \quad C^*\underline{v} = \underline{h}. \tag{1.4.7}$$

Then it follows from (1.4.5) and (1.4.6) that

$$(u, F)_{\mathbb{R}_+} = (Lu, v)_{\mathbb{R}_+} + \big((\mathcal{D}u)(0),\, \underline{g} - T\underline{\psi}\big)_{\mathbb{C}^{2m}} \quad \text{for } u \in W_2^l(\mathbb{R}_+). \tag{1.4.8}$$

This together with (1.4.7) and with the equality $B = Q \cdot \mathcal{D}$ implies (1.4.4). Hence $(v, \underline{v})$ is a solution of the equation $\mathcal{A}^* (v, \underline{v}) = (F, \underline{h})$.

2) Analogously, if $(v, \underline{v}) \in W_2^{l-2m}(\mathbb{R}_+)^* \times \mathbb{C}^{m+J}$ is a solution of the equation $\mathcal{A}^*(v, \underline{v}) = (F, \underline{h})$ and $\underline{\psi} = T^{-1}(\underline{g} - Q^*\underline{v})$, we obtain (1.4.7) and (1.4.8). Using the representation (1.4.5) for the functional F, we arrive at (1.4.6). Thus, $(v, \underline{\psi}, \underline{v})$ is a solution of the equation $\mathcal{A}^+(v, \underline{\psi}, \underline{v}) = (f, \underline{g}, \underline{h})$. The proof is complete. ■

1.4.2. Bijectivity of the adjoint operator. If the boundary value problem (1.1.2), (1.1.3) is regular, then the operator (1.4.3) is an isomorphism for arbitrary integer $l \geq 2m$. Now we consider the restriction of the adjoint operator $\mathcal{A}^*$ to Sobolev spaces of nonnegative order.

Motivated by Theorem 1.4.1, we introduce the space $D_2^{l,k}(\mathbb{R}_+)$ as follows. Let l, k be arbitrary integers, $k \geq 0$, $l \geq -k$. Then we define the space $D_2^{l,k}(\mathbb{R}_+)$ as the set of all functionals $F \in W_2^k(\mathbb{R}_+)^*$ which have the form

$$(u, F)_{\mathbb{R}_+} = (u, f)_{\mathbb{R}_+} + \big((\mathcal{D}^{(k)}u)(0), \underline{g}\big)_{\mathbb{C}^l}, \quad u \in W_2^k(\mathbb{R}_+), \tag{1.4.9}$$

where $f \in \tilde{W}_2^{l,0}(\mathbb{R}_+)$ (i.e. $f \in W_2^l(\mathbb{R}_+)$ for $l \geq 0$, $f \in W_2^{-l}(\mathbb{R}_+)^*$ for $l < 0$), $\underline{g} \in \mathbb{C}^k$, and $\mathcal{D}^{(k)}$ denotes the vector

$$\mathcal{D}^{(k)} = (1, D_t, \dots, D_t^{k-1}) \text{ if } k = 1, 2, \dots, \quad \mathcal{D}^{(0)} = 0\,. \tag{1.4.10}$$

The norm of the functional F in $D_2^{l,k}(\mathbb{R}_+)$ is defined in a natural way as the infimum of the sum

$$\|f\|_{\tilde{W}_2^{l,0}(\mathbb{R}_+)} + |\underline{g}|_{\mathbb{C}^k}\,,$$

where f and $\underline{g}$ satisfy (1.4.9).

REMARK 1.4.1. If l is a negative integer, then the functional

$$u \to \sum_{j=1}^{-l} (D_t^{j-1}u)(0)\,\overline{g_j}$$

belongs to $W_2^{-l}(\mathbb{R}_+)^*$ and the space $D_2^{l,k}(\mathbb{R}_+)$ can be even defined as the set of all functionals $F \in W_2^k(\mathbb{R}_+)^*$ which have the form

$$(u, F)_{\mathbb{R}_+} = (u, f)_{\mathbb{R}_+} + \sum_{j=-l+1}^{k} (D_t^{j-1}u)(0)\,\overline{g_j}\,,$$

where $f \in W_2^{-l}(\mathbb{R}_+)^*$, $g_j \in \mathbb{C}$.

In the case $l \le -k$, $k \ge 0$ we set $D_2^{l,k}(\mathbb{R}_+) = W_2^{-l}(\mathbb{R}_+)^*$. In particular, with this notation, we have

$$D_2^{l,0}(\mathbb{R}_+) = \tilde{W}_2^{l,0}(\mathbb{R}_+) = \begin{cases} W_2^l(\mathbb{R}_+) & \text{if } l \ge 0, \\ W_2^{-l}(\mathbb{R}_+)^* & \text{if } l < 0. \end{cases}$$

It can be easily verified that the space $D_2^{l_1,k}(\mathbb{R}_+)$ is continuously imbedded and dense in $D_2^{l,k}(\mathbb{R}_+)$ if $l_1 > l$.

THEOREM 1.4.2. *Suppose that $\mu_k < 2m$ for $k = 1, \dots, m+J$ and the boundary value problem (1.1.2), (1.1.3) is regular. Then the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ realizes an isomorphism from $D_2^{l,0}(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ onto $D_2^{l-2m,2m}(\mathbb{R}_+) \times \mathbb{C}^J$ for arbitrary integer l.*

Proof: If $l \le 0$, then $D_2^{l,0}(\mathbb{R}_+) = W_2^{-l}(\mathbb{R}_+)^*$, $D_2^{l-2m,2m}(\mathbb{R}_+) = W_2^{2m-l}(\mathbb{R}_+)^*$ and the assertion follows immediately from Theorem 1.2.1.

Now let l be a positive integer. We show first that the operator $\mathcal{A}^*$ continuously maps the space $W_2^l(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ into $D_2^{l-2m,2m}(\mathbb{R}_+) \times \mathbb{C}^J$. Let $(v, \underline{v})$ be an arbitrary element of $W_2^l(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ and let $\psi \in \mathbb{C}^{2m}$ be the vector with the components

$$\psi_j = \begin{cases} (D_t^{j-1}v)(0) & \text{for } j = 1, 2, \dots, \min(l, 2m), \\ 0 & \text{for } \min(l, 2m) < j \le 2m. \end{cases}$$

Then $(v, \underline{\psi})$ is an element of the space $\tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J}$. Moreover, it follows from Theorem 1.3.1 that $(f, \underline{g}, \underline{h}) = \mathcal{A}^+(v, \underline{\psi}, \underline{v}) \in \tilde{W}_2^{l-2m,0}(\mathbb{R}_+) \times \mathbb{C}^{2m} \times \mathbb{C}^J$ and

$$\|f\|_{\tilde{W}_2^{l-2m,0}(\mathbb{R}_+)} + \|\underline{g}\|_{\mathbb{C}^{2m}} + \|\underline{h}\|_{\mathbb{C}^J} \le c\left(\|v\|_{W_2^l(\mathbb{R}_+)} + \|\underline{v}\|_{\mathbb{C}^{m+J}}\right) \tag{1.4.11}$$

with a constant c independent of v and $\underline{v}$. Furthermore, by Theorem 1.4.1, we have $\mathcal{A}^*(v, \underline{v}) = (F, \underline{h})$, where the functional $F \in W_2^{2m}(\mathbb{R}_+)^*$ has the form

$$(u, F)_{\mathbb{R}_+} = (u, f)_{\mathbb{R}_+} + \big((\mathcal{D}u)(0),\, \underline{g}\big)_{\mathbb{C}^{2m}}\,, \quad u \in W_2^{2m}(\mathbb{R}_+), \tag{1.4.12}$$

i.e., $F \in D_2^{l-2m,2m}(\mathbb{R}_+)$. Due to (1.4.11), the norm of F in $D_2^{l-2m,2m}(\mathbb{R}_+)$ can be estimated by the norms of v and $\underline{v}$. This proves the continuity of the operator $\mathcal{A}^*$ from $W_2^l(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ into $D_2^{l-2m,2m}(\mathbb{R}_+) \times \mathbb{C}^J$.

Now we prove that $\mathcal{A}^*$ maps $W_2^l(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ onto $D_2^{l-2m,2m}(\mathbb{R}_+) \times \mathbb{C}^J$ for $l > 0$. Let $(F, \underline{h})$ be an arbitrary element of the space $D_2^{l-2m,2m}(\mathbb{R}_+) \times \mathbb{C}^J$. According to the definition of the space $D_2^{l-2m,2m}(\mathbb{R}_+)$, the functional F has the form (1.4.12), where $f \in \tilde{W}_2^{l-2m,0}(\mathbb{R}_+)$. By Theorem 1.3.2, there exists a solution $(v, \underline{\psi}, \underline{v}) \in$

$\tilde{W}_2^{l,2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ of the equation $\mathcal{A}^+ (v, \underline{\psi}, \underline{v}) = (f, \underline{g}, \underline{h})$. Using Theorem 1.4.1, we conclude that $(v, \underline{v})$ is a solution of the equation $\mathcal{A}^*(v, \underline{v}) = (F, \underline{h})$.

Thus, we have proved that $\mathcal{A}^*$ is a continuous mapping from $W_2^l(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ onto $D_2^{l-2m,2m}(\mathbb{R}_+) \times \mathbb{C}^J$ for $l > 0$. The injectivity of $\mathcal{A}^*$ is obvious. ∎

1.4.3. Regularity of the solution of the adjoint problem. Since the space $D_2^{l-2m,2m}(\mathbb{R}_+)$ is continuously imbedded into $D_2^{q-2m,2m}(\mathbb{R}_+)$ for $l \geq q$, Theorem 1.4.2 yields the following regularity assertion for the solution $(v, \underline{v})$ of the adjoint problem

$$\mathcal{A}^* (v, \underline{v}) = (F, \underline{h}) \tag{1.4.13}$$

THEOREM 1.4.3. *Let the assumptions of Theorem 1.4.2 be satisfied and let $(v, \underline{v}) \in W_2^k(\mathbb{R}_+)^* \times \mathbb{C}^{m+J}$, $k \geq 0$, be a solution of the adjoint problem (1.4.13), where $F \in D_2^{l-2m,2m}(\mathbb{R}_+)$. Then $(v, \underline{v}) \in D_2^{l,0}(\mathbb{R}_+) \times \mathbb{C}^J$.*

Moreover, if $l \geq 2m$ and F has the form (1.4.9) with a function $f \in W_2^{l-2m}(\mathbb{R}_+)$ and $\underline{g} \in \mathbb{C}^{2m}$, then $(v, \underline{v})$ is a solution of the formally adjoint problem (1.1.15), (1.1.16).

Proof: The first assertion follows immediately from Theorem 1.4.2. It remains to show that $(v, \underline{v})$ is a solution of the corresponding formally adjoint problem if $l \geq 2m$. Let $(v, \underline{v}) \in W_2^l(\mathbb{R}_+) \times \mathbb{C}^{m+J}$ be a solution of the equation (1.4.13) with a functional $F \in D_2^{l-2m,2m}(\mathbb{R}_+)$, $l \geq 2m$. Then $(v, (\mathcal{D}v)(0))$ belongs to the space $\tilde{W}_2^{l,2m}(\mathbb{R}_+)$, and Theorem 1.4.1 yields

$$\mathcal{A}^+ \big(v, (\mathcal{D}v)(0), \underline{v}\big) = (f, \underline{g}, \underline{h}) \ .$$

Thus, $(v, \underline{v})$ satisfies the equations (1.1.15), (1.1.16). ∎

REMARK 1.4.2. By Theorem 1.4.2, the adjoint operator $\mathcal{A}^*$ realizes an isomorphism

$$D_2^{l,0}(\mathbb{R}_+) \times \mathbb{C}^{m+J} \to D_2^{l-2m,2m}(\mathbb{R}_+) \times \mathbb{C}^J$$

for arbitrary integer l if the boundary value problem (1.1.2), (1.1.3) is regular and $\mu_k < 2m$ for $k = 1, \dots, m + J$. It can be shown in the same way that $\mathcal{A}^*$ realizes an isomorphism

$$D_2^{l,\kappa-2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J} \to D_2^{l-2m,\kappa}(\mathbb{R}_+) \times \mathbb{C}^J$$

if problem (1.1.2), (1.1.3) is regular and $\kappa \geq \max(2m, \mu_1 + 1, \dots, \mu_{m+J} + 1)$. Furthermore, a regularity assertion analogous to that given in Theorem 1.4.3 holds.

CHAPTER 2

Elliptic boundary value problems in the half-space

This chapter deals with elliptic partial differential equations in the Euclidean space $\mathbb{R}^n$ and elliptic boundary value problems in the half-space $\mathbb{R}^n_+$. We investigate the solvability in Sobolev spaces of periodic functions and derive a priori estimates for the solutions. Since every smooth domain is diffeomorphic to a half-space in a neighbourhood of any boundary point, this is the decisive step in the proof of the Fredholm property of operators of elliptic boundary value problems in smooth bounded domains. As in Chapter 1, we consider solutions in Sobolev spaces of both positive and negative integer orders.

2.1. Periodic solutions of partial differential equations

In this section necessary and sufficient conditions for the unique solvability of partial differential equations with constant coefficients in Sobolev spaces of periodic functions are obtained. Furthermore, we prove regularity assertions and a priori estimates for solutions of differential equations with variable coefficients and show that the ellipticity is necessary for the validity of this a priori estimate.

2.1.1. Sobolev spaces of periodic functions. We consider the set of all smooth functions on $\mathbb{R}^n$ which are 2π-periodic, i.e., the equality

$$u(x) = u(x + 2\pi \cdot k)$$

is satisfied for every $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$, where $\mathbb{Z}$ denotes the set of the integer numbers. Every such function is uniquely determined by its values in the cube

$$\mathbb{Q}^n = \{x = (x_1, \dots , x_n) \in \mathbb{R}^n \, : \, |x_j| \le \pi \text{ for } j = 1, \dots , n\} .$$

and can be written as a Fourier series

$$u(x) = (2\pi)^{-n/2} \sum_{k\in\mathbb{Z}^n} \dot{u}(k)\, e^{ik\cdot x}$$

with the Fourier coefficients

$$\dot{u}(k) = (2\pi)^{-n/2} \int_{\mathbb{Q}^n} e^{-ik\cdot x}\, u(x)\, dx.$$

Here $k \cdot x$ denotes the sum $k_1x_1 + \cdots + k_nx_n$. We define the space $W^l_{2,per}(\mathbb{R}^n)$ for arbitrary real l as the closure of the set of all smooth 2π-periodic functions with respect to the norm

$$\|u\|_{W^l_{2,per}(\mathbb{R}^n)} = \Big(\sum_{k\in\mathbb{Z}^n} (1 + |k|^2)^l \, |\dot{u}(k)|^2 \Big)^{1/2} . \tag{2.1.1}$$

In particular, $L_{2,per}(\mathbb{R}^n) = W^0_{2,per}(\mathbb{R}^n)$ is the Hilbert space with the scalar product

$$(u,v)_{\mathbb{Q}^n} = \int\limits_{\mathbb{Q}^n} u(x)\,\overline{v(x)}\,dx = \sum_{k\in\mathbb{Z}^n} \dot{u}(k)\,\overline{\dot{v}(k)}\,.$$

If l is a nonnegative integer, then $W^l_{2,per}(\mathbb{R}^n)$ is the set of all 2π-periodic functions which have quadratically integrable generalized derivatives up to order l on every compact subset of $\mathbb{R}^n$.

2.1.2. Solvability of elliptic differential equations with constant coefficients. For an arbitrary multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ let $|\alpha| = \alpha_1 + \dots + \alpha_n$ denote the length of this multi-index. Furthermore, we set $D^\alpha_x = D^{\alpha_1}_{x_1} \cdots D^{\alpha_n}_{x_n}$, where $D_{x_j} = -i\,\partial/\partial x_j$.

If

$$L(D_x) = \sum_{|\alpha|\le 2m} a_\alpha\, D^\alpha_x \tag{2.1.2}$$

is a differential operator of order $2m$, then we denote by

$$L^\circ(D_x) = \sum_{|\alpha|= 2m} a_\alpha\, D^\alpha_x \tag{2.1.3}$$

the *principal part* of L. The operator L is said to be *elliptic,* if

$$L^\circ(\xi) = \sum_{|\alpha|= 2m} a_\alpha\, \xi^\alpha \ne 0 \tag{2.1.4}$$

for each $\xi \in \mathbb{R}^n$, $\xi \ne 0$. From the ellipticity it follows that $|L^\circ(\xi)| \ge c\,|\xi|^{2m}$ for all $\xi \in \mathbb{R}^n$, where c is a positive constant independent of ξ.

THEOREM 2.1.1. *The operator (2.1.2) is an isomorphism from $W^l_{2,per}(\mathbb{R}^n)$ onto the space $W^{l-2m}_{2,per}(\mathbb{R}^n)$ if and only if the following conditions are satisfied:*
(i) *L is elliptic.*
(ii) *$L(k) \ne 0$ for each $k \in \mathbb{Z}^n$.*

Proof: 1) We assume that L is elliptic and satisfies condition (ii). From the ellipticity it follows that

$$|L^\circ(\xi)| \ge c\,|\xi|^{2m} \ge c\,\big((1+|\xi|^2)/2\big)^m \quad \text{for } |\xi| \ge 1.$$

Since $|L(\xi) - L^\circ(\xi)| \le c\,(1+|\xi|^{2m-1})$, it holds

$$|L(\xi)| \ge c\,(1+|\xi|^2)^m$$

with a constant c independent of ξ if $|\xi|$ is sufficiently large. If, additionally, condition (ii) is satisfied, we get

$$|L(k)| \ge c\,(1+|k|^2)^m \tag{2.1.5}$$

for each $k \in \mathbb{Z}^n$, where the constant c is independent of k. Let f be an arbitrary function from $W^{l-2m}_{2,per}(\mathbb{R}^n)$ with the Fourier coefficients $\dot{f}(k)$. Then the function

$$u(x) = (2\pi)^{-n/2} \sum_{k\in\mathbb{Z}^n} L(k)^{-1}\,\dot{f}(k)\,e^{ik\cdot x}\,,$$

is the uniquely determined solution of the equation $L(D_x)\,u = f$. By (2.1.5), this solution satisfies the estimate

$$\|u\|_{W^l_{2,per}(\mathbb{R}^n)} \le c\,\|f\|_{W^{l-2m}_{2,per}(\mathbb{R}^n)}\,. \tag{2.1.6}$$

with a constant c independent of f.

2) If the operator L is an isomorphism from $W^l_{2,per}(\mathbb{R}^n)$ onto $W^{l-2m}_{2,per}(\mathbb{R}^n)$, then (2.1.6) is satisfied for each $u \in W^l_{2,per}(\mathbb{R}^n)$, $f = Lu$. Setting $u = e^{ik\cdot x}$, we get the inequality

$$(1+|k|^2)^{l/2} = \|u\|_{W^l_{2,per}(\mathbb{R}^n)} \le c\|Lu\|_{W^{l-2m}_{2,per}(\mathbb{R}^n)} = c\,|L(k)|\,(1+|k|^2)^{(l-2m)/2}$$

which yields (2.1.5) with a constant c independent of $k \in \mathbb{Z}^n$. Consequently, condition (ii) is satisfied. Furthermore, (2.1.5) yields

$$|L^\circ(k/|k|)| = |k|^{-2m}|L^\circ(k)| \ge c/2$$

if $|k| \ge \rho$ and ρ is sufficiently large. Since the set $\{\xi = k/|k| \,:\, k \in \mathbb{Z}^n,\ k \ne 0\}$ is dense on the unit sphere and $L^\circ(\xi)$ is continuous, we obtain the inequality

$$|L^\circ(\xi)| \ge c/2$$

for each $\xi \in \mathbb{R}^n$, $|\xi| = 1$. This implies the ellipticity of L. ■

REMARK 2.1.1. Let $L^\circ(D_x)$ be the operator (2.1.3). We assume that this operator is elliptic. Then condition (ii) of Theorem 2.1.1 is satisfied for the operator

$$L^\circ(D_x + \tfrac{1}{2}\vec{1}) = \sum_{|\alpha|=2m} a_\alpha\,(D_{x_1}+\tfrac{1}{2})^{\alpha_1}\cdots(D_{x_n}+\tfrac{1}{2})^{\alpha_n}\,.$$

Here and elsewhere $\vec{1}$ denotes the vector $(1,1,\dots,1)$.

2.1.3. Regularity of periodic solutions of elliptic differential equations. Now let

$$L(x, D_x) = \sum_{|\alpha|\le 2m} a_\alpha(x)\,D_x^\alpha$$

be a differential operator with 2π-periodic coefficients $a_\alpha \in C^\infty(\mathbb{R}^n)$. We assume additionally that the following conditions are fulfilled:

a) The polynomial $L^\circ(0,\xi) = \sum_{|\alpha|=2m} a_\alpha(0)\,\xi^\alpha$ satisfies condition (2.1.4).

b) The coefficients a_α of L satisfy the estimate

$$|\,a_\alpha(x) - a_\alpha(0)\,| < \varepsilon \quad \text{for } |\alpha| = 2m,$$

where ε is a sufficiently small positive number.

Our aim is to obtain a regularity assertion for the solution of the equation

$$L(x, D_x)u = f. \tag{2.1.7}$$

For this we need the following lemma which follows immediately from the definition of the space $W^l_{2,per}(\mathbb{R}^n)$.

LEMMA 2.1.1. *The operator*

$$(S_\rho u)(x) = (2\pi)^{-n/2} \sum_{k\in\mathbb{Z}^n,\,|k|>\rho} \dot{u}(k)\,e^{ik\cdot x}\,, \quad u \in W^l_{2,per}(\mathbb{R}^n)\,, \tag{2.1.8}$$

where ρ is a positive real parameter, continuously maps $W^l_{2,per}(\mathbb{R}^n)$ into $W^l_{2,per}(\mathbb{R}^n)$. Furthermore, the inequality

$$\|S_\rho u\|_{W^{l-1}_{2,per}(\mathbb{R}^n)} \le (1+\rho^2)^{-1/2}\,\|u\|_{W^l_{2,per}(\mathbb{R}^n)}$$

is satisfied for every $u \in W^l_{2,per}(\mathbb{R}^n)$. *The operator* $I - S_\rho$, *where* I *is the identity in* $W^l_{2,per}(\mathbb{R}^n)$, *realizes a continuous mapping from* $W^l_{2,per}(\mathbb{R}^n)$ *into* $W^{l+1}_2(\mathbb{R}^n)$ *for arbitrary integer* l.

THEOREM 2.1.2. *If conditions a) and b) are satisfied and* $u \in W^l_{2,per}(\mathbb{R}^n)$ *is a solution of the equation (2.1.7) with* $f \in W^{l-2m+1}_{2,per}(\mathbb{R}^n)$, *then* u *belongs to the space* $W^{l+1}_{2,per}(\mathbb{R}^n)$ *and the estimate*

$$\|u\|_{W^{l+1}_{2,per}(\mathbb{R}^n)} \le c\left(\|Lu\|_{W^{l-2m+1}_{2,per}(\mathbb{R}^n)} + \|u\|_{W^l_{2,per}(\mathbb{R}^n)}\right) \tag{2.1.9}$$

is valid. Here the constant c *is independent of* u.

Proof: Let $L^\circ(0, D_x)$ be the principal part of the operator L with coefficients frozen at the origin. We denote by $L^{(1)}(x, D_x)$ the operator

$$L^{(1)}(x, D_x) = L(x, D_x) - L^\circ(0, D_x + \tfrac{1}{2}\vec{1}).$$

and rewrite (2.1.7) as

$$\left(L^\circ(0, D_x + \tfrac{1}{2}\underline{1}) + L^{(1)}(x, D_x)\, S_\rho\right) u = f - L^{(1)}(x, D_x)\,(I - S_\rho)\, u. \tag{2.1.10}$$

By Theorem 2.1.1, Remark 2.1.1, the operator $L^\circ(0, D_x + \frac{1}{2}\underline{1})$ is invertible. Furthermore, by condition b) and Lemma 2.1.1, we have

$$\begin{aligned}\|L^{(1)}(x, D_x)\, S_\rho\, u\|_{W^{l-2m}_{2,per}(\mathbb{R}^n)} &\le c\left(\varepsilon\,\|S_\rho u\|_{W^l_{2,per}(\mathbb{R}^n)} + \|S_\rho u\|_{W^{l-1}_{2,per}(\mathbb{R}^n)}\right)\\ &\le c\,(\varepsilon + (1+\rho^2)^{-1/2})\,\|u\|_{W^l_{2,per}(\mathbb{R}^n)}\end{aligned}$$

for each $u \in W^l_{2,per}(\mathbb{R}^n)$, where the constant c is independent of u, ρ and ε. Consequently, the operator on the left side of (2.1.10) is an isomorphism $W^l_{2,per}(\mathbb{R}^n) \to W^{l-2m}_{2,per}(\mathbb{R}^n)$ for every given integer l if ε is sufficiently small and ρ is sufficiently large. Using the fact that the operator $L^{(1)}(x, D_x)\,(I - S_\rho)$ on the right side of (2.1.10) is continuous from $W^l_{2,per}(\mathbb{R}^n)$ into $W^{l-2m+1}_{2,per}(\mathbb{R}^n)$ for arbitrary ρ, we get the assertion of the theorem. ■

Next we show that (2.1.9) implies the ellipticity of $L(0, D_x)$.

LEMMA 2.1.2. *Let* L *be a differential operator of order* $2m$ *with smooth* 2π-*periodic coefficients and let* $\mathcal{U}$ *be an arbitrary neighbourhood of the point* $x = 0$. *If the inequality (2.1.9) with a constant* c *independent of* u *is satisfied for all* $u \in W^{l+1}_{2,per}(\mathbb{R}^n)$, $\operatorname{supp} u \cap \mathbb{Q}^n \subset \mathcal{U}$, *then* $L(0, D_x)$ *is elliptic.*

Proof: In a sufficiently small neighbourhood of the points $2\pi k$, $k \in \mathbb{Z}^n$, the coefficients a_α of L satisfy the inequality $|a_\alpha(x) - a_\alpha(0)| < \varepsilon$. Therefore, for every function u equal to zero outside a sufficiently small neighbourhood of the points $2\pi k$, the inequality

$$\|(L(x, D_x) - L^\circ(0, D_x))u\|_{W^{l+1-2m}_{2,per}(\mathbb{R}^n)} \le c\left(\varepsilon\,\|u\|_{W^{l+1}_{2,per}(\mathbb{R}^n)} + \|u\|_{W^l_{2,per}(\mathbb{R}^n)}\right)$$

is valid. From this and (2.1.9) we conclude that

$$\|u\|_{W^{l+1}_{2,per}(\mathbb{R}^n)} \le c\left(\|L^\circ(0, D_x)u\|_{W^{l-2m+1}_{2,per}(\mathbb{R}^n)} + \|u\|_{W^l_{2,per}(\mathbb{R}^n)}\right) \tag{2.1.11}$$

for all $u \in W^{l+1}_{2,per}(\mathbb{R}^n)$ equal to zero outside a sufficiently small neighbourhood of the points $2\pi k$. Since the operator $L^\circ(0, D_x)$ is translation invariant, the inequality (2.1.11) is satisfied for all $u \in W^{l+1}_{2,per}(\mathbb{R}^n)$.

Inserting $u = e^{ik\cdot x}$ into (2.1.11), we get

$$(1+|k|^2)^{(l+1)/2} \le c\Big((1+|k|^2)^{(l-2m+1)/2}\,|L^\circ(0,k)| + (1+|k|^2)^{l/2}\Big).$$

Consequently, for sufficiently large ρ, $|k| > \rho$ we obtain

$$|L^\circ(0,k)| \ge c_0\,(1+|k|^2)^m \ge c_0\,|k|^{2m}$$

and

$$|L^\circ(0,k/|k|)| \ge c_0.$$

Since the set $\{k/|k| \,:\, k \in \mathbb{Z}^n,\ |k| > \rho\}$ is dense on the unit sphere in $\mathbb{R}^n$, this implies $|L^\circ(0,\xi)| \ge c_0$ for all $\xi \in \mathbb{R}^n$, $|\xi| = 1$, i.e., the operator $L(0, D_x)$ is elliptic. ■

2.2. Solvability of elliptic boundary value problems in the half-space

This and the following sections are concerned with boundary value problems in the half-space $x_n > 0$. The goal of this section is to obtain necessary and sufficient conditions for the bijectivity of the operator of the boundary value problem. Here we restrict ourselves to differential operators with constant coefficients.

As in the foregoing section, we use Sobolev spaces of periodic functions. However, in contrast to Section 2.1, now the functions are periodic only with respect to the variables $x_1, \dots, x_{n-1}$.

2.2.1. Sobolev spaces of periodic functions. Let $\mathbb{R}^n_+$ be the half-space $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,:\, x_n > 0\}$. For the sake of brevity, we will use the letter t for the last variable x_n and the letter y for the tuple $(x_1, \dots, x_{n-1})$. We consider the set of all smooth functions on $\overline{\mathbb{R}^n_+}$ which are 2π-periodic with respect to the variable y, i.e.,

$$u(y,t) = u(y + 2\pi\cdot q, t)$$

is satisfied for every $y \in \mathbb{R}^{n-1}$, $t > 0$, and every tuple $q = (q_1, \dots, q_{n-1}) \in \mathbb{Z}^{n-1}$. By $W^l_{2,per}(\mathbb{R}^n_+)$ we denote the closure of the set of these functions with respect to the norm

$$\|u\|_{W^l_{2,per}(\mathbb{R}^n_+)} = \Big(\int_{\mathbb{R}_+}\sum_{j=0}^{l}\|D_t^j u(\cdot,t)\|^2_{W^{l-j}_{2,per}(\mathbb{R}^{n-1})}\,dt\Big)^{1/2}. \tag{2.2.1}$$

In particular, $L_{2,per}(\mathbb{R}^n_+) = W^0_{2,per}(\mathbb{R}^n_+)$ is a Hilbert space with the scalar product

$$(u,v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} = \int_{\mathbb{R}_+}\int_{\mathbb{Q}^{n-1}} u(y,t)\,\overline{v(y,t)}\,dy\,dt. \tag{2.2.2}$$

Here $\mathbb{Q}^{n-1}$ denotes the cube $[-\pi,\pi]^{n-1}$.

Using the representation (2.1.1) for the norm in $W^{l-j}_{2,per}(\mathbb{R}^{n-1})$, we get

$$\|u\|_{W^l_{2,per}(\mathbb{R}^n_+)} = \Big(\int_{\mathbb{R}_+}\sum_{j=0}^{l}\sum_{q\in\mathbb{Z}^{n-1}}(1+|q|^2)^{l-j}\,|D_t^j\hat{u}(q,t)|^2\,dt\Big)^{1/2},$$

where

$$\dot{u}(q,t) = (2\pi)^{(n-1)/2} \int_{\mathbb{Q}^{n-1}} e^{-iq\cdot y}\, u(y,t)\, dt$$

are the Fourier coefficients of $u(\cdot,t)$. It is more convenient to use the function

$$U(q,t) \stackrel{def}{=} \dot{u}(q, \langle q\rangle^{-1} t) \tag{2.2.3}$$

instead of $\dot{u}$, where $\langle q\rangle = (1+|q|^2)^{1/2}$. Then we have

$$\|u\|_{W^l_{2,per}(\mathbb{R}^n_+)} = \Big(\sum_{q\in\mathbb{Z}^{n-1}} \langle q\rangle^{2l-1} \, \|U(q,\cdot)\|^2_{W^l_2(\mathbb{R}_+)} \Big)^{1/2}. \tag{2.2.4}$$

Analogously, we define the space $W^l_{2,per}(\mathbb{R}^{n-1}\times\mathbb{R})$ as the closure of the set of all smooth 2π-periodic (relative to y) functions on $\mathbb{R}^n$ with respect to the norm

$$\|u\|_{W^l_{2,per}(\mathbb{R}^{n-1}\times\mathbb{R})} = \Big(\sum_{q\in\mathbb{Z}^{n-1}} \langle q\rangle^{2l-1} \, \|U(q,\cdot)\|^2_{W^l_2(\mathbb{R})} \Big)^{1/2}. \tag{2.2.5}$$

Using the Fourier transformation (1.2.2), we obtain the following equivalent norm to (2.2.5):

$$\|u\| = \Big(\sum_{q\in\mathbb{Z}^{n-1}} \langle q\rangle^{2l-1} \int_{\mathbb{R}} (1+\tau^2)^l \, |(\mathcal{F}_{t\to\tau} U)(q,\tau)|^2 \, d\tau \Big)^{1/2}. \tag{2.2.6}$$

Obviously, the restriction of any function $u \in W^l_{2,per}(\mathbb{R}^{n-1}\times\mathbb{R})$ to the half-space $\mathbb{R}^n_+$ belongs to $W^l_{2,per}(\mathbb{R}^n_+)$. Furthermore, every function $u \in W^l_{2,per}(\mathbb{R}^n_+)$ can be extended to a function $v \in W^l_{2,per}(\mathbb{R}^{n-1}\times\mathbb{R})$ as follows:

$$v(y,t) = \sum_{q\in\mathbb{Z}^{n-1}} (e_+ U)(q, \langle q\rangle t)\, e^{iq\cdot y}$$

where e_+ is an arbitrary continuous extension operator $W^l_2(\mathbb{R}_+) \to W^l_2(\mathbb{R})$.

Finally, the space $\overset{\circ}{W}{}^l_{2,per}(\mathbb{R}^n_+)$ is defined as the closure with respect to the norm (2.2.1) of the set of all smooth functions on $\mathbb{R}^n_+$ which are 2π-periodic in y and equal to zero in a neighbourhood of the plane $t=0$.

2.2.2. The trace operator. Let γ_μ be the operator

$$\gamma_\mu : u(y,t) \to \big(u(y,0), (D_t u)(y,0), \ldots, (D_t^{\mu-1} u)(y,0) \big)$$

defined on the set of all smooth and 2π-periodic (with respect to y) functions on $\mathbb{R}^n_+$. The operator γ_1 is called the *trace operator* and the function $(\gamma_1 u)(y) = u(y,0)$ is called the *trace* of u on the boundary plane $x_n = 0$.

LEMMA 2.2.1. 1) *The operator γ_μ can be continuously extended to a bounded mapping*

$$W^l_{2,per}(\mathbb{R}^n_+) \to \prod_{j=1}^{\mu} W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1})$$

if $l \ge \mu$.

2) *For arbitrary integer $l \ge \mu$ there exists a linear and continuous right inverse*

$$e_\mu : \prod_{j=1}^{\mu} W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1}) \to W^l_{2,per}(\mathbb{R}^n_+)$$

to the operator γ_μ.

Proof: 1) Let u be an arbitrary 2π-periodic (with respect to y) smooth function on $\mathbb{R}^n_+$ and let U be defined by (2.2.3). Then the equality

$$u(y,t) = \sum_{q\in\mathbb{Z}^{n-1}} U(q,\langle q\rangle t)\, e^{iq\cdot y}$$

yields

$$(D_t^{j-1}u)(y,0) = \sum_{q\in\mathbb{Z}^{n-1}} \big(-i\langle q\rangle\big)^{j-1}\,(D_t^{j-1}U)(q,0)\, e^{iq\cdot y}\,.$$

Since the space $W_2^l(\mathbb{R}_+)$ is continuously imbedded into $C^{l-1}(\overline{\mathbb{R}}_+)$, we have

$$|(D_t^{j-1}U)(q,0)| \le c\,\|U(q,\cdot)\|_{W_2^l(\mathbb{R}_+)}$$

for $j = 1,\dots,l$. Consequently, we obtain

$$\|(D_t^{j-1}u)(\cdot,0)\|^2_{W_{2,per}^{l-j+1/2}(\mathbb{R}^{n-1})} \le c \sum_{q\in\mathbb{Z}^{n-1}} \langle q\rangle^{2l-1}\|U(q,\cdot)\|^2_{W_2^l(\mathbb{R}_+)} = c\,\|u\|^2_{W_{2,per}^l(\mathbb{R}^n_+)}$$

for $j \le l$. This proves the first assertion.

2) Let g_j be arbitrary functions from $W_{2,per}^{l-j+1/2}(\mathbb{R}^{n-1})$, $j = 1,\dots,\mu$, and let $\dot{g}_j(q)$ denote the Fourier coefficients of g_j. Furthermore, let χ be a smooth cut-off function on $\mathbb{R}_+$ equal to one in the interval $0 \le t \le 1/2$ and to zero in the interval $1 \le t < \infty$. Then it can be easily shown that the function

$$u(y,t) = \sum_{q\in\mathbb{Z}^{n-1}} \chi(\langle q\rangle t) \sum_{j=1}^{\mu} \dot{g}_j(q)\,\frac{(it)^{j-1}}{(j-1)!}\, e^{iq\cdot y}$$

satisfies the equations $(D_t^{j-1}u)(y,0) = g_j(y)$ for $j = 1,\dots,\mu$ and the inequality

$$\|u\|_{W_{2,per}^l(\mathbb{R}^n_+)} \le c \sum_{j=1}^{\mu} \|g_j\|_{W_{2,per}^{l-j+1/2}(\mathbb{R}^{n-1})}\,.$$

This proves the second part of the lemma. ∎

2.2.3. Ellipticity of boundary value problems in the half-space. Let again

$$L(D_x) = L(D_y,D_t) = \sum_{|\beta|+j\le 2m} a_{\beta,j}\, D_y^\beta\, D_t^j$$

be an elliptic differential operator of order $2m$ with constant coefficients. We consider the boundary value problem

(2.2.7) $$L(D_y,D_t)\,u(y,t) = f(y,t)\,,\quad (y,t)\in\mathbb{R}^n_+\,,$$

(2.2.8) $$B(D_y,D_t)\,u(y,t)\Big|_{t=0} + C(D_y)\,\underline{u}(y) = \underline{g}(y)\,,\quad y\in\mathbb{R}^{n-1},$$

where $B(D_y,D_t)$ is a vector of linear differential operators

$$B_k(D_y,D_t) = \sum_{|\beta|+j\le\mu_k} b_{k;\beta,j}\, D_y^\beta\, D_t^j\,,\quad k = 1,\dots,m+J,$$

with constant coefficients, $\operatorname{ord} B_k \le \mu_k$, and

$$C(D_y) = \Big(C_{k,j}(D_y)\Big)_{1\le k\le m+J,\ 1\le j\le J}$$

is a matrix of linear differential operators

$$C_{k,j}(D_y) = \sum_{|\beta|\le\mu_k+\tau_j} c_{k,j;\beta}\, D_y^\beta$$

with constant coefficients. Here μ_k and τ_j are given integer numbers. If $\mu_k < 0$ or $\mu_k + \tau_j < 0$, then the corresponding operators B_k and $C_{k,j}$ are assumed to be identically equal to zero. We suppose that $\tau_j + \max\mu_k \ge 0$ for $j = 1, \dots, J$. Otherwise, the component u_j of the vector-function $\underline{u}$ does not appear in the boundary condition (2.2.8).

Under our assumptions on L, B, and C, the operator $(u, \underline{u}) \to (f, \underline{g})$ of the boundary value problem (2.2.7), (2.2.8) realizes a linear and continuous mapping

$$(2.2.9)\quad \mathcal{A} : W^l_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1}) \to W^{l-2m}_{2,per}(\mathbb{R}^n_+) \times W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})$$

for arbitrary integer $l \ge 2m$, $l > \max\mu_k$. Here

$$W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1}) \stackrel{def}{=} \prod_{j=1}^{J} W^{l+\tau_j-1/2}_{2,per}(\mathbb{R}^{n-1}) \quad \text{and}$$

$$W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1}) \stackrel{def}{=} \prod_{k=1}^{m+J} W^{l-\mu_k-1/2}_{2,per}(\mathbb{R}^{n-1}).$$

We denote by

$$B^\circ_k(D_y, D_t) = \sum_{|\beta|+j=\mu_k} b_{k;\beta,j}\, D_y^\beta D_t^j$$

and

$$C^\circ_{k,j}(D_y) = \sum_{|\beta|=\mu_k+\tau_j} c_{k,j;\beta}\, D_y^\beta$$

($k = 1, \dots, m+J$; $j = 1, \dots, J$) the "principal parts" of B_k and $C_{k,j}$, respectively. The operators B°_k and $C^\circ_{k,j}$ depend on the choice of the numbers μ_k and τ_j. In particular, we have $B^\circ_{k,j} \equiv 0$ if $\operatorname{ord} B_k < \mu_k$ or $\mu_k < 0$, and analogously, $C^\circ_{k,j} \equiv 0$ if $\operatorname{ord} C_{k,j} < \mu_k + \tau_j$ or $\mu_k + \tau_j < 0$.

In contrast to the problem in $\mathbb{R}^n$, the ellipticity of L does not suffice for the bijectivity of the operator (2.2.9).

DEFINITION 2.2.1. The operator L is said to be *properly elliptic* if for every $\eta \in \mathbb{R}^{n-1}$, $\eta \ne 0$, exactly m zeros (counting multiplicity) of the polynomial

$$\tau \to L^\circ(\eta, \tau) = \sum_{|\beta|+j=2m} a_{\beta,j}\, \eta^\beta\, \tau^j$$

lie both in the upper half-plane $\operatorname{Im}\tau > 0$ and in the lower half-plane $\operatorname{Im}\tau < 0$.

It is evident that every properly elliptic operator is elliptic. The operator $L = D^2_{x_1} - 2i\, D_{x_1} D_{x_2} - D^2_{x_2}$ is an example for an elliptic operator in the case $n = 2$ which is not properly elliptic.

Definition 2.2.1 has the disadvantage that it depends on the choice of the Cartesian coordinate system. For this reason, we give another equivalent definition.

LEMMA 2.2.2. *The following assertions are equivalent:*

1) *The operator L is properly elliptic.*

2) *For every pair* ξ, ζ *of linearly independent vectors in* $\mathbb{R}^n$ *exactly* m *zeros (counting multiplicity) of the polynomial*

$$\tau \to L^\circ(\xi + \tau\zeta) \tag{2.2.10}$$

lie both in the upper half-plane $\operatorname{Im}\tau > 0$ *and in the lower half-plane* $\operatorname{Im}\tau < 0$.

Proof: Obviously assertion 2) implies 1). We assume now that L is properly elliptic. The polynomial (2.2.10) can be written in the form

$$L^\circ(\xi + \tau\zeta) = L^\circ(\zeta)\tau^{2m} + L^{(1)}(\xi,\zeta)\tau^{2m-1} + \cdots + L^{(2m-1)}(\xi,\zeta)\,\tau + L^\circ(\xi)\,,$$

where $L^{(j)}(\xi,\zeta)$ are polynomials of ξ and ζ ($j = 1,\dots,2m-1$). By the proper ellipticity, exactly m zeros of the polynomial (2.2.10) with the vectors $\zeta = (0,\dots,0,1)$ and $\xi = (\xi_1,\dots,\xi_{n-1},0)$ lie in the upper half-plane. Furthermore, the zeros of the polynomial (2.2.10) continuously depend on the vectors ξ and ζ, $\zeta \neq 0$. Since the polynomial (2.2.10) has no real zeros for arbitrary linearly independent vectors ξ and ζ, it follows that exactly m zeros of this polynomial lie in the upper half-plane if ξ and ζ are linearly independent. The lemma is proved. ■

As in Chapter 1, we denote by $\mathcal{M}^+(\eta)$ the set of the stable solutions of the equation

$$L^\circ(\eta, D_t)\,u(t) = \sum_{|\beta|+j=2m} a_{\beta,j}\,\eta^\beta\, D_t^j\, u(t) = 0 \quad \text{for } t > 0, \tag{2.2.11}$$

where $\eta \in \mathbb{R}^{n-1}$ is an arbitrary parameter.

LEMMA 2.2.3. 1) *In the case* $n \geq 3$ *every elliptic operator is properly elliptic.*
2) *If* $n = 2$, L *is elliptic and the system of the homogeneous boundary conditions*

$$B_k^\circ(\eta, D_t)\,u(t)\Big|_{t=0} + \sum_{j=1}^{J} C_{k,j}^\circ(\eta)\,u_j = 0 \quad \textit{for } k = 1,\dots,m+J \tag{2.2.12}$$

has only the trivial solution $(u,\underline{u}) = (u,u_1,\dots,u_J) = 0$ *in* $\mathcal{M}^+(\eta)\times\mathbb{C}^J$ *for each* $\eta \in \mathbb{R}^{n-1}\backslash\{0\}$, *then* L *is properly elliptic.*

Proof: Let η be a point in $\mathbb{R}^{n-1}\backslash\{0\}$ and let $\tau_1(\eta),\dots,\tau_\mu(\eta)$ be the different zeros of the polynomial $\tau \to L^\circ(\eta,\tau)$ with the multiplicities $r_1(\eta),\dots,r_\mu(\eta)$, $r_1(\eta)+\cdots+r_\mu(\eta) = 2m$. Since $L^\circ(\eta,\tau)$ has no real zeros, the set of all solutions of the equation (2.2.11) is the direct sum

$$\mathcal{M}(\eta) = \mathcal{M}^+(\eta) \oplus \mathcal{M}^-(\eta)$$

where $\mathcal{M}^\pm(\eta)$ are the linear spaces spanned by the functions

$$t^\nu\, e^{i\tau_j(\eta)t}$$

with $\pm\operatorname{Im}\tau_j(\eta) > 0$, $\nu = 0,\dots,r_j(\eta)-1$. The dimensions of $\mathcal{M}^+(\eta)$ and $\mathcal{M}^-(\eta)$ depend continuously on $\eta \neq 0$.

Furthermore, the relation $L^\circ(\eta,\tau) = L^\circ(-\eta,-\tau)$ yields $\mathcal{M}^-(\eta) = \mathcal{M}^+(-\eta)$ and, consequently,

$$\dim\,\mathcal{M}^+(\eta) + \dim\,\mathcal{M}^+(-\eta) = \dim\,\mathcal{M}(\eta) = 2m\,. \tag{2.2.13}$$

If $n \geq 3$, then there exists a continuous curve in $\mathbb{R}^{n-1}\backslash\{0\}$ connecting η with $-\eta$. Hence by the continuity of $\dim\,\mathcal{M}^+(\eta)$ with respect to η, we have $\dim\,\mathcal{M}^+(\eta) = \dim\,\mathcal{M}^+(-\eta)$. Thus, (2.2.13) implies $\dim\,\mathcal{M}^+(\eta) = m$.

If $n = 2$ and the system of the equations (2.2.12) has only the trivial solution in $\mathcal{M}^+(\eta) \times \mathbb{C}^J$ for $\eta \in \mathbb{R}^{n-1}\backslash\{0\}$, then we have $\dim \mathcal{M}^+(\eta) \le m$ for each $\eta \in \mathbb{R}^{n-1}$, $\eta \ne 0$. Together with (2.2.13) this implies $\dim \mathcal{M}^+(\eta) = m$. This means that L is properly elliptic. ∎

Now we introduce the notion of ellipticity for the boundary value problem (2.2.7), (2.2.8).

DEFINITION 2.2.2. The boundary value problem (2.2.7), (2.2.8) is said to be *elliptic* if the boundary value problem on the half-axis

$$L^\circ(\eta, D_t)\, u(t) = f(t) \qquad \text{for } t > 0\,, \tag{2.2.14}$$

$$B^\circ(\eta, D_t)u(t)\Big|_{t=0} + C^\circ(\eta)\, \underline{u} = \underline{g} \tag{2.2.15}$$

is regular in the sense of Definition 1.2.1 for every $\eta \in \mathbb{R}^{n-1}\backslash\{0\}$. This means that

(i) the operator L is properly elliptic,
(ii) the system of the homogeneous boundary conditions (2.2.12) has only the trivial solution in $\mathcal{M}^+(\eta) \times \mathbb{C}^J$ for each $\eta \in \mathbb{R}^{n-1}$, $\eta \ne 0$.

REMARK 2.2.1. The conditions of ellipticity for boundary value problems were introduced first by Z. Ya. Shapiro [**231**] (in the case of the Dirichlet problem) and Ya. B. Lopatinskiĭ [**127**]. For this reason, condition (ii) is often called *Lopatinskiĭ condition* or *Shapiro-Lopatinskiĭ condition.* Furthermore, it is sometimes said that the boundary conditions (2.2.8) *cover* the differential operator L or satisfy the *complementary condition.*

REMARK 2.2.2. The operators B° and C° in (2.2.15) and, therefore, the validity of condition (ii) in Definition 2.2.2 depend on the choice of the numbers μ_k and τ_j. As the following example shows, a boundary value problem may be elliptic for different choices of the numbers μ_k and τ_j.

Example. We consider the boundary value problem

$$\Delta u = f \qquad \text{for } t > 0, \tag{2.2.16}$$

$$u|_{t=0} + u_1 = g_1\,, \qquad u_1 = g_2. \tag{2.2.17}$$

First let $\mu_1 = \mu_2 = \tau_1 = 0$. Then we obtain the following problem on the half-axis (cf. (2.2.14), (2.2.15)):

$$(-\eta^2 - D_t^2)u(t) = f(t) \qquad \text{for } t > 0,$$
$$u(0) + u_1 = g_1\,, \qquad u_1 = g_2.$$

If we choose $\mu_1 = 0$, $\mu_2 = -1$, $\tau_1 = 1$, then the principal parts of the operators generate the boundary value problem

$$(-\eta^2 - D_t^2)u(t) = f(t) \qquad \text{for } t > 0,$$
$$u(0) = g_1\,, \qquad u_1 = g_2.$$

In both cases condition (ii) of Definition 2.2.2 is satisfied. In the first case the operator $(u, u_1) \to (f, g_1, g_2)$ of the boundary value problem (2.2.16), (2.2.17) is considered as a mapping

$$W^l_{2,per}(\mathbb{R}^n_+) \times W^{l-1/2}_{2,per}(\mathbb{R}^{n-1}) \to W^{l-2}_{2,per}(\mathbb{R}^n_+) \times W^{l-1/2}_{2,per}(\mathbb{R}^{n-1}) \times W^{l-1/2}_{2,per}(\mathbb{R}^{n-1}),$$

$l \geq 2$, while in the second case this operator is considered as a mapping

$$W^l_{2,per}(\mathbb{R}^n_+) \times W^{l+1/2}_{2,per}(\mathbb{R}^{n-1}) \to W^{l-2}_{2,per}(\mathbb{R}^n_+) \times W^{l-1/2}_{2,per}(\mathbb{R}^{n-1}) \times W^{l+1/2}_{2,per}(\mathbb{R}^{n-1}).$$

2.2.4. Existence and uniqueness of the solution. We are interested in the solvability of the problem (2.2.7), (2.2.8) The following theorem gives a necessary and sufficient condition for the existence and uniqueness of the solution.

THEOREM 2.2.1. *Let l be an arbitrary integer, $l \geq 2m$, $l > \max \mu_k$. The operator (2.2.9) of the boundary value problem (2.2.7), (2.2.8) is an isomorphism if and only if the following conditions are satisfied:*

(i) *The boundary value problem (2.2.7), (2.2.8) is elliptic.*

(ii) *The boundary value problem on the half-axis*

$$L(q, D_t)\, v(t) = \phi(t)\,, \quad t > 0\,, \tag{2.2.18}$$

$$B(q, D_t) v(t)\Big|_{t=0} + C(q)\, \underline{v} = \underline{\psi} \tag{2.2.19}$$

is regular (in the sense of Definition 1.2.1) for every $q \in \mathbb{Z}^{n-1}$.

Proof: The proof consists of the following three steps. In the first step we will show that the conditions (i), (ii) are equivalent to the following conditions (iii) and (iv):

(iii) The boundary value problem

$$L(q, \langle q \rangle D_t)\, v(t) = \phi(t)\,, \quad t > 0\,, \tag{2.2.20}$$

$$B(q, \langle q \rangle D_t) v(t)\Big|_{t=0} + C(q)\, \underline{v} = \underline{\psi} \tag{2.2.21}$$

has a unique solution $(v, \underline{v}) \in W^l_2(\mathbb{R}_+) \times \mathbb{C}^J$ for every $q \in \mathbb{Z}^{n-1}$, $\phi \in W^{l-2m}_2(\mathbb{R}_+)$, $\underline{\psi} \in \mathbb{C}^{m+J}$.

(iv) Every solution $(v, \underline{v}) \in W^l_2(\mathbb{R}_+) \times \mathbb{C}^J$ of (2.2.20), (2.2.21) satisfies the estimate

(2.2.22)

$$\|v\|^2_{W^l_2(\mathbb{R}_+)} + \sum_{j=1}^{J} \langle q \rangle^{2\tau_j} |v_j|^2 \leq c \left(\langle q \rangle^{-4m} \|\phi\|^2_{W^{l-2m}_2(\mathbb{R}_+)} + \sum_{k=1}^{m+J} \langle q \rangle^{-2\mu_k} |\psi_k|^2 \right)$$

with a constant c independent of q.

Under the assumption that conditions (iii), (iv) are satisfied, we will prove in the second step that problem (2.2.7), (2.2.8) is solvable in $W^l_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1})$ for any given f and $\underline{g}$ from the corresponding Sobolev spaces and that every solution satisfies the estimate

(2.2.23)

$$\|u\|^2_{W^l_{2,per}(\mathbb{R}^n_+)} + \|\underline{u}\|^2_{W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1})} \leq c \left(\|f\|^2_{W^{l-2m}_{2,per}(\mathbb{R}^n_+)} + \|\underline{g}\|^2_{W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})} \right).$$

In the last step it will be proved that the bijectivity of the operator (2.2.9) implies (iii) and (iv).

Step 1. Applying the transformation $t = \tau/\langle q \rangle$ to problem (2.2.18), (2.2.19), one gets problem (2.2.20), (2.2.21). Consequently, both problems are simultaneously solvable. We assume that additionally condition (i) is satisfied. Then the operator of the problem

$$L^\circ\Big(\frac{q}{\langle q\rangle}, D_t\Big)\, v(t) = \phi(t)\,, \quad t>0\,, \tag{2.2.24}$$

$$B^\circ\Big(\frac{q}{\langle q\rangle}, D_t\Big) v(t)\Big|_{t=0} + C^\circ\Big(\frac{q}{\langle q\rangle}\Big)\, \underline{v} = \underline{\psi} \tag{2.2.25}$$

is an isomorphism

$$W_2^l(\mathbb{R}_+) \times \mathbb{C}^J \to W_2^{l-2m}(\mathbb{R}_+) \times \mathbb{C}^{m+J} \tag{2.2.26}$$

for every $q \in \mathbb{Z}^{n-1}$, $q \neq 0$. Since the set of all points $\eta = q/\langle q\rangle$ with $q \neq 0$ is contained in the set $\{\eta \in \mathbb{R}^{n-1} : \frac{1}{2} < |\eta| < 1\}$, the norm of the inverse operator is less than a constant c_1 independent of q, $q \neq 0$ (see Lemma 1.2.2). For large $|q|$ the operator of the problem (2.2.24), (2.2.25) is close the operator of the problem

$$\langle q\rangle^{-2m}\, L(q, \langle q\rangle D_t)\, v(t) = \phi(t)\,, \quad t>0\,,$$

$$\langle q\rangle^{-\mu_k}\, B_k(q, \langle q\rangle D_t) v(t)\Big|_{t=0} + \sum_{j=1}^{J} \langle q\rangle^{-\mu_k-\tau_j}\, C_{k,j}(q)\, v_j = \psi_k\,, \quad k = 1, \dots, m+J,$$

in the operator norm (2.2.26). Hence the norm of the inverse to this operator is bounded by the constant $2c_1$ if $|q| > \rho$ and ρ is sufficiently large. This implies the estimate (2.2.22) for the solution of (2.2.20), (2.2.21) if $|q| > \rho$. For $|q| \le \rho$ this estimate follows immediately from (iii).

If conversely conditions (iii) and (iv) are satisfied, then by the same arguments, we obtain the unique solvability of the problem (2.2.24), (2.2.25) for sufficiently large $|q| \ge \rho$. Since every point η on the unit sphere in $\mathbb{R}^{n-1}$ is an accumulation point of the set $\{q/\langle q\rangle : q \in \mathbb{Z}^{n-1}, q \neq 0\}$, this implies the unique solvability of the problem (2.2.14), (2.2.15) for $|\eta| = 1$. Therefore, by the homogeneity of the operators L°, B° and C°, this problem is uniquely solvable for each $\eta \in \mathbb{R}^{n-1}$, $\eta \neq 0$, i.e., problem (2.2.7), (2.2.8) is elliptic.

Step 2. Let $(u, \underline{u}) \in W^l_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1})$ be a solution of the problem (2.2.7), (2.2.8). Then the Fourier coefficients $(\dot u(q,\cdot), \underline{\dot u}(q))$ lie in $W_2^l(\mathbb{R}_+) \times \mathbb{C}^J$ and satisfy the equations (2.2.18), (2.2.19) with the Fourier coefficients $\dot f(q,\cdot)$ of f and $\underline{\dot g}(q)$ of g on the right-hand side. If we set $U(q,t) = \dot u(q, \langle q\rangle^{-1} t)$, then the tuple $(U(q,\cdot), \underline{\dot u})$ satisfies (2.2.20), (2.2.21) with $F(q,t) = \dot f(q, \langle q\rangle^{-1} t)$ and $\underline{\dot g}(q)$ on the right-hand side and condition (iv) yields

$$\begin{aligned} \|U(q,\cdot)\|^2_{W_2^l(\mathbb{R}_+)} &+ \sum_{j=1}^{j} \langle q\rangle^{2\tau_j}\, |\dot u_j(q)|^2 \\ &\le c\Big(\langle q\rangle^{-4m}\, \|F(q,\cdot)\|^2_{W_2^{l-2m}(\mathbb{R}_+)} + \sum_{k=1}^{m+J} \langle q\rangle^{-2\mu_k}\, |\dot g_k(q)|^2 \Big)\,. \end{aligned} \tag{2.2.27}$$

Multiplying this inequality by $\langle q\rangle^{2l-1}$ and summing up over all $q \in \mathbb{Z}^{n-1}$, one gets (2.2.23). Consequently, the solution of the problem (2.2.7), (2.2.8) is uniquely determined. Furthermore, the solvability of the problem (2.2.18), (2.2.19) ensures the existence of a solution of the problem (2.2.7), (2.2.8).

Step 3. We assume that (2.2.9) is an isomorphism. Then the inequality (2.2.27) is satisfied for each $u \in W^l_{2,per}(\mathbb{R}^n)$, $\underline{u} \in W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1})$. If we set

$$u(y,t) = v(\langle q\rangle t)\, e^{iq\cdot y}\,, \quad \underline{u} = \underline{v}\, e^{iq\cdot y}$$

with arbitrary $v \in W^l_2(\mathbb{R}_+)$, $\underline{v} \in \mathbb{C}^J$ and $q \in \mathbb{Z}^{n-1}$, we get (2.2.22), where ϕ and $\underline{\psi}$ are the right-hand sides of (2.2.20), (2.2.21). This proves the uniqueness of the solution of (2.2.20), (2.2.21). Now let ϕ be an arbitrary function from $W^{l-2m}_2(\mathbb{R}_+)$ and $\underline{\psi} \in \mathbb{C}^J$ an arbitrary vector. Then by our assumption, there exists a solution $(u, \underline{u})$ of the problem (2.2.7), (2.2.8) with the right-hand sides

$$f(y,t) = \phi(\langle q\rangle t)\, e^{iq\cdot y} \quad \underline{g} = \underline{\psi}\, e^{iq\cdot y}\,.$$

The Fourier coefficients $\dot{u}(q,\cdot)$ and $\underline{\dot{u}}(q)$ of u and $\underline{u}$ satisfy the equations

$$L(q,D_t)\,\dot{u}(q,t) = \phi(\langle q\rangle t) \ \text{ for } t = 0,$$
$$B(q,D_t)\dot{u}(q,t)\Big|_{t=0} + C(q)\underline{\dot{u}}(q) = \underline{\psi}\,.$$

Hence the pair $(v, \underline{v})$ with $v(t) = \underline{u}(q, \langle q\rangle^{-1}t)$, $\underline{v} = \underline{\dot{u}}(q)$ is a solution of the problem (2.2.20), (2.2.21). The proof of the theorem is complete. ∎

REMARK 2.2.3. The conditions (i), (ii) in Theorem 2.2.1 are satisfied, e.g., for the boundary value problem

$$L^\circ(D_y + \tfrac{1}{2}\,\vec{1}, D_t + \tfrac{1}{2})\, u = f \ \text{ in } \mathbb{R}^n_+\,, \tag{2.2.28}$$

$$B^\circ(D_y + \tfrac{1}{2}\,\vec{1}, D_t + \tfrac{1}{2})u\Big|_{t=0} + C^\circ(D_y + \tfrac{1}{2}\,\vec{1})\underline{u} = \underline{g} \ \text{ on } \mathbb{R}^{n-1} \tag{2.2.29}$$

if the boundary value problem (2.2.7), (2.2.8) is elliptic.

Indeed, the principal parts of the differential operators of the boundary value problems (2.2.7), (2.2.8) and (2.2.28), (2.2.29) coincide. Hence condition (i) in Theorem 2.2.1 is obviously satisfied for the problem (2.2.28), (2.2.29). Furthermore, it follows from the ellipticity of the problem (2.2.7), (2.2.8) that the boundary value problem

$$L^\circ(q + \tfrac{1}{2}\,\vec{1}, D_t)\, u = \phi \ \text{ in } \mathbb{R}_+\,,$$
$$B^\circ(q + \tfrac{1}{2}\,\vec{1}, D_t)u\Big|_{t=0} + C^\circ(q + \tfrac{1}{2}\,\vec{1})\underline{u} = \underline{\psi}$$

is uniquely solvable in $W^l_2(\mathbb{R}_+)\times\mathbb{C}^J$ for every $q \in \mathbb{Z}^{n-1}$, $\phi \in W^{l-2m}_2(\mathbb{R}_+)$, $\underline{\psi} \in \mathbb{C}^{m+J}$ (see Theorem 1.2.1). This problem is equivalent to

$$L^\circ(q + \tfrac{1}{2}\,\vec{1}, D_t + \tfrac{1}{2})\, v = e^{it/2}\phi \ \text{ in } \mathbb{R}_+\,,$$
$$B^\circ(q + \tfrac{1}{2}\,\vec{1}, D_t + \tfrac{1}{2})v\Big|_{t=0} + C^\circ(q + \tfrac{1}{2}\,\vec{1})\underline{u} = \underline{\psi}$$

if we set $u = e^{it/2}v$. Consequently, condition (ii) in Theorem 2.2.1 is satisfied for the boundary value problem (2.2.28), (2.2.29).

2.3. Solvability of elliptic boundary value problems in the half-space in Sobolev spaces of arbitrary integer order

In this section the results of Section 2.2 are extended to Sobolev spaces of arbitrary integer order. We prove the validity of a Green formula and introduce the formally adjoint boundary value problem. Here we restrict ourselves to the case when the boundary operators B_k are of order less than $2m$.

The extension of the operator of the boundary value problem to Sobolev spaces of lower order is constructed in the same way as in the one-dimensional case. We show that conditions (i) and (ii) in Theorem 2.1.2 are also necessary and sufficient for the unique solvability of the boundary value problem in Sobolev spaces of small positive and negative order.

Furthermore, this section contains a regularity assertion and a priori estimates for solutions of elliptic boundary value problems with variable coefficients in the half-space.

2.3.1. Sobolev spaces of negative order. For $l = 1, 2, \dots$ let $W^l_{2,per}(\mathbb{R}^n_+)^*$ be the dual space of $W^l_{2,per}(\mathbb{R}^n_+)$ equipped with the norm

(2.3.1)
$$\|u\|_{W^l_{2,per}(\mathbb{R}^n_+)^*} = \sup \left\{ |(u,v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}| \,:\, v \in W^l_{2,per}(\mathbb{R}^n_+),\ \|v\|_{W^l_{2,per}(\mathbb{R}^n_+)} = 1 \right\}.$$

Here $(\cdot,\cdot)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}$ denotes the extension of the scalar product (2.2.2) to the product of the spaces $W^l_{2,per}(\mathbb{R}^n_+)^*$ and $W^l_{2,per}(\mathbb{R}^n_+)$. If u is an arbitrary element of $W^l_{2,per}(\mathbb{R}^n_+)^*$, then the Fourier coefficients $\dot{u}(q,\cdot)$ are functionals on $W^l_2(\mathbb{R}_+)$ which are defined by the equality

$$(\dot{u}(q,\cdot)\,,\,\phi(\cdot))_{\mathbb{R}_+} = (2\pi)^{-(n-1)/2}\,(u\,,\,e^{iq\cdot y}\,\phi)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}\,, \qquad \phi \in W^l_2(\mathbb{R}_+). \tag{2.3.2}$$

LEMMA 2.3.1. *The norm (2.3.1) is equal to*

$$\|u\| = \Big(\sum_{q\in\mathbb{Z}^{n-1}} \langle q\rangle^{-2l-1} \|U(q,\cdot)\|^2_{W^l_2(\mathbb{R}_+)^*}\Big)^{1/2}, \tag{2.3.3}$$

where U is given by (2.2.3).

Proof: For arbitrary $u \in W^l_{2,per}(\mathbb{R}^n_+)^*$, $v \in W^l_{2,per}(\mathbb{R}^n_+)$ we have

$$\begin{aligned} |(u,v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}| &= \Big|\sum_q \big(\dot{u}(q,\cdot)\,,\,\dot{v}(q,\cdot)\big)_{\mathbb{R}_+}\Big| = \Big|\sum_q \langle q\rangle^{-1}\,\big(U(q,\cdot)\,,\,V(q,\cdot)\big)_{\mathbb{R}_+}\Big| \\ &\le \sum_q \langle q\rangle^{-1}\,\|U(q,\cdot)\|_{W^l_2(\mathbb{R}_+)^*}\,\|V(q,\cdot)\|_{W^l_2(\mathbb{R}_+)}\,, \end{aligned}$$

where $U(q,t) = \dot{u}(q,\langle q\rangle^{-1}t)$, $V(q,t) = \dot{v}(q,\langle q\rangle^{-1}t)$. Using the Schwartz inequality and the representation (2.2.4) for the norm of v in $W^l_{2,per}(\mathbb{R}^n_+)$, we get

$$|(u,v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}| \le \Big(\sum_q \langle q\rangle^{-2l-1}\,\|U(q,\cdot)\|^2_{W^l_2(\mathbb{R}_+)^*}\Big)^{1/2} \|v\|_{W^l_{2,per}(\mathbb{R}^n_+)}\,.$$

Consequently, the norm (2.3.1) does not exceed the norm (2.3.3).

On the other hand, for every given $U(q,\cdot) \in W^l_2(\mathbb{R}_+)^*$ and every positive number $\varepsilon < 1$ there exists a function $V(q,\cdot) \in W^l_2(\mathbb{R}_+)$ such that

$$(U(q,\cdot)\,,\,V(q,\cdot))_{\mathbb{R}_+} \ge (1-\varepsilon)\,\|U(q,\cdot)\|_{W^l_2(\mathbb{R}_+)^*}\,\|V(q,\cdot)\|_{W^l_2(\mathbb{R}_+)}$$

and

$$\|V(q,\cdot)\|_{W^l_2(\mathbb{R}_+)} = \langle q\rangle^{-2l}\,\|U(q,\cdot)\|_{W^l_2(\mathbb{R}_+)^*}\,. \tag{2.3.4}$$

We define the function $v \in W^l_{2,per}(\mathbb{R}^n_+)$ as

$$v(y,t) = \sum_q V\big(q,\langle q\rangle\,t\big)\,e^{iq\cdot y}\,.$$

Then it follows from (2.3.4) that the norm of v in $W^l_{2,per}(\mathbb{R}^n_+)$ is equal to the right side of (2.3.3). Hence the inequality

$$\sum_q \langle q\rangle^{-2l-1} \|U(q,\cdot)\|^2_{W^l_2(\mathbb{R}_+)^*} = \sum_q \langle q\rangle^{-1} \|U(q,\cdot)\|_{W^l_2(\mathbb{R}_+)^*} \|V(q,\cdot)\|_{W^l_2(\mathbb{R}_+)}$$
$$\le (1-\varepsilon)^{-1} \sum_q \langle q\rangle^{-1} \big(U(q,\cdot), V(q,\cdot)\big)_{\mathbb{R}_+} = (1-\varepsilon)^{-1} (u,v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}$$
$$\le (1-\varepsilon)^{-1} \|u\|_{W^l_{2,per}(\mathbb{R}^n_+)^*} \|v\|_{W^l_{2,per}(\mathbb{R}^n_+)} .$$

yields

$$\Big(\sum_q \langle q\rangle^{-2l-1} \|U(q,\cdot)\|^2_{W^l_2(\mathbb{R}_+)^*}\Big)^{1/2} \le (1-\varepsilon)^{-1} \|u\|_{W^l_{2,per}(\mathbb{R}^n_+)^*} .$$

This proves the lemma. ■

Analogously to the one-dimensional case, we define the space $\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)$ for integer k, l, $k \ge 0$, as follows. If $l \ge 0$, then $\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)$ is the set of all pairs $(u, \underline{\phi})$, where $u \in W^l_{2,per}(\mathbb{R}^n_+)$ and $\underline{\phi} = (\phi_1, \dots, \phi_k)$ is a vector-function with components $\phi_j \in W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1})$ satisfying the condition

$$\phi_j(y) = (D_t^{j-1} u)(y, 0) \quad \text{for } j \le l.$$

The norm in $\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)$ is defined as

$$\|(u,\underline{\phi})\|_{\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)} = \|u\|_{W^l_{2,per}(\mathbb{R}^n_+)} + \sum_{j=1}^k \|\phi_j\|_{W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1})} .$$

Since only the components ϕ_j with $j > l$ can be chosen independent of u, the space $\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)$ can be identified with $W^l_{2,per}(\mathbb{R}^n_+)$ if $l \ge k$ and with

$$W^l_{2,per}(\mathbb{R}^n_+) \times \prod_{j=l+1}^k W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1})$$

if $0 \le l < k$. In the case $l \le 0$ we set

$$\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+) = W^{-l}_{2,per}(\mathbb{R}^n_+)^* \times \prod_{j=1}^k W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1})$$

and

$$\|(u,\underline{\phi})\|_{\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)} = \|u\|_{W^{-l}_{2,per}(\mathbb{R}^n_+)^*} + \sum_{j=1}^k \|\phi_j\|_{W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1})} .$$

In particular, we have $\tilde{W}^{l,0}_{2,per}(\mathbb{R}^n_+) = W^{-l}_{2,per}(\mathbb{R}^n_+)^*$ if $l < 0$ and $\tilde{W}^{l,0}_{2,per}(\mathbb{R}^n_+) = W^l_{2,per}(\mathbb{R}^n_+)$ if $l \ge 0$.

Obviously, the space $\tilde{W}^{l+1,k}_{2,per}(\mathbb{R}^n_+)$ is continuously imbedded into $\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)$. Furthermore, it can be shown that $\tilde{W}^{l+1,k}_{2,per}(\mathbb{R}^n_+)$ is dense in $\tilde{W}^{l,k}_{2,per}(\mathbb{R}^n_+)$ (cf. Lemma 1.3.1).

2.3.2. The Green formula in the half-space. Now we consider the boundary value problem

$$L(y,t,D_y,D_t)\,u(y,t) = f(y,t) \quad \text{in } \mathbb{R}^n_+\,, \tag{2.3.5}$$

$$B(y,t,D_y,D_t)u(y,t)\Big|_{t=0} + C(y,D_y)\underline{u}(y) = \underline{g}(y)\,, \quad y \in \mathbb{R}^{n-1}, \tag{2.3.6}$$

where, in contrast to the previous section, the coefficients of the operators L, B and C may be variable. More precisely,

$$L(y,t,D_y,D_t) = \sum_{|\beta|+j\le 2m} a_{\beta,j}(y,t)\,D_y^\beta\,D_t^j \tag{2.3.7}$$

is a differential operator of order $2m$ with infinitely differentiable coefficients $a_{\beta,j}$ which are 2π-periodic with respect to the variable y and have bounded t-derivatives of arbitrary order. Furthermore, B is a vector of differential operators

$$B_k(y,t,D_y,D_t) = \sum_{|\beta|+j\le \mu_k} b_{k;\beta,j}(y,t)\,D_y^\beta\,D_t^j\,, \quad k=1,\dots,m+J, \tag{2.3.8}$$

and C is a matrix of differential operators

$$C_{k,j}(y,D_y) = \sum_{|\beta|\le \mu_k+\tau_j} c_{k,j;\beta}(y)\,D_y^\beta\,, \quad k=1,\dots,m+J,\ j=1,\dots,J, \tag{2.3.9}$$

with infinitely differentiable 2π-periodic (with respect to y) coefficients. It will be assumed in this section that $\operatorname{ord} B_k < 2m$ for $k=1,\dots,m+J$.

In order to derive a Green formula for problem (2.3.5), (2.3.6), it is useful to write the operator L in the form

$$L(y,t,D_y,D_t) = \sum_{j=0}^{2m} A_j(y,t,D_y)\,D_t^j\,,$$

where

$$A_j(y,t,D_y) = \sum_{|\beta|\le 2m-j} a_{\beta,j}(y,t)\,D_y^\beta\,.$$

Analogously, the operators B_k have the form

$$B_k(y,t,D_y,D_t) = \sum_{j=0}^{\mu_k} B_{k,j}(y,t,D_y)\,D_t^j$$

where $B_{k,j}(y,D_y)$ are linear differential operators of order $\le \mu_k - j$. Since $\operatorname{ord} B_k < 2m$ for $k=1,\dots,m+J$, the operator B admits the representation

$$B(y,t,D_y,D_t) = Q(y,t,D_y)\cdot\mathcal{D}\,, \tag{2.3.10}$$

where, as in Chapter 1, $\mathcal{D}$ denotes the column vector with the components D_t^{j-1}, $j=1,\dots,2m$, and

$$Q(y,t,D_y) = \Big(Q_{k,j}(y,t,D_y)\Big)_{1\le k\le m+J,\,1\le j\le 2m}$$

is the matrix with the elements

$$Q_{k,j}(y,t,D_y) = \begin{cases} B_{k,j-1}(y,t,D_y) & \text{for } \ j\le \mu_k+1,\\ 0 & \text{for } \ \mu_k+1 < j \le 2m.\end{cases}$$

Let L^+ be the formally adjoint operator to the differential operator (2.3.7), i.e.,

$$L^+(y,t,D_y,D_t)\,v = \sum_{|\beta|+j\le 2m} D_y^\beta\, D_t^j\,(\overline{a_{\beta,j}(y,t)}\,v).$$

Analogously, the formally adjoint operators to A_j, $B_{k,j}$ and $C_{k,j}$ are defined. The formally adjoint operator to the matrix $C(y,D_y)$ is the $J\times(m+J)$-matrix

$$C^+(y,D_y) = \Big(C^+_{k,j}(y,D_y)\Big)_{1\le j\le J,\ 1\le k\le m+J}.$$

Using this notation, we obtain the following Green formula which has the same form as formula (1.1.6).

THEOREM 2.3.1. *Let $u=u(y,t)$, $v=v(y,t)$ be arbitrary smooth functions on $\overline{\mathbb{R}^n_+}$ which are 2π-periodic with respect to y and equal to zero for large t. Furthermore, let $\underline{u}=(u_1,\dots,u_J)$, $\underline{v}=(v_1,\dots,v_{m+J})$ be arbitrary smooth 2π-periodic vector-functions on $\mathbb{R}^{n-1}$. Then the Green formula*

$$\begin{aligned}(2.3.11)\quad &\int_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} Lu\cdot\overline{v}\,dy\,dt + \int_{\mathbb{Q}^{n-1}} \Big((Bu)(y,0)+C\,\underline{u}(y)\,,\,\underline{v}(y)\Big)_{\mathbb{C}^{m+J}}\,dy \\ &= \int_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} u\cdot\overline{L^+v}\,dy\,dt + \int_{\mathbb{Q}^{n-1}} \Big((\mathcal{D}u)(y,0)\,,\,(Pv)(y,0)+Q^+\,\underline{v}(y)\Big)_{\mathbb{C}^{2m}}\,dy \\ &\quad + \int_{\mathbb{Q}^{n-1}} \big(\underline{u}(y),C^+\underline{v}(y)\big)_{\mathbb{C}^J}\,dy\end{aligned}$$

is valid. Here $Pv=P(y,t,D_y,D_t)\,v$ denotes the vector with the components

$$(2.3.12)\qquad P_j(y,t,D_y,D_t)\,v = -i\sum_{s=0}^{2m-j} D_t^s\Big(A^+_{j+s}(y,t,D_y)\,v\Big),\quad j=1,\dots,2m.$$

Proof: Analogously to the proof of Theorem 1.1.1, it can be shown by induction that

$$\begin{aligned}(2.3.13)\quad \int_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} u\cdot\overline{L^+v}\,dy\,dt &= \int_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} \big(L_ju\cdot\overline{v} - i\,D_t^ju\cdot\overline{P_jv}\big)\,dy\,dt \\ &\quad -\sum_{s=1}^{j}\int_{\mathbb{Q}^{n-1}} (D_t^{s-1}u)(y,0)\cdot\overline{(P_sv)(y,0)}\,dy\end{aligned}$$

for $j=1,\dots,2m$ and for all infinitely differentiable functions u, v on $\overline{\mathbb{R}^n_+}$ which are 2π-periodic with respect to the variable y and equal to zero for large t. Here L_j denotes the differential operator

$$(2.3.14)\qquad L_j = \sum_{s=0}^{j-1} A_s(y,t,D_y)\,D_t^s$$

For $j=2m$ we get

$$(2.3.15)\quad \int_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} \big(u\cdot\overline{L^+v} - Lu\cdot\overline{v}\big)\,dy\,dt = -\int_{\mathbb{Q}^{n-1}} \big((\mathcal{D}u)(y,0)\,,\,(Pv)(y,0)\big)_{\mathbb{C}^{2m}}\,dy.$$

Furthermore, we have

$$\int\limits_{\mathbb{Q}^{n-1}} \big(C\underline{u}(y), \underline{v}(y)\big)_{\mathbb{C}^{m+J}}\, dy = \int\limits_{\mathbb{Q}^{n-1}} \big(\underline{u}(y), C^+\underline{v}(y)\big)_{\mathbb{C}^J}\, dy$$

and, by the representation (2.3.10) for B,

$$\int\limits_{\mathbb{Q}^{n-1}} \big((Bu)(y,0), \underline{v}(y)\big)_{\mathbb{C}^{m+J}}\, dy = \int\limits_{\mathbb{Q}^{n-1}} \big((\mathcal{D}u)(y,0), Q^+\underline{v}\big)_{\mathbb{C}^{2m}}\, dy.$$

Hence we get (2.3.11). ■

REMARK 2.3.1. The vector P given in Theorem 2.3.1 can be written in the form $P = T(y,t,D_y)\,\mathcal{D}$, where

$$T(y,t,D_y) = \Big(T_{j,s}(y,t,D_y)\Big)_{1\le j,s\le 2m}$$

is a triangular matrix of differential operators $T_{j,s}$ with smooth coefficients, $T_{j,s} = 0$ if $j+s > 2m+1$, $T_{j,2m+1-j} = -i\,\overline{a}_{0,2m}(y,t)$ for $j = 1,\dots,2m$, and $\operatorname{ord} T_{j,s} \le 2m+1-j-s$ for $j+s \le 2m$. If $a_{0,2m}(y,0) \neq 0$ for $y \in \mathbb{R}^{n-1}$, then the mapping $\underline{w} \to T(y,0,D_y)\underline{w}$ is an isomorphism

$$\prod_{j=1}^{2m} W_{2,per}^{l-j+1/2}(\mathbb{R}^{n-1}) \to \prod_{j=1}^{2m} W_{2,per}^{l-2m+j-1/2}(\mathbb{R}^{n-1})$$

for arbitrary l. The inverse to T is a matrix

$$T^{-1}(y,t,D_y) = \Big(S_{j,s}(y,t,D_y)\Big)_{1\le j,s\le 2m}$$

of differential operators $S_{j,s}$ with $S_{j,s} = 0$ for $j+s < 2m+1$, $S_{j,2m+1-j} = i\,\overline{a}_{0,2m}(y,t)^{-1}$ for j=1, ... ,2m, $\operatorname{ord} S_{j,s} \le j+s-2m-1$ for $j+s > 2m+1$.

Analogously to the one-dimensional case, we define the formally adjoint boundary value problem to problem (2.3.5), (2.3.6) as follows.

DEFINITION 2.3.1. Let P be the vector with the components (2.3.12). Then the boundary value problem

$$L^+ v = f \quad \text{in } \mathbb{R}^n_+\,, \tag{2.3.16}$$

$$Pv|_{t=0} + Q^+\underline{v} = \underline{g}\,, \quad C^+\underline{v} = \underline{h} \quad \text{on } \mathbb{R}^{n-1} \tag{2.3.17}$$

is said to be *formally adjoint* to the boundary value problem (2.3.5), (2.3.6).

REMARK 2.3.2. If the operators L, B and C have constant coefficients, then the coefficients of the operators in the formally adjoint problem (2.3.16), (2.3.17) are also constant. Furthermore, as a consequence of Theorem 1.2.2, the formally adjoint problem is elliptic if and only if the original problem (2.3.5), (2.3.6) is elliptic.

2.3.3. Extension of the operator of the boundary value problem. We consider the boundary value problem (2.3.5), (2.3.6). If we identify every function $u \in W^l_{2,per}(\mathbb{R}^n_+)$, $l \geq 2m$, with the corresponding pair $(u, \mathcal{D}u|_{t=0}) \in \tilde{W}^{l,2m}_{2,per}(\mathbb{R}^n_+)$, we can consider the operator $\mathcal{A}$ of this problem as a linear and continuous mapping from

$$\tilde{W}^{l,2m}_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1}) \tag{2.3.18}$$

into

$$\tilde{W}^{l-2m,0}_{2,per}(\mathbb{R}^n_+) \times W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1}) . \tag{2.3.19}$$

Using the Green formula (2.3.11) and the formula

$$\int\limits_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} Lu \cdot \overline{v}\, dy\, dt = \int\limits_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} \big(L_l u \cdot \overline{v} - i\, D_t^l u \cdot \overline{P_l v}\big)\, dy\, dt + \sum_{j=l+1}^{2m} \int\limits_{\mathbb{Q}^{n-1}} (D_t^{j-1}u)(y,0) \cdot \overline{(P_j v)(y,0)}\, dy \tag{2.3.20}$$

which follows from (2.3.13) and (2.3.15), we can construct an extension of the operator $\mathcal{A}$ to the space (2.3.18) with arbitrary integer $l < 2m$. Analogously to Theorem 1.3.1, the following statement holds.

THEOREM 2.3.2. *The operator*

$$\tilde{W}^{2m,2m}_{2,per}(\mathbb{R}^n_+) \times W^{2m+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1}) \ni \big(u, \mathcal{D}u|_{t=0}, \underline{u}\big) \to \big(Lu, Bu|_{t=0} + C\underline{u}\big) \in L_{2,per}(\mathbb{R}^n_+) \times W^{2m-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1}) \tag{2.3.21}$$

can be uniquely extended to a continuous operator from (2.3.18) into (2.3.19) with $l < 2m$. This extension has the form

$$(u, \underline{\phi}, \underline{u}) \to \big(L(u,\underline{\phi}),\ Q\underline{\phi} + C\underline{u}\big)$$

where the functional $f = L(u,\underline{\phi}) \in W^{2m-l}_{2,per}(\mathbb{R}^n_+)^*$ is defined

a) *in the case $l \leq 0$ by the equality*

$$(f, v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} = (u, L^+ v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} + \big(\underline{\phi},\ Pv|_{t=0}\big)_{\mathbb{Q}^{n-1}} , \tag{2.3.22}$$

b) *in the case $0 < l < 2m$ by the equality*

$$(f, v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} = \int\limits_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} \big(L_l u \cdot \overline{v} - i\, D_t^l u \cdot \overline{P_l v}\big)\, dy\, dt + \sum_{j=l+1}^{2m} (\phi_j\, ,\, P_j v|_{t=0})_{\mathbb{Q}^{n-1}} , \tag{2.3.23}$$

where v is an arbitrary function from $W^{-l+2m}_{2,per}(\mathbb{R}^n_+)$.

Here $(\cdot,\cdot)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}$ denotes the scalar product (2.2.2), while $(\cdot,\cdot)_{\mathbb{Q}^{n-1}}$ is the scalar product in $L_{2,per}(\mathbb{R}^{n-1})$ and $L_{2,per}(\mathbb{R}^{n-1})^{2m}$.

Note that the mappings

$$Q(y, 0, D_y)\ :\ \prod_{j=1}^{2m} W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1}) \to W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})$$

and

$$C(y, D_y) \,:\, W_{2,per}^{l+\underline{\tau}-1/2}(\mathbb{R}^{n-1}) \to W_{2,per}^{l-\underline{\mu}-1/2}(\mathbb{R}^{n-1})$$

are continuous for arbitrary l. We denote the operator (2.3.21) and its extension to the space (2.3.18) with $l < 2m$ also by $\mathcal{A}$.

If $(u, \underline{\phi}) \in \tilde{W}_{2,per}^{l,2m}(\mathbb{R}_+^n)$, $l \geq 2m$, then $\underline{\phi} = (\phi_1, \dots, \phi_{2m}) = \mathcal{D}u|_{t=0}$ and, consequently,

$$\sum_{j=1}^{2m} \|\phi_j\|_{W_{2,per}^{l-j+1/2}(\mathbb{R}^{n-1})} \leq c\, \|u\|_{W_{2,per}^{l}(\mathbb{R}_+^n)} \,.$$

This is not true if $l < 2m$. However, the following assertion holds.

LEMMA 2.3.2. *Let ζ be a smooth 2π-periodic function on $\mathbb{R}^{n-1}$ and let $\eta = \eta(y,t)$ be a smooth function on $\overline{\mathbb{R}_+^n}$ which is 2π-periodic with respect to the variable y such that $\eta(y,t) = 0$ for $t > 2$, $\eta(y,t) = 1$ if $t < 1$ and y lies in a neighbourhood of* $\operatorname{supp}\zeta$. *Suppose that the coefficient $a_{0,2m}$ of L does not vanish. Then there exists a constant $c > 0$ such that the inequality*

$$\sum_{j=1}^{2m} \|\zeta\phi_j\|_{W_{2,per}^{l-j+1/2}(\mathbb{R}^{n-1})} \leq c\left(\|\eta u\|_{\tilde{W}_{2,per}^{l,0}(\mathbb{R}_+^n)} + \|\eta L(u,\underline{\phi})\|_{\tilde{W}_{2,per}^{l-2m,0}(\mathbb{R}_+^n)}\right) \tag{2.3.24}$$

is satisfied for every $(u, \underline{\phi}) \in \tilde{W}_{2,per}^{l,2m}(\mathbb{R}_+^n)$.

Proof: For $l \geq 2m$ this estimate is obvious. In this case the norm of $\eta L(u, \underline{\phi})$ can be even omitted. Now let l be a nonpositive integer. Then the functional $f = L(u, \underline{\phi})$ is given by (2.3.22). Here the trace of Pv on the plane $t = 0$ is a vector of the form $T(y, 0, D_y) \cdot \mathcal{D}v$ with an invertible matrix $T(y, 0, D_y)$ of differential operators on $\mathbb{R}^{n-1}$ (see Remark 2.3.1). Consequently, for every vector-function $\underline{\psi} = (\psi_1, \dots, \psi_{2m})$, $\psi_j \in W_{2,per}^{-l+j-1/2}(\mathbb{R}^{n-1})$, there exists a function $w = e_{2m}T^{-1}\psi \in W_{2,per}^{-l}(\mathbb{R}_+^n)$ such that $Pw|_{t=0} = \underline{\psi}$. Here e_{2m} denotes the extension operator of Lemma 2.2.1. Let $\zeta^{(1)}$ be a smooth 2π-periodic function on $\mathbb{R}^{n-1}$ such that $\zeta(y)\,\zeta^{(1)}(y) = \zeta(y)$, $\zeta^{(1)}(y)\,\eta(y,0) = \zeta^{(1)}(y)$. We insert $v = \overline{\zeta^{(1)}}\, e_{2m}\, T^{-1}\,(\overline{\zeta}\underline{\psi})$ into (2.3.22). Since $\eta \equiv 1$ on $\operatorname{supp} v$, we obtain

$$\begin{aligned}(\zeta\underline{\phi}, \underline{\psi})_{\mathbb{Q}^{n-1}} &= (\underline{\phi}, \overline{\zeta}\underline{\psi})_{\mathbb{Q}^{n-1}} = (f, v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} - (u, L^+v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} \\ &= (\eta f, v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} - (\eta u, L^+v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} \,.\end{aligned}$$

This implies (2.3.24). If $0 < l < 2m$, then this inequality can be analogously proved by means of (2.3.23). The proof is complete. ■

Naturally, in Lemma 2.3.2 the function ζ can be identically equal to one. Then as a consequence of this lemma we get the inequality (2.3.24) without the functions ζ and η.

2.3.4. Existence and uniqueness of the solutions of elliptic boundary value problems with constant coefficients in the half-space. We return to the problem (2.2.7), (2.2.8) with constant coefficients. As in the previous subsections, we assume that $\operatorname{ord} B_k < 2m$ for $k = 1, \dots, m + J$. Then the Green formula (2.3.11) is valid and the coefficients of the operators L^+, P, Q^+ and C^+ are constant.

THEOREM 2.3.3. *The boundary value problem (2.2.7), (2.2.8) is elliptic if and only if the formally adjoint problem (2.3.16), (2.3.17) is elliptic. Furthermore, condition (ii) in Theorem 2.2.1 is satisfied for problem (2.2.7), (2.2.8) if and only if it is satisfied for the formally adjoint problem.*

This assertion follows immediately from Theorem 1.2.2.

COROLLARY 2.3.1. *If conditions (i) and (ii) in Theorem 2.2.1 are satisfied for the boundary value problem (2.2.7), (2.2.8), then the operator*

$$\begin{aligned}(2.3.25)\qquad \mathcal{A}^+ : W^l_{2,per}(\mathbb{R}^n_+) \times \prod_{k=1}^{m+J} W^{l-2m+\mu_k+1/2}_{2,per}(\mathbb{R}^{n-1}) \\ \to W^{l-2m}_{2,per}(\mathbb{R}^n_+) \times \prod_{j=1}^{2m} W^{l-2m+j-1/2}_{2,per}(\mathbb{R}^{n-1}) \times \prod_{j=1}^{J} W^{l-2m-\tau_j+1/2}_{2,per}(\mathbb{R}^{n-1})\end{aligned}$$

of the formally adjoint problem is an isomorphism for arbitrary $l \geq 2m$.

Using this result and the relations between the operators $\mathcal{A}$ and $\mathcal{A}^+$, we obtain the following generalization of Theorem 2.2.1.

THEOREM 2.3.4. *Suppose that conditions (i), (ii) of Theorem 2.2.1 are satisfied. Then the operator $\mathcal{A}$ is an isomorphism from (2.3.18) into (2.3.19) for arbitrary integer l.*

Proof: For $l \geq 2m$ the assertion of the theorem has been already proved in the foregoing section. By Theorem 2.3.2, the mapping

$$\begin{aligned}\tilde{W}^{l,2m}_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1}) \ni (u, \underline{\phi}, \underline{u}) \\ \to (f, \underline{g}) = \mathcal{A}\,(u, \underline{\phi}, \underline{u}) \in \tilde{W}^{l-2m,0}_{2,per}(\mathbb{R}^n_+) \times W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})\end{aligned}$$

is defined in the case $l \leq 0$ by the equality

$$\begin{aligned}(2.3.26)\qquad & (f, v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} + (\underline{g}, \underline{v})_{\mathbb{Q}^{n-1}} \\ & = (u, L^+v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} + \big(\underline{\phi}\,,\, Pv|_{t=0} + Q^+\underline{v}\big)_{\mathbb{Q}^{n-1}} + (\underline{u}, C^+\underline{v})_{\mathbb{Q}^{n-1}}\,,\end{aligned}$$

where $v \in W^{-l+2m}_{2,per}(\mathbb{R}^n_+)$, $\underline{v} \in W^{-l+\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})$. (Here, for the sake of brevity, we have used the same notation $(\cdot,\cdot)_{\mathbb{Q}^{n-1}}$ for the scalar products in $L_2(\mathbb{Q}^{n-1})^{2m}$, $L_2(\mathbb{Q}^{n-1})^J$, and $L_2(\mathbb{Q}^{n-1})^{m+J}$).) This means, the operator $\mathcal{A}$ is adjoint to the operator (2.3.25) if we replace the number l in (2.3.25) by $2m-l$. Therefore, for $l \leq 0$ the assertion of the theorem is an immediate consequence of Corollary 2.3.1.

We consider the case $0 < l < 2m$. Let f be an arbitrary functional from $W^{2m-l}_{2,per}(\mathbb{R}^n_+)^*$, and let $\underline{g}$ be an arbitrary vector-function from $W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})$. For every $q \in \mathbb{Z}^{n-1}$ we denote by $\mathcal{A}_q$ the operator of the boundary value problem (2.2.18), (2.2.19) and also its extension to the space $\tilde{W}^{l,2m}_2(\mathbb{R}_+) \times \mathbb{C}^J$. By Theorem 1.3.1, the operator $\mathcal{A}_q$ is an isomorphism from $\tilde{W}^{l,2m}_2(\mathbb{R}_+) \times \mathbb{C}^J$ onto $W^{2m-l}_2(\mathbb{R}_+)^* \times \mathbb{C}^{m+J}$. Consequently, for each $q \in \mathbb{Z}^{n-1}$ there exists a unique solution $(\dot{u}(q,\cdot), \underline{\dot{\phi}}(q), \underline{\dot{u}}(q)\,) \in \tilde{W}^{l,2m}_2(\mathbb{R}_+) \times \mathbb{C}^J$ of the equation

$$(2.3.27)\qquad \mathcal{A}_q\,\big(\dot{u}(q,\cdot), \underline{\dot{\phi}}(q), \underline{\dot{u}}(q)\big) = \big(\dot{f}(q,\cdot), \underline{\dot{g}}(q)\big),$$

where $\dot{f}(q,\cdot)$, $\dot{\underline{g}}(q)$ denote the Fourier coefficients of f and $\underline{g}$. Analogously to the proof of Theorem 2.2.1, it can be shown that $U(q,t) = \dot{u}(q, \langle q\rangle^{-1}t)$, $\dot{\underline{\phi}}(q)$ and $\dot{\underline{u}}(q)$ satisfy the estimate

$$\langle q\rangle^{2l-1}\,\|U(q,\cdot)\|^2_{W_2^l(\mathbb{R}_+)} + \sum_{j=1}^{2m}\langle q\rangle^{2(l-j)+1}\,|\dot{\phi}_j(q)|^2 + \sum_{j=1}^{J}\langle q\rangle^{2(l+\tau_j)-1}\,|\dot{u}_j(q)|^2 \tag{2.3.28}$$
$$\le c\Big(\langle q\rangle^{2(l-2m)-1}\,\|F(q,\cdot)\|^2_{W_2^{2m-l}(\mathbb{R}_+)^*} + \sum_{k=1}^{m+J}\langle q\rangle^{2(l-\mu_k)-1}\,|\dot{g}_k(q)|^2\Big)$$

with a constant c independent of f, $\underline{g}$ and q. Consequently, the function

$$u(y,t) = \sum_{q\in\mathbb{Z}^{n-1}} \dot{u}(q,t)\, e^{iq\cdot y}$$

belongs to $W^l_{2,per}(\mathbb{R}^n_+)$ and the functions ϕ_j, u_j with the Fourier coefficients $\dot{\phi}_j(q)$, $\dot{u}_j(q)$ belong to the spaces $W^{l-j+1/2}_{2,per}(\mathbb{R}^{n-1})$ and $W^{l+\tau_j-1/2}_{2,per}(\mathbb{R}^{n-1})$, respectively. Furthermore, from $\dot{\phi}_j(q) = (D_t^{j-1}\dot{u})(q,0)$ for $j=1,\dots,l$ it follows that $\phi_j(y) = (D_t^{j-1}u)(y,0)$ for $j=1,\dots,l$ and, therefore, $(u,\underline{\phi}) \in \tilde{W}_2^{l,2m}(\mathbb{Q}^n_+)$. By (2.3.27), $(u,\underline{\phi},\underline{u})$ is a solution of the equation

$$\mathcal{A}\,(u,\underline{\phi},\underline{u}) = (f,\underline{g})\,.$$

The uniqueness of the solution follows from (2.3.28). ∎

2.3.5. A regularity assertion for solutions of elliptic boundary value problems in the half-space. Now we consider problem (2.3.5), (2.3.6) with variable coefficients in the half-space. Our goal is to obtain a regularity assertion like Theorem 2.1.2 for this problem. Analogously to Theorem 2.1.2, we assume that the following conditions are satisfied:

a) The boundary value problem

$$L(0,D_y,D_t)\,u = f \quad \text{in } \mathbb{R}^n_+\,, \tag{2.3.29}$$

$$B(0,D_y,D_t)u\Big|_{t=0} + C(0,D_y)\underline{u} = \underline{g} \tag{2.3.30}$$

with coefficients frozen in the origin is elliptic.

b) The coefficients of L, B_k and $C_{k,j}$ satisfy the inequalities

$$|\,a_{\beta,j}(y,t) - a_{\beta,j}(0)\,| < \varepsilon \qquad \text{for } |\beta| + j = 2m,$$
$$|\,b_{k;\beta,j}(y) - b_{k;\beta,j}(0)\,| < \varepsilon \qquad \text{for } |\beta| + j = \mu_k,$$
$$|\,c_{k,j;\beta}(y) - c_{k,j;\beta}(0)\,| < \varepsilon \qquad \text{for } |\beta| + j = \mu_k + \tau_j,$$

where ε is a sufficiently small positive number.

We denote by $\mathcal{A}$ the operator of the boundary value problem (2.3.5), (2.3.6) and by $\mathcal{A}^{(0)}$ the operator of the problem

$$L^\circ(0, D_y + \tfrac12\,\vec{1}, D_t + \tfrac12)\,u = f \quad \text{in } \mathbb{R}^n_+\,,$$
$$B^\circ(0, D_y + \tfrac12\,\vec{1}, D_t + \tfrac12)u\Big|_{t=0} + C^\circ(0, D_y + \tfrac12\vec{1})\,\underline{u} = \underline{g} \quad \text{in } \mathbb{R}^{n-1}\,.$$

Then condition a) implies that $\mathcal{A}^{(0)}$ is an isomorphism from the space (2.3.18) onto (2.3.19) (see Theorem 2.3.4, Remark 2.2.3). This will be used in the proof of the following theorem.

THEOREM 2.3.5. *Let* $(u,\underline{\phi},\underline{u}) \in \tilde{W}^{l,2m}_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1})$ *be a solution of the equation*

$$\mathcal{A}\,(u,\underline{\phi},\underline{u}) = (f,\underline{g})$$

We assume that $f \in \tilde{W}^{l-2m+1,0}_{2,per}(\mathbb{R}^n_+)$, $\underline{g} \in W^{l-\underline{\mu}+1/2}_{2,per}(\mathbb{R}^{n-1})$, *and conditions a), b) are satisfied. Then the solution belongs to the space*

$$\tilde{W}^{l+1,2m}_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}+1/2}_{2,per}(\mathbb{R}^{n-1}) \tag{2.3.31}$$

and satisfies the estimate

$$\|(u,\underline{\phi},\underline{u})\|_{l+1} \le c\left(\|f\|_{\tilde{W}^{l-2m+1,0}_{2,per}(\mathbb{R}^n_+)} + \|\underline{g}\|_{W^{l-\underline{\mu}+1/2}_{2,per}(\mathbb{R}^{n-1})} + \|(u,\underline{\phi},\underline{u})\|_l\right), \tag{2.3.32}$$

where $\|\cdot\|_l$ *denotes the norm in the space (2.3.18) and* $\|\cdot\|_{l+1}$ *denotes the norm in (2.3.31).*

Proof: First let $l \ge 2m$. Then $\mathcal{A}$ can be identified with the operator

$$W^l_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1}) \ni (u,\underline{u})$$
$$\to \big(Lu, Bu|_{t=0} + C\underline{u}\big) \in W^{l-2m}_{2,per}(\mathbb{R}^n_+) \times W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})$$

and the proof proceeds similarly to Theorem 2.1.2. We introduce the following operator $S_{\rho,1}$ defined on the space $W^{s-1/2}_{2,per}(\mathbb{R}^{n-1})$ with arbitrary integer s :

$$(S_{\rho,1}\,\phi)(y) = (2\pi)^{-n/2} \sum_{|q|\ge\rho} \dot{\phi}(q)\, e^{iq\cdot y}$$

(cf. (2.1.8)). Analogously, we define the operator $S_{\rho,2}$ on $W^s_{2,per}(\mathbb{R}^n_+)$, $s \ge 0$, as

$$S_{\rho,2}\,u = (2\pi)^{-(n-1)/2} \sum_{q\in\mathbb{Z}^{n-1}} \Big(\mathcal{F}^{-1}_{\tau\to t}\,\chi\big(\frac{(1+\tau^2)^{1/2}\langle q\rangle}{\rho}\big)\,\mathcal{F}_{t\to\tau}\,e_+U\Big)(q,\langle q\rangle t)\; e^{iq\cdot y}\,.$$

Here $\mathcal{F}_{t\to\tau}$ denotes the Fourier transformation (1.2.2), $\mathcal{F}^{-1}_{\tau\to t}$ its inverse, χ is an arbitrary smooth function on $\mathbb{R}$ equal to zero in the interval $-1 < t < +1$ and to one outside the interval $-2 < t < +2$, e_+ is a continuous extension operator $W^s_2(\mathbb{R}_+) \to W^s_2(\mathbb{R})$ and U is defined by (2.2.3).

Obviously, the solution $(u,\underline{u})$ of (2.3.5), (2.3.6) satisfies the equation

$$\begin{aligned}\mathcal{A}^{(0)}(u,\underline{u}) &+ (\mathcal{A}-\mathcal{A}^{(0)})(S_{\rho,2}u, S_{\rho,1}\underline{u})\\ &= (f,\underline{g}) - (\mathcal{A}-\mathcal{A}^{(0)})\,(u - S_{\rho,2}u\,,\,\underline{u} - S_{\rho,1}\underline{u})\,.\end{aligned} \tag{2.3.33}$$

It can be easily verified that $S_{\rho,1}$ and $S_{\rho,2}$ have analogous properties to the operator S_ρ in Lemma 2.1.1. In particular, the operator

$$(u,\underline{u}) \to (\mathcal{A}-\mathcal{A}^{(0)})(S_{\rho,2}u, S_{\rho,1}\underline{u})$$

has a small norm if ε is small and ρ is large. Hence for sufficiently large ρ the operator on the left-hand side of (2.3.33) is an isomorphism from

$$W^l_{2,per}(\mathbb{R}^n_+) \times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1}) \tag{2.3.34}$$

onto $W^{l-2m}_{2,per}(\mathbb{R}^n_+) \times W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})$ for arbitrary given $l \ge 2m$. Moreover, the operator

$$(u,\underline{u}) \to (\mathcal{A}-\mathcal{A}^{(0)})(u - S_{\rho,2}u\,,\,\underline{u} - S_{\rho,1}\underline{u})$$

on the right side of (2.3.33) continuously maps the space (2.3.34) into $W^{l+1}_{2,per}(\mathbb{R}^n_+)\times W^{l+\underline{\tau}+1/2}_{2,per}(\mathbb{R}^{n-1})$. This proves the assertion of the theorem in the case $l\ge 2m$.

Now let $l\le 0$. We rewrite the equation $\mathcal{A}(u,\underline{\phi},\underline{u})=(f,\underline{g})$ as

$$\left(\mathcal{A}^{(0)}+\mathcal{A}^{(1)}\right)(u,\underline{\phi},\underline{u})=(f,\underline{g})-\left(\mathcal{A}-\mathcal{A}^{(0)}-\mathcal{A}^{(1)}\right)(u,\underline{\phi},\underline{u})\,.$$

Here the functional $\mathcal{A}^{(1)}(u,\underline{\phi},\underline{u})=(f^{(1)},\underline{g}^{(1)})\in W^{2m-l}_{2,per}(\mathbb{R}^n_+)^*\times W^{l-\underline{\mu}-1/2}_{2,per}(\mathbb{R}^{n-1})$ is defined for given $(u,\underline{\phi},\underline{u})$ from the space (2.3.18) by the equalities

$$\begin{aligned}(f^{(1)},v)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+} &= \Big(u\,,\,(L^+-(L^\circ)^+(0,D_y+\tfrac12\underline{1},D_t+\tfrac12))S_{\rho,2}v\Big)_{\mathbb{Q}^{n-1}\times\mathbb{R}_+}\,,\\ &\quad+\Big(\underline{\phi},\,(P-P^\circ(0,D_y+\tfrac12\underline{1},D_t+\tfrac12))S_{\rho,2}v|_{t=0}\Big)_{\mathbb{Q}^{n-1}}\\ (\underline{g}^{(1)},\underline{v})_{\mathbb{Q}^{n-1}} &= \Big(\underline{\phi},(Q^+-(Q^\circ)^+(0,D_y+\tfrac12\underline{1}))S_{\rho,1}\underline{v}\Big)_{\mathbb{Q}^{n-1}}\\ &\quad+\Big(\underline{u}\,,\,(C^+-(C^\circ)^+(0,D_y+\tfrac12\underline{1}))S_{\rho,1}\underline{v}\Big)_{\mathbb{Q}^{n-1}}\,.\end{aligned}$$

Since the norm of the operator $\mathcal{A}^{(1)}$ is small for small ε and large ρ, the operator $\mathcal{A}^{(0)}+\mathcal{A}^{(1)}$ is an isomorphism from (2.3.18) onto (2.3.19) for arbitrary l if ε is sufficiently small and ρ is sufficiently large. Furthermore, the operator $\mathcal{A}-\mathcal{A}^{(0)}-\mathcal{A}^{(1)}$ continuously maps the space (2.3.18) into $W^{2m-l-1}_{2,per}(\mathbb{R}^n_+)^*\times W^{l-\underline{\mu}+1/2}_{2,per}(\mathbb{R}^{n-1})$. Consequently, the assertion of the theorem is true for $l<0$.

Similarly the theorem can be proved in the case $0<l<2m$. ■

COROLLARY 2.3.2. *Let $(u,\underline{u})\in W^l_{2,per}(\mathbb{R}^n_+)\times W^{l+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1})$, $l\ge 2m$, be a solution of the boundary value problem (2.3.5), (2.3.6). If $f\in W^{l-2m+1}_{2,per}(\mathbb{R}^n_+)$, $\underline{g}\in W^{l-\underline{\mu}+1/2}_{2,per}(\mathbb{R}^{n-1})$, then $(u,\underline{u})$ belongs to the space $W^{l+1}_{2,per}(\mathbb{R}^n_+)\times W^{l+\underline{\tau}+1/2}_{2,per}(\mathbb{R}^{n-1})$ and satisfies the estimate*

$$\|(u,\underline{u})\|_{l+1}\le c\left(\|f\|_{W_2^{l-2m+1}(\mathbb{Q}^n_+)}+\|\underline{g}\|_{W_2^{l-\underline{\mu}+1/2}(\mathbb{Q}^{n-1})}+\|(u,\underline{u})\|_l\right), \tag{2.3.35}$$

where $\|\cdot\|_k$ denotes the norm in the space $W^k_{2,per}(\mathbb{R}^n_+)\times W^{k+\underline{\tau}-1/2}_{2,per}(\mathbb{R}^{n-1})$.

Proof: By Theorem 2.3.5, the triple $(u,\underline{\phi},\underline{u})=(u,\mathcal{D}u|_{t=0},\underline{u})$ belongs to the space (2.3.31) and satisfies the estimate (2.3.32). Since

$$\|u\|_{W^l_{2,per}(\mathbb{R}^n_+)}\le\|(u,\mathcal{D}u|_{t=0})\|_{\tilde W^{l,2m}_{2,per}(\mathbb{R}^n_+)}\le c\,\|u\|_{W^l_{2,per}(\mathbb{R}^n_+)}$$

for $l\ge 2m$, we get (2.3.35). ■

2.3.6. Necessity of the ellipticity. Analogously to Lemma 2.1.2, we show now that (2.3.32) implies Condition a).

LEMMA 2.3.3. *Let $\mathcal{U}$ be an arbitrary neighbourhood of the origin. Suppose that estimate (2.3.32) is satisfied for all $(u,\underline{\phi})\in\tilde W^{l+1,2m}_{2,per}(\mathbb{R}^n_+)$, $\underline{u}\in W^{l+\underline{\tau}+1/2}_{2,per}(\mathbb{R}^{n-1})$ such that* $\operatorname{supp}u\cap\mathbb{Q}^n\subset\mathcal{U}$, $\operatorname{supp}\underline{\phi}\cap\mathbb{Q}^{n-1}\subset\mathcal{U}\cap\mathbb{Q}^{n-1}$, $\operatorname{supp}\underline{u}\cap\mathbb{Q}^{n-1}\subset\mathcal{U}\cap\mathbb{Q}^{n-1}$. *Then problem (2.3.29), (2.3.30) is elliptic.*

Proof: If $\underline{u}$, $\underline{\phi}$ are equal to zero outside a sufficiently small neighbourhood of every point $q\in\mathbb{Z}^{n-1}$ and u is equal to zero outside a sufficiently small neighbourhood of the set $\{(q,0):\,q\in\mathbb{Z}^{n-1}\}$, then, by the same arguments as in the proof of

Lemma 2.1.2, we can deduce the inequality

(2.3.36) $$\|(u,\underline{\phi},\underline{u})\|_{l+1} \le c\Big(\|L^\circ(0,D_y,D_t)\,(u,\underline{\phi})\|_{\tilde{W}^{l+1-2m,0}_{2,per}(\mathbb{R}^n_+)} + \|B^\circ(0,D_y,D_t)\,(u,\underline{\phi}) + C^\circ(0,D_y)\underline{u}\|_{W^{l-\underline{\mu}+1/2}_{2,per}(\mathbb{R}^{n-1})} + \|(u,\underline{\phi},\underline{u})\|_l\Big)$$

from (2.3.32), where $\|\cdot\|_l$, $\|\cdot\|_{l+1}$ denote the norms in (2.3.18) and (2.3.31), respectively. Since the operators $L^\circ(0,D_y,D_t)$, $B^\circ_k(0,D_y,D_t)$, $C^\circ_{k,j}(0,D_y,D_t)$ are translation invariant, we obtain (2.3.36) for functions having arbitrary support with respect to the variable y.

Let $(v,\underline{\psi}) = (v,\psi_1,\dots,\psi_{2m})$ be an arbitrary element of the space $\tilde{W}^{l+1,2m}_2(\mathbb{R}_+)$ with compact support, $\underline{v} = (v_1,\dots,v_J)$ an arbitrary vector in $\mathbb{C}^J$, and $q \in \mathbb{Z}^{n-1}$, $|q| > \rho$, where ρ is sufficiently large. We set

$$\begin{aligned} u(y,t) &= v(\langle q\rangle t)\,e^{iq\cdot y}, \\ \phi_j(y) &= \langle q\rangle^{j-1}\,\psi_j\,e^{iq\cdot y}, \quad j=1,\dots,2m, \\ u_j(y) &= \langle q\rangle^{-\tau_j}\,v_j\,e^{iq\cdot y}, \quad j=1,\dots,J, \end{aligned}$$

Obviously, $(u,\underline{\phi}) = (u,\phi_1,\dots,\phi_{2m})$ is an element of the space $\tilde{W}^{l+1,2m}_{2,per}(\mathbb{R}^n_+)$. Since the function $t \to v(\langle q\rangle t)$ has small support, the inequality (2.3.36) with a constant c independent of q, $|q| > \rho$, is satisfied for the just introduced u, $\underline{\phi}$, $\underline{u}$.

For $k = l$, $k = l+1$ we have

(2.3.37) $$\|(u,\underline{\phi})\|_{\tilde{W}^{k,2m}_{2,per}(\mathbb{R}^n_+)} = (2\pi)^{(n-1)/2}\,\langle q\rangle^{k-1/2}\,\|(v,\underline{\psi})\|_{\tilde{W}^{k,2m}_2(\mathbb{R}_+)}$$

and

(2.3.38) $$\|u_j\|_{W^{k+\tau_j-1/2}_{2,per}(\mathbb{R}^{n-1})} = (2\pi)^{(n-1)/2}\,\langle q\rangle^{k-1/2}\,|v_j|, \quad j=1,\dots,J.$$

We show that

(2.3.39) $$\|L^\circ(0,D_y,D_t)(u,\underline{\phi})\|_{\tilde{W}^{k-2m,0}_{2,per}(\mathbb{R}^n_+)} = (2\pi)^{(n-1)/2}\,\langle q\rangle^{k-1/2}\,\|L^\circ(0,\langle q\rangle^{-1}q,D_t)\,(v,\underline{\psi})\|_{\tilde{W}^{k-2m,0}_2(\mathbb{R}_+)}$$

for all $(u,\underline{\phi}) \in \tilde{W}^{k,2m}_{2,per}(\mathbb{R}^n_+)$. In the case $k \ge 2m$ this follows immediately from the representation (2.2.4) for the norm in $W^l_{2,per}(\mathbb{R}^n_+)$ and from the equality

$$L^\circ(0,D_y,D_t)\,(u,\underline{\phi}) = L^\circ(0,D_y,D_t)u = \langle q\rangle^{2m}\,e^{iq\cdot y}\,L^\circ(0,\langle q\rangle^{-1}q,D_{\langle q\rangle t})\,v(\langle q\rangle t).$$

In the case $k \le 0$ we have

$$L^\circ(0,D_y,D_t)\,(u,\underline{\phi}) = f = (2\pi)^{-(n-1)/2}\,\dot{f}(q,t)\,e^{iq\cdot y},$$

where, according to (2.3.2), (2.3.22), the functional $\dot{f}(q,\cdot) \in W^{2m-k}_2(\mathbb{R}_+)^*$ is defined by the equality

$$\begin{aligned} (2\pi)^{(n-1)/2}\,\big(\dot{f}(q,\cdot),\,w(\cdot)\big)_{\mathbb{R}_+} &= \big(f,\,e^{iq\cdot y}w\big)_{\mathbb{Q}^n_+} \\ &= \big(u,\,(L^\circ)^+(0,D_y,D_t)\,e^{iq\cdot y}w\big)_{\mathbb{Q}^n_+} + \big(\underline{\phi},\,P^\circ(0,D_y,D_t)\,e^{iq\cdot y}w|_{t=0}\big)_{\mathbb{Q}^{n-1}} \\ &= (2\pi)^{n-1}\Big(\big(v(\langle q\rangle t),\,(L^\circ)^+(0,q,D_t)w\big)_{\mathbb{R}_+} + \sum_{j=1}^{2m}\langle q\rangle^{j-1}\,\psi_j\,\overline{P^\circ_j(0,q,D_t)w}\big|_{t=0}\Big), \end{aligned}$$

$w \in W^{2m-k}_2(\mathbb{R}_+)$. Hence the norm of f in $W^{2m-k}_{2,per}(\mathbb{R}^n_+)^*$ is equal to

$$\langle q\rangle^{k-2m-1/2}\,\|F(q,\cdot)\|_{W^{2m-k}_2(\mathbb{R}_+)^*}$$

(see Lemma 2.3.1), where the functional F is defined as follows:

$$(F,w)_{\mathbb{R}_+} = \big(\dot f(q,\langle q\rangle^{-1}t),\, w\big)_{\mathbb{R}_+} = \langle q\rangle \,\big(\dot f(q,t),\, w(\langle q\rangle t)\big)_{\mathbb{R}_+}$$

$$= (2\pi)^{(n-1)/2}\,\langle q\rangle^{2m}\Big(\big(v,\,(L^\circ)^+(0,q/\langle q\rangle,D_t)w\big)_{\mathbb{R}_+} + \big(\underline{\psi},\, P^\circ(0,q/\langle q\rangle,D_t)w|_{t=0}\big)_{\mathbb{C}^{2m}}\Big)$$

$$= (2\pi)^{(n-1)/2}\,\langle q\rangle^{2m}\,\big(L^\circ(0,q/\langle q\rangle,D_t)\,(v,\underline{\psi})\,,\, w\big)_{\mathbb{R}_+}\,, \quad w\in W_2^{2m-k}(\mathbb{R}_+),$$

i.e.,

$$F(q,t) = (2\pi)^{(n-1)/2}\,\langle q\rangle^{2m}\,L^\circ(0,\langle q\rangle^{-1}q,D_t)\,(v,\underline{\psi}).$$

This proves (2.3.39) in the case $k\le 0$. Analogously, (2.3.39) can be proved in the case $0<k<2m$ by means of the representation (2.3.23) for the functional $L^\circ(0,D_y,D_t)\,(u,\underline{\phi})$.

Furthermore, using the equality

$$B^\circ(0,D_y,D_t)\,(u,\underline{\phi}) = Q^\circ(0,D_y)\underline{\phi},$$

we obtain

$$\text{(2.3.40)}\quad \|B^\circ(0,D_y,D_t)\,(u,\underline{\phi}) + C^\circ(0,D_y)\underline{u}\|_{W^{k-\mu-1/2}_{2,per}(\mathbb{R}^{n-1})}$$
$$= (2\pi)^{(n-1)/2}\,\langle q\rangle^{k-1/2}\,\|B^\circ(0,q/\langle q\rangle,D_t)\,(v,\underline{\psi}) + C^\circ(0,q/\langle q\rangle)\underline{v}\|_{\mathbb{C}^{m+J}}\,.$$

From (2.3.36) - (2.3.40) it follows that

$$\text{(2.3.41)}\quad \|(v,\underline{\psi},\underline{v})\|_{\tilde W_2^{l+1,2m}(\mathbb{R}_+)\times\mathbb{C}^J} \le c\Big(\|L^\circ(0,\langle q\rangle^{-1}q,D_t)(v,\underline{\psi})\|_{\tilde W_2^{l+1-2m,0}(\mathbb{R}_+)}$$
$$+\Big|B^\circ(0,\langle q\rangle^{-1}q,D_t)(v,\underline{\psi}) + C^\circ(0,\langle q\rangle^{-1}q)\underline{v}\Big|_{\mathbb{C}^{m+J}}$$
$$+\langle q\rangle^{-1}\,\|(v,\underline{\psi},\underline{v})\|_{\tilde W_2^{l,2m}(\mathbb{R}_+)\times\mathbb{C}^J}\Big)$$

for all $(v,\underline{\psi})\in \tilde W_2^{l+1,2m}(\mathbb{R}_+)$ with support contained in an arbitrary given finite interval $[0,T]$, $\underline{v}\in\mathbb{C}^{m+J}$, $q\in\mathbb{Z}^{n-1}$, $|q|>\rho$. Here the constant c is independent of $(v,\underline{\psi},\underline{v})$ and q, $|q|>\rho$. If ρ is sufficiently large, then the term $\langle q\rangle^{-1}\,\|(v,\underline{\psi},\underline{v})\|$ on the right-hand side of (2.3.41) can be omitted. Since every η on the unit sphere in $\mathbb{R}^{n-1}$ can be approximated by a sequence of elements $\langle q\rangle^{-1}q$, where $q\in\mathbb{Z}^{n-1}$, $|q|>\rho$, and the set of all $(v,\underline{\psi})\in\tilde W_2^{l+1,2m}(\mathbb{R}_+)$ with compact support is dense in $\tilde W_2^{l+1,2m}(\mathbb{R}_+)$, we get

$$\text{(2.3.42)}\quad \|(v,\underline{\psi},\underline{v})\|_{\tilde W_2^{l+1,2m}(\mathbb{R}_+)\times\mathbb{C}^J} \le c\Big(\|L^\circ(0,\eta,D_t)\,(v,\underline{\psi})\|_{\tilde W_2^{l+1-2m,0}(\mathbb{R}_+)}$$
$$+\Big|B^\circ(0,\eta,D_t)\,(v,\underline{\psi}) + C^\circ(0,\eta)\underline{v}\Big|_{\mathbb{C}^{m+J}}\Big),$$

for all $(v,\underline{\psi})\in\tilde W_2^{l+1,2m}(\mathbb{R}_+)$, $\underline{v}\in\mathbb{C}^J$, $\eta\in\mathbb{R}^{n-1}$, $|\eta|=1$. Consequently, by the second part of Theorem 1.3.2, the polynomial $L^\circ(0,\eta,\cdot)$ has no real zeros for $|\eta|=1$, and condition (ii) in Definion 1.2.1 is satisfied for the problem

$$\text{(2.3.43)}\qquad L^\circ(0,\eta,D_t)\,v = f,\quad t>0,$$

$$\text{(2.3.44)}\qquad B^\circ(0,\eta,D_t)v\big|_{t=0} + C^\circ(0,\eta)\underline{v} = \underline{g}$$

with $|\eta|=1$. Applying the transformation $t=|\eta|\,t'$, we obtain the validity of these conditions for problem (2.3.43), (2.3.44) with arbitrary $\eta\in\mathbb{R}^{n-1}\backslash\{0\}$. From Lemma 2.2.3 we conclude that the operator $L^\circ(0,D_x)$ is properly elliptic and, therefore, problem (2.3.43), (2.3.44) is regular for every $\eta \in \mathbb{R}^{n-1}\backslash\{0\}$. This proves the ellipticity of the boundary value problem (2.3.29), (2.3.30). ∎

CHAPTER 3

Elliptic boundary value problems in smooth domains

In this chapter we explore elliptic boundary value problems for $2m$ order differential equations in a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Throughout this chapter, it is assumed that the orders of the differential operators B_k in the boundary conditions are less than $2m$. The generalization to arbitrary boundary conditions and to boundary value problems for systems of differential equation is one of the goals of the next chapter.

In the same way as in Chapters 1 and 2, we extend the operator of the boundary value problem to Sobolev spaces of arbitrary integer order. Furthermore, we introduce the formally adjoint problem, using a Green formula which is analogous to that in the foregoing chapter. The main result of this chapter is the proof of the Fredholm property for the operators of elliptic boundary value problems. To this end, we derive a priori estimates for the solutions and investigate the adjoint operator.

In Section 3.5 we study the Green functions for elliptic boundary value problems, while the last section in this chapter is dedicated to elliptic boundary value problems with parameter. We show that these problems are uniquely solvable if the parameter λ is situated near the imaginary axis and its modulus is sufficiently large.

3.1. The boundary value problem and its formally adjoint

We derive a Green formula for boundary value problems in an arbitrary smooth domain. This enables us again to introduce a formally adjoint problem. Under additional assumptions on the boundary conditions, other formally adjoint problems can be defined by means of the *classical Green formula.* We deal with the question of equivalence of boundary conditions and show that, in particular, all formally adjoint problems are equivalent. Furthermore, we prove that equivalent boundary value problems are simultaneously elliptic.

3.1.1. Formulation of the problem. Let Ω be a domain, i.e., an open and connected subset of the Euclidean space $\mathbb{R}^n$. We assume that the boundary $\partial\Omega$ is smooth (from the class C^∞) and consider the following boundary value problem

$$Lu = f \quad \text{in } \Omega\,, \tag{3.1.1}$$

$$Bu + C\underline{u} = \underline{g} \quad \text{on } \partial\Omega\,, \tag{3.1.2}$$

where,

$$L(x, D_x) = \sum_{|\alpha|\le 2m} a_\alpha(x)\, D_x^\alpha \tag{3.1.3}$$

is a differential operator of order $2m$ with coefficients a_α infinitely differentiable up to the boundary, B is a vector of differential operators

$$B_k(x, D_x) = \sum_{|\alpha|\le\mu_k} b_{k;\alpha}(x)\, D_x^\alpha\,, \quad k = 1, \dots, m+J, \tag{3.1.4}$$

$\operatorname{ord} B_k \le \mu_k$ ($B_k \equiv 0$ if $\mu_k < 0$), with infinitely differentiable coefficients $b_{k;\alpha}$, and C is a matrix of *tangential* (see Definition 3.1.1 below) operators

$$C_{k,j}(x, D_x) = \sum_{|\alpha|\le\mu_k+\tau_j} c_{k,j;\alpha}(x)\, D_x^\alpha\,, \quad k = 1, \dots, m+J,\ j = 1, \dots, J, \tag{3.1.5}$$

on $\partial\Omega$, $\operatorname{ord} C_{k,j} \le \mu_k + \tau_j$ ($C_{k,j} \equiv 0$ if $\mu_k + \tau_j < 0$), with coefficients $c_{k,j}$ which are infinitely differentiable in a neighbourhood of $\partial\Omega$.

Here μ_k and τ_j are given integer numbers such that

$$\tau_j + \max \mu_k \ge 0 \quad \text{for} \quad j = 1, \dots, J.$$

(If this condition fails for one j, then the operators $C_{1,j}, \dots, C_{m+J,j}$ are zero and, therefore, the component u_j of the vector-function $\underline{u}$ does not appear in the boundary value problem.)

Note that the orders of the operators B_k, $C_{k,j}$ may be strictly less than μ_k and $\mu_k + \tau_j$, respectively.

DEFINITION 3.1.1. A differential operator $P(x, D_x)$ of order k with infinitely differentiable coefficients in a neighbourhood of $\partial\Omega$ is said to be *tangential on* $\partial\Omega$ if $P(x, D_x)v|_{\partial\Omega}$ depends only on $v|_{\partial\Omega}$ for every smooth function v in a neighbourhood of $\partial\Omega$.

If the differential operator $P(x, D_x)$ is tangential on $\partial\Omega$ and u is an infinitely differentiable function on $\partial\Omega$, then $P(x, D_x)u$ is defined as the restriction of $P(x, D_x)v$ to $\partial\Omega$, where v is an arbitrary infinitely differentiable extension of u.

REMARK 3.1.1. Let x', x_n be local coordinates in a neighbourhood of $x^{(0)} \in \partial\Omega$ such that $\partial\Omega$ is given by $x_n = 0$. Then every tangential operator has the form $P(x', x_n, D_{x'})$ in this neighbourhood.

Example. Let $\partial\Omega$ be the circle $x_1^2 + x_2^2 = 1$ and let r, θ be the polar coordinates in the (x_1, x_2)-plane. Then the operator

$$\frac{\partial^k}{\partial\theta^k} = \Big(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\Big)^k$$

is tangential on $\partial\Omega$.

In problem (3.1.1), (3.1.2) the function f on Ω and the vector-function $\underline{g} = (g_1, \dots, g_{m+J})$ on $\partial\Omega$ are given, while u is an unknown function on Ω and $\underline{u} = (u_1, \dots, u_J)$ is an unknown vector-function on $\partial\Omega$.

3.1.2. Ellipticity of the boundary value problem. As in Chapter 2, we denote by $L^\circ(x, D_x)$ the principal part of $L(x, D_x)$ which contains only the derivatives of order $2m$. Analogously, the operators B_k° and $C_{k,j}^\circ$ consist of the terms in (3.1.4), (3.1.5) with derivatives of order μ_k and $\mu_k + \tau_j$, respectively. If $\operatorname{ord} B_k < \mu_k$ or $\operatorname{ord} C_{k,j} < \mu_k + \tau_j$, then we set $B_k^\circ = 0$ and $C_{k,j}^\circ = 0$, respectively. Thus, B_k° and $C_{k,j}^\circ$ depend on the choice of the numbers μ_k and τ_j.

In the following, let $\nu = \nu(x)$ be the exterior normal to the boundary $\partial\Omega$ at the point x.

DEFINITION 3.1.2. The boundary value problem (3.1.1), (3.1.2) is said to be *elliptic* if

(i) The operator L is elliptic in $\overline{\Omega}$. This means, for every $x^{(0)} \in \overline{\Omega}$ the operator $L(x^{(0)}, D_x)$ with coefficients frozen in $x^{(0)}$ is elliptic (see Section 2.1.2).

(ii) The boundary value problem

$$L^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})D_t\big)\, u(t) = f(t) \quad \text{for } t > 0,$$
$$B^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})D_t\big)\, u(t)\,\big|_{t=0} + C^\circ(x^{(0)}, \xi')\,\underline{u} = \underline{g}$$

is regular (see Definition 1.2.1) for every $x^{(0)} \in \partial\Omega$ and every vector $\xi' \neq 0$ tangential to $\partial\Omega$ at $x^{(0)}$.

3.1.3. The Green formula. Let $D_\nu = -i\,\partial/\partial\nu$ be the derivative in the direction of the exterior normal ν to $\partial\Omega$. We denote by $\mathcal{D}$ the column vector with the components $1, D_\nu, \dots, D_\nu^{2m-1}$. In the following we will assume always that the orders of the operators B_k are less than $2m$. Then the vector B admits the representation

$$Bu|_{\partial\Omega} = Q \cdot \mathcal{D}u|_{\partial\Omega} \tag{3.1.6}$$

for all $u \in C^\infty(\overline{\Omega})$, where

$$Q = \big(Q_{k,j}\big)_{1\le k\le m+J,\ 1\le j\le J}$$

is a matrix of tangential differential operators $Q_{k,j}$, $\operatorname{ord} Q_{k,j} \le \mu_k + 1 - j$, $Q_{k,j} \equiv 0$ if $\mu_k + 1 - j < 0$.

Let L^+ be the formally adjoint operator to L. Then for all $u, v \in C_0^\infty(\overline{\Omega})$ we have

$$\int_\Omega Lu \cdot \overline{v}\, dx = \int_\Omega u \cdot \overline{L^+ v}\, dx + \int_{\partial\Omega} \big(\mathcal{D}u|_{\partial\Omega}\,,\, Pv|_{\partial\Omega}\big)_{\mathbb{C}^{2m}}\, d\sigma \tag{3.1.7}$$

(cf. formula (2.3.15)), where P is a vector of differential operators $P_j(x, D_x)$ $j = 1, \dots, 2m$, with smooth coefficients, $\operatorname{ord} P_j \le 2m - j$. The restrictions of the coefficients of the operators P_j to the boundary $\partial\Omega$ are uniquely determined by L.

REMARK 3.1.2. There is the following representation for the vector P :

$$Pv|_{\partial\Omega} = T \cdot \mathcal{D}v|_{\partial\Omega}\,, \tag{3.1.8}$$

where $T = (T_{j,s})_{1\le j,s\le 2m}$ is a triangular matrix of tangential differential operators $T_{j,s}$ on $\partial\Omega$, $\operatorname{ord} T_{j,s} \le 2m + 1 - j - s$ for $j + s \le 2m + 1$, $T_{j,s} \equiv 0$ if $j + s > 2m + 1$. The elements $T_{j,2m-j+1}$ are functions,

$$T_{j,2m-j+1}(x^{(0)}) = -i\,\overline{L^\circ(x^{(0)}, \nu(x^{(0)}))} \quad \text{for } x^{(0)} \in \partial\Omega,\ j = 1, \dots, 2m.$$

(The last formula holds if we write formula (3.1.7) in local Cartesian coordinates y, t where $y = (y_1, \dots, y_{n-1})$ are coordinates on the tangent plane to $\partial\Omega$ in $x^{(0)}$ see Remark 2.3.1). The functions $T_{j,2m-j+1}$ do not vanish on $\partial\Omega$ if, e.g., the operator L is elliptic in $\overline{\Omega}$.

Furthermore, let C^+, Q^+ the formally adjoint operators to C and Q, respectively. The operator C^+ is the $J \times (m + J)$−matrix of the tangential differential operators $C^+_{k,j}(x, D_x)$, $j = 1, \dots, J$, $k = 1, \dots, m+J$, on $\partial\Omega$. If ϕ and ψ are arbitrary smooth functions on $\partial\Omega$ with compact supports, then

$$\int\limits_{\partial\Omega} C_{k,j}\phi \cdot \overline{\psi}\, d\sigma = \int\limits_{\partial\Omega} \phi \cdot \overline{C^+_{k,j}\psi}\, d\sigma.$$

THEOREM 3.1.1. *Let L, B and C be the operators given by (3.1.3)–(3.1.6). Then the following Green formula is satisfied for each u, $v \in C_0^\infty(\overline{\Omega})$, $\underline{u} \in C_0^\infty(\partial\Omega)^J$ and $\underline{v} \in C_0^\infty(\partial\Omega)^{m+J}$:*

$$(3.1.9)\quad \int\limits_{\Omega} Lu \cdot \overline{v}\, dx + \int\limits_{\partial\Omega} \big(Bu|_{\partial\Omega} + C\underline{u}\,, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma$$
$$= \int\limits_{\Omega} u \cdot \overline{L^+ v}\, dx + \int\limits_{\partial\Omega} \big(\mathcal{D}u|_{\partial\Omega}\,,\, Pv|_{\partial\Omega} + Q^+\underline{v}\big)_{\mathbb{C}^{2m}} d\sigma + \int\limits_{\partial\Omega} \big(\underline{u}, C^+\underline{v}\big)_{\mathbb{C}^J} d\sigma.$$

Proof: By the definition of the operators C^+, Q^+, we have

$$\int\limits_{\partial\Omega} \big(C\underline{u}, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma = \int\limits_{\partial\Omega} \big(\underline{u}, C^+\underline{v}\big)_{\mathbb{C}^J}\, d\sigma$$

and

$$\int\limits_{\partial\Omega} \big(Bu|_{\partial\Omega}\,, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma = \int\limits_{\partial\Omega} \big(Q \cdot \mathcal{D}u|_{\partial\Omega}\,, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma = \int\limits_{\partial\Omega} \big(\mathcal{D}u|_{\partial\Omega}\,, Q^+\underline{v}\big)_{\mathbb{C}^{2m}}\, d\sigma\,.$$

The last two equalities together with (3.1.7) imply (3.1.9). ∎

DEFINITION 3.1.3. The boundary value problem

$$(3.1.10)\qquad L^+ v = f \quad \text{in } \Omega\,,$$

$$(3.1.11)\qquad Pv + Q^+\underline{v} = \underline{g}\,, \quad C^+\underline{v} = \underline{h}$$

is said to be *formally adjoint* to (3.1.1), (3.1.2).

THEOREM 3.1.2. *The boundary value problem (3.1.1), (3.1.2) is elliptic if and only if the formally adjoint problem (3.1.10), (3.1.11) is elliptic.*

Proof: Obviously, the differential operators L and L^+ are simultaneously properly elliptic. Let $x^{(0)}$ be an arbitrary point on the boundary $\partial\Omega$. Then the Green formula (2.3.11) in a half-space is valid for the principal parts at $x^{(0)}$ of the operators of the boundary value problem (3.1.1), (3.1.2). Consequently, the problem

$$(L^\circ)^+(x^{(0)}, \xi' + \nu(x^{(0)})D_t)\, v(t) = f(t) \quad \text{for } t > 0,$$
$$P^\circ(x^{(0)}, \xi' + \nu(x^{(0)})D_t)\, v(t)\,\big|_{t=0} + (Q^\circ)^+(x^{(0)}, \xi')\, \underline{v} = \underline{g}\,, \quad (C^\circ)^+(x^{(0)}, \xi')\, \underline{v} = \underline{h}$$

is formally adjoint to the boundary value problem

$$L^\circ(x^{(0)}, \xi' + \nu(x^{(0)})D_t)\, u(t) = f(t) \qquad \text{for } t > 0,$$
$$B^\circ(x^{(0)}, \xi' + \nu(x^{(0)})D_t)\, u(t)\,\big|_{t=0} + C^\circ(x^{(0)}, \xi')\, \underline{u} = \underline{g}$$

for every $x^{(0)} \in \partial\Omega$ and every vector $\xi' \neq 0$ tangential to $\partial\Omega$ in $x^{(0)}$. By Theorem 1.2.2, every of these two problems is regular if and only if the other one is elliptic. Therefore, condition (ii) in Definition 3.1.2 is simultaneously satisfied for problem (3.1.1), (3.1.2) and for the formally adjoint problem. ∎

3.1.4. The classical Green formula. Now we consider a special class of boundary value problems, namely the classical boundary value problems

$$Lu = f \quad \text{in}\ \Omega, \tag{3.1.12}$$

$$B_k u = g_k \quad \text{on}\ \partial\Omega,\ k = 1, \dots, m, \tag{3.1.13}$$

where $J = 0$ and the vector $\underline{u} = (u_1, \dots, u_J)$ does not occur.

DEFINITION 3.1.4. The system of the boundary operators $B_1, \dots, B_m$ is said to be *normal* on $\partial\Omega$ if

(i) $\operatorname{ord} B_k \neq \operatorname{ord} B_j$ for $k \neq j$,
(ii) $B_k^\circ(x^{(0)}, \nu(x^{(0)})) \neq 0$ for each $x^{(0)} \in \partial\Omega$.

If additionally the orders of the operators B_k are less than m, then this system is called a *Dirichlet system* of order m on $\partial\Omega$.

Note that, by Remark 3.1.2, the operators $P_1, \dots, P_{2m}$ in (3.1.7) form a Dirichlet system of order $2m$ on $\partial\Omega$ if L is elliptic.

We suppose that $B_1, \dots, B_m$ is a normal system of boundary operators of orders $\mu_k < 2m$ ($k = 1, \dots, m$). From condition (ii) it follows that the elements Q_{k,μ_k+1} of the matrix Q in (3.1.6) are functions not vanishing on $\partial\Omega$. The system of the operators $B_1, \dots, B_m$ can be completed by operators B_k ($k = m+1, \dots, 2m$) of order $\mu_k < 2m$ to a Dirichlet system of order $2m$ on $\partial\Omega$. Then every of the operators B_k, $k = 1, \dots, 2m$, has the representation

$$B_k u\Big|_{\partial\Omega} = \sum_{j=1}^{2m} Q_{k,j}\, D_\nu^{j-1} u\Big|_{\partial\Omega}, \tag{3.1.14}$$

where $Q_{k,j}$ are tangential differential operators, $Q_{k,j} \equiv 0$ for $j > \mu_k + 1$, $\operatorname{ord} Q_{k,j} \le \mu_k + 1 - j$ for $j \le \mu_k$, and Q_{k,μ_k+1} are nonvanishing functions on $\partial\Omega$. The matrix

$$\big(Q_{k,j}(x, D_x)\big)_{1 \le k,j \le 2m}$$

has an inverse

$$\big(\Lambda_{j,k}(x, D_x)\big)_{1 \le j,k \le 2m},$$

where $\Lambda_{j,k}$ are tangential differential operators on $\partial\Omega$ of orders $j-1-\mu_k$ if $j \ge \mu_k+1$, $\Lambda_{j,k} \equiv 0$ if $j < \mu_k + 1$. From this it follows that the normal derivatives D_ν^{j-1} can be written in the form

$$D_\nu^{j-1} u\Big|_{\partial\Omega} = \sum_{k=1}^{2m} \Lambda_{j,k}(x, D_x)\, B_k u\Big|_{\partial\Omega}, \quad j = 1, \dots, 2m, \tag{3.1.15}$$

Hence by (3.1.7), we get

$$\begin{aligned}
\int_\Omega Lu \cdot \overline{v}\, dx &= \int_\Omega u \cdot \overline{L^+ v}\, dx + \sum_{j=1}^{2m} \int_{\partial\Omega} \sum_{k=1}^{2m} \Lambda_{j,k}\, B_k u \cdot \overline{P_j v}\, d\sigma \\
&= \int_\Omega u \cdot \overline{L^+ v}\, dx + \sum_{k=1}^{2m} \int_{\partial\Omega} B_k u \cdot \overline{\sum_{j=1}^{2m} \Lambda_{j,k}^+ P_j v}\, d\sigma .
\end{aligned}$$

If we set

$$\sum_{j=1}^{2m} \Lambda_{j,k}^+ P_j = \begin{cases} -B'_{k+m} & \text{for}\quad k = 1, \dots, m, \\ B'_{k-m} & \text{for}\quad k = m+1, \dots, 2m, \end{cases} \tag{3.1.16}$$

we obtain the *classical Green formula*

(3.1.17)
$$\int_\Omega Lu \cdot \overline{v}\, dx + \sum_{k=1}^{m} \int_{\partial\Omega} B_k u \cdot \overline{B'_{k+m} v}\, d\sigma = \int_\Omega u \cdot \overline{L^+ v}\, dx + \sum_{k=1}^{m} \int_{\partial\Omega} B_{k+m} u \cdot \overline{B'_k v}\, d\sigma.$$

The boundary value problem

$$L^+ v = f \quad \text{in } \Omega, \tag{3.1.18}$$

$$B'_k v = h_k \quad \text{on } \partial\Omega \;\; (k = 1, \dots, m) \tag{3.1.19}$$

is said to be *formally adjoint* to (3.1.12), (3.1.13) with respect to the Green formula (3.1.17). The classical Green formula has the advantage that the number of the unknowns in the formally adjoint problem is the same as in the starting problem. However, one has to suppose that the boundary conditions in the starting problem are normal. The following lemma describes the connection between this formally adjoint problem and the boundary value problem which is formally adjoint to (3.1.12), (3.1.13) with respect to the more general Green formula (3.1.9).

LEMMA 3.1.1. *Let B_k, $k = 1, \dots, m$, be operators of order $\mu_k < 2m$ which form a normal system on $\partial\Omega$. Then $(v, \underline{v}) = (v, v_1, \dots, v_m)$ is a solution of the formally adjoint problem*

$$L^+ v = f \quad \text{in } \Omega, \tag{3.1.20}$$

$$P_j v + \sum_{k=1}^{m} Q^+_{k,j} v_k = g_j \quad \text{on } \partial\Omega, \quad j = 1, \dots, 2m, \tag{3.1.21}$$

if and only if

(i) *v is a solution of problem (3.1.18), (3.1.19), where*

$$h_k = \sum_{j=1}^{2m} \Lambda^+_{j,k+m}\, g_j,$$

(ii) *the functions v_k are given by the equality*

$$v_k = B'_{k+m} v\Big|_{\partial\Omega} + \sum_{j=1}^{2m} \Lambda^+_{j,k} g_j\,. \tag{3.1.22}$$

Proof: From (3.1.14) and (3.1.15) it follows that

$$\sum_{j=1}^{2m} \Lambda_{k,j}\, Q_{j,s} = \sum_{j=1}^{2m} Q_{k,j}\, \Lambda_{j,s} = \delta_{k,s} \quad \text{for } k, s = 1, \dots, 2m, \tag{3.1.23}$$

where $\delta_{k,s}$ denotes the Kronecker symbol. We assume that $(v, \underline{v})$ is a solution of the problem (3.1.20), (3.1.21). Multiplying (3.1.21) by $\Lambda^+_{j,s}$ and summing up over $j = 1, \dots, 2m$, we get

$$\sum_{j=1}^{2m} \Lambda^+_{j,s}\, P_j v + \sum_{k=1}^{m} \Big(\sum_{j=1}^{2m} \Lambda^+_{j,s}\, Q^+_{k,j} \Big) v_k = \sum_{j=1}^{2m} \Lambda^+_{j,s}\, g_j \quad \text{on } \partial\Omega. \tag{3.1.24}$$

By (3.1.16) and (3.1.23), the left-hand side of (3.1.24) is equal to $-B'_{s+m} v + v_s$ for $s = 1, \dots, m$ and to $B'_{s-m} v$ for $s = m+1, \dots, 2m$. Consequently, v is a solution of (3.1.18), (3.1.19), and $v_1, \dots, v_m$ are determined by (3.1.22).

If conversely v is a solution of (3.1.18), (3.1.19) with $h_k = \sum_{j=1}^{2m} \Lambda^+_{j,k+m}\, g_j$ and $v_1, \dots, v_m$ are given by (3.1.22), then $(v, \underline{v})$ satisfies (3.1.24) for $s = 1, \dots, 2m$. Multiplying this equation by $Q^+_{s,\mu}$ and summing up over $s = 1, \dots, 2m$, then by means of (3.1.23), we get (3.1.21). This proves the lemma. ■

3.1.5. Examples. Let $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial/\partial x_n^2$ be the *Laplace operator* in $\mathbb{R}^n$.

Example 1. We consider the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \tag{3.1.25}$$

$$b\, u + c\, u_1 = g \quad \text{on } \partial\Omega, \tag{3.1.26}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial u_1}{\partial \tau} = 0 \quad \text{on } \partial\Omega, \tag{3.1.27}$$

where Ω is a plane domain with smooth boundary, b, c, g are given real-valued continuously differentiable functions on $\partial\Omega$, $\partial/\partial\nu$ denotes the derivative in the direction of the exterior normal, and $\partial/\partial\tau$ denotes the derivative in the direction tangential to $\partial\Omega$.

This boundary value problem is equivalent to the *Cauchy-Hilbert problem:* Find a function $u+iv$ which is holomorphic in Ω, continuous in $\overline{\Omega}$, and satisfies the boundary condition $bu + cv = g$ on $\partial\Omega$. (If we set $u_1 = v|_{\partial\Omega}$, the boundary condition (3.1.27) holds from the Cauchy-Riemann equations.)

The following two examples illustrate the assertion of Lemma 3.1.1.

Example 2. We consider the *Dirichlet problem* for the Laplace operator

$$\Delta u = f \quad \text{in } \Omega, \tag{3.1.28}$$

$$u = g \quad \text{on } \partial\Omega. \tag{3.1.29}$$

Here the system of the boundary operators consists only of the operator $B = 1$ and is obviously normal on $\partial\Omega$. The formally adjoint problem with respect to the classical Green formula

$$\int\limits_\Omega \Delta u \cdot \overline{v}\, dx + \int\limits_{\partial\Omega} u \cdot \frac{\partial \overline{v}}{\partial \nu}\, d\sigma = \int\limits_\Omega u \cdot \Delta\overline{v}\, dx + \int\limits_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \overline{v}\, d\sigma$$

coincides with the starting problem (3.1.28), (3.1.29). If we use the more general Green formula (3.1.9) in Theorem 3.1.1

$$\int\limits_\Omega \Delta u \cdot \overline{v}\, dx + \int\limits_{\partial\Omega} u \cdot \overline{v}_1\, d\sigma = \int\limits_\Omega u \cdot \Delta\overline{v}\, dx + \int\limits_{\partial\Omega} u \cdot (\overline{v}_1 - \frac{\partial \overline{v}}{\partial \nu})\, d\sigma + \int\limits_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot \overline{v}\, d\sigma\,,$$

we obtain the formally adjoint (in the sense of Definition 3.1.3) problem

$$\Delta v = f \quad \text{in } \Omega,$$
$$v = g_1\,, \quad v_1 - \frac{\partial v}{\partial \nu} = g_2 \quad \text{on } \partial\Omega$$

which is obviously equivalent to problem (3.1.28), (3.1.29).

Example 3. Let Ω be a plane domain with smooth boundary $\partial\Omega$. We consider the problem with oblique derivative

$$\Delta u = f \quad \text{in } \Omega, \tag{3.1.30}$$

$$Bu = b_1(x)\,\frac{\partial u}{\partial \nu} + b_2(x)\frac{\partial u}{\partial \tau} = g \quad \text{on } \partial\Omega, \tag{3.1.31}$$

where b_1, b_2 are smooth functions satisfying the condition $(b_1, b_2) \neq 0$ on $\partial\Omega$. The last condition ensures the ellipticity of the problem (3.1.30), (3.1.31). In the special case $b_1 = 1$, $b_2 = 0$ we obtain the *Neumann problem*

$$\Delta u = f \ \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = g \ \text{ on } \partial\Omega.$$

The boundary condition (3.1.31) is normal if $b_1 \neq 0$ on $\partial\Omega$. Then the classical Green formula

$$\int_\Omega \Delta u \cdot \overline{v}\, dx - \int_{\partial\Omega} Bu \cdot b_1^{-1}\overline{v}\, d\sigma = \int_\Omega u \cdot \Delta\overline{v}\, dx - \int_{\partial\Omega} u \cdot \Big(\frac{\partial \overline{v}}{\partial \nu} - \frac{\partial}{\partial \tau}(b_1^{-1} b_2 \overline{v})\Big)\, d\sigma$$

holds and the formally adjoint problem with respect to this Green formula is

$$\Delta v = f \quad \text{in } \Omega, \tag{3.1.32}$$

$$-\frac{\partial v}{\partial \nu} + \frac{\partial}{\partial \tau}\big(\overline{b_1^{-1}\, b_2}\, v\big) = g \quad \text{on } \partial\Omega. \tag{3.1.33}$$

The coefficients in the boundary condition are bounded if $b_1 \neq 0$ on $\partial\Omega$.

Now we construct the formally adjoint problem with respect to the Green formula (3.1.9). In our example formula (3.1.9) has the form

$$\int_\Omega \Delta u \cdot \overline{v}\, dx + \int_{\partial\Omega} Bu \cdot \overline{v_1}\, d\sigma$$
$$= \int_\Omega u \cdot \Delta\overline{v}\, dx - \int_{\partial\Omega} u \cdot \Big(\frac{\partial \overline{v}}{\partial \nu} + \frac{\partial}{\partial \tau}(b_2\, \overline{v_1})\Big)\, d\sigma + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \cdot (\overline{v} + b_1 \overline{v_1})\, d\sigma,$$

and the formally adjoint boundary value problem to (3.1.30), (3.1.31) with respect to this Green formula is

$$\Delta v = f \quad \text{in } \Omega, \tag{3.1.34}$$

$$-\frac{\partial v}{\partial \nu} - \frac{\partial}{\partial \tau}(\overline{b_2}\, v_1) = g_1\,, \qquad v + \overline{b_1}\, v_1 = g_2 \quad \text{on } \partial\Omega. \tag{3.1.35}$$

Obviously, in the case $b_1 \neq 0$ the pair (v, v_1) is a solution of (3.1.34), (3.1.35) if and only if v is a solution of the problem (3.1.32), (3.1.33) with

$$g = g_1 + \frac{\partial}{\partial \tau}\big(\overline{b_1^{-1}\, b_2\, g_2}\big)$$

and v_1 coincides with the function $\overline{b_1}^{-1}(g_2 - v|_{\partial\Omega})$. Therefore, (3.1.34), (3.1.35) can be considered as an equivalent problem to (3.1.32), (3.1.33) if $b_1 \neq 0$ on $\partial\Omega$. In the case, when $b_1 = 0$ on a non-vanishing subset of $\partial\Omega$, the problem (3.1.34), (3.1.35) cannot be reduced to a boundary value problem with only one unknown.

3.1.6. Equivalence of boundary conditions. We have seen that the formally adjoint problems with respect to different Green formulas are equivalent in certain sense. Now we want to give a precise definition for the equivalence of boundary conditions.

Let two boundary value problems for the same differential operator L of order $2m$ be given:

$$\text{Problem 1.}\quad \begin{aligned} &Lu = f \ \text{ in } \Omega, \\ &Bu + C\underline{u} = \underline{g} \ \text{ on } \partial\Omega, \end{aligned}$$

$$\text{Problem 2.}\quad \begin{aligned} &Lu = f \ \text{ in } \Omega, \\ &B'u + C'\underline{v} = \underline{h} \ \text{ on } \partial\Omega. \end{aligned}$$

Here B is a vector of differential operators B_k $(k = 1, \dots, m+J)$, $\operatorname{ord} B_k \le \mu_k$, B' is a vector of differential operators B'_k $(k = 1, \dots, m+J')$, $\operatorname{ord} B'_k \le \mu'_k$, C is a $(m+J)\times J-$matrix of tangential differential operators $C_{k,j}$, $\operatorname{ord} C_{k,j} \le \mu_k + \tau_j$, and C' is a $(m+J')\times J'-$ matrix of tangential differential operators $C'_{k,j}$, $\operatorname{ord} C'_{k,j} \le \mu'_k + \tau'_j$. We suppose that

$$\sum_{k=1}^{m+J} \mu_k + \sum_{j=1}^{J} \tau_j = \sum_{k=1}^{m+J'} \mu'_k + \sum_{j=1}^{J'} \tau'_j \tag{3.1.36}$$

DEFINITION 3.1.5. Problem 1 and Problem 2 are said to be *equivalent* if there exist

a) a vector $\mathcal{B}$ of differential operators B_k, $k = m+J+1, \dots, d$, and a vector $\mathcal{B}'$ of differential operators B'_k, $k = m+J'+1, \dots, d$, where $\operatorname{ord} B_k \le \mu_k$, $\operatorname{ord} B'_k \le \mu'_k$,

$$\sum_{k=1}^{d} \mu_k = \sum_{k=1}^{d} \mu'_k\,. \tag{3.1.37}$$

b) an invertible matrix

$$S = \big(S_{k,j}\big)_{1\le k,j\le d}$$

of tangential differential operators $S_{k,j}$ on $\partial\Omega$, $\operatorname{ord} S_{k,j} \le \mu'_k - \mu_j$,

c) an invertible matrix

$$S' = \big(S'_{k,j}\big)_{1\le k,j\le d-m}$$

of tangential differential operators $S'_{k,j}$, $\operatorname{ord} S'_{k,j} \le \tau_j - \tau'_k$, where $\tau_j = -\mu_{m+j}$ for $j = J+1, \dots, d-m$, $\tau'_j = -\mu'_{m+j}$ for $j = J'+1, \dots, d-m$

such that

$$S\begin{pmatrix} C & 0 \\ 0 & I_{d-m-J} \end{pmatrix} = \begin{pmatrix} C' & 0 \\ 0 & I_{d-m-J'} \end{pmatrix} S' \quad \text{and} \quad S\begin{pmatrix} Bu \\ \mathcal{B}u \end{pmatrix}\bigg|_{\partial\Omega} = \begin{pmatrix} B'u \\ \mathcal{B}'u \end{pmatrix}\bigg|_{\partial\Omega} \tag{3.1.38}$$

for $u \in C^\infty(\overline{\Omega})$. Here I_s denotes the identity matrix of size $s \times s$.

REMARK 3.1.3. By (3.1.37), the determinant of S is a function on $\partial\Omega$. The invertibility of the matrix S means that the determinant of S does not vanish on $\partial\Omega$. Then the inverse matrix to S is a matrix of tangential differential operators of order not greater than $\mu_k - \mu'_j$.

Analogously, from (3.1.36) and (3.1.37) it follows that

$$\sum_{j=1}^{d-m} \tau_j = \sum_{j=1}^{d-m} \tau'_j$$

and, consequently, the determinant of $\mathcal{S}'$ is a function. If this function does not vanish on $\partial\Omega$, then $\mathcal{S}'$ is invertible. The inverse to S' is a matrix of tangential differential operators of order not greater than $\tau'_j - \tau_k$.

Hence the equivalence relation for boundary value problems given above is symmetric. Furthermore, it can be easily shown that this relation is transitive.

Example 1. We consider the formally adjoint boundary value problem (3.1.10), (3.1.11) and construct the formally adjoint problem to it. Obviously, the Green formula

$$\text{(3.1.39)}\quad \int\limits_\Omega u\cdot\overline{L^+v}\,dx + \int\limits_{\partial\Omega} \big(\underline{w}, (Pv+Q^+\underline{v})\big)_{\mathbb{C}^{2m}}\,d\sigma + \int\limits_{\partial\Omega} \big(\underline{u}, C^+\underline{v}\big)_{\mathbb{C}^J}\,d\sigma$$
$$= \int\limits_\Omega Lu\cdot\overline{v}\,dx + \int\limits_{\partial\Omega} \big(T^+(\underline{w}-\mathcal{D}u), \mathcal{D}v\big)_{\mathbb{C}^{2m}} d\sigma + \int\limits_{\partial\Omega} \big(Q\underline{w}+C\underline{u}, \underline{v}\big)_{\mathbb{C}^{m+J}}\,d\sigma,$$

is valid for all $u, v \in C_0^\infty(\overline{\Omega})$, $\underline{u} \in C_0^\infty(\partial\Omega)^J$, $\underline{v} \in C_0^\infty(\partial\Omega)^{m+J}$, $\underline{w} \in C^\infty(\partial\Omega)^{2m}$, where T is the matrix defined in Remark 3.1.2. Hence the boundary value problem

$$\text{(3.1.40)}\qquad Lu = f \quad \text{in } \Omega,$$

$$\text{(3.1.41)}\qquad -T^+\mathcal{D}u + T^+\underline{w} = \underline{g}^{(1)}, \qquad Q\underline{w} + C\underline{u} = \underline{g}^{(2)} \quad \text{on } \partial\Omega$$

is formally adjoint to (3.1.10), (3.1.11). We show that this problem is equivalent to problem (3.1.1), (3.1.2).

The boundary value problem (3.1.40), (3.1.41) contains $3m+J$ boundary conditions. The numbers μ'_k, τ'_j which determine the orders of the differential operators in (3.1.41) are

$$\mu'_k = \begin{cases} 2m-k & \text{for} \quad k = 1,\dots,2m, \\ \mu_{k-2m} & \text{for} \quad k = 2m+1,\dots,3m+J \end{cases}$$

and

$$\tau'_j = \begin{cases} 1-j & \text{for} \quad j = 1,\dots,2m, \\ \tau_{j-2m} & \text{for} \quad j = 2m+1,\dots,2m+J. \end{cases}$$

Obviously,

$$\sum_{k=1}^{3m+J} \mu'_k + \sum_{j=1}^{2m+J} \tau'_j = \sum_{k=1}^{m+J} \mu_k + \sum_{j=1}^{J} \tau_j$$

If L is elliptic, then the matrix T is invertible. Consequently,

$$S = \begin{pmatrix} 0 & -T^+ \\ I_{m+J} & -Q \end{pmatrix}$$

is an invertible $(3m+J)\times(3m+J)-$matrix. The order of the differential operator in the k-th row and j-th column of S is less or equal to $\mu'_k - \mu_j$, where $\mu_k = k-m-J-1$

for $k = m + J + 1, \dots, 3m + J$. Since $B = Q \cdot \mathcal{D}$, we have

$$S \cdot \begin{pmatrix} B \\ \mathcal{D} \end{pmatrix} = \begin{pmatrix} -T^+\mathcal{D} \\ 0 \end{pmatrix} \quad \text{and} \quad S \cdot \begin{pmatrix} C & 0 \\ 0 & I_{2m} \end{pmatrix} = \begin{pmatrix} T^+ & 0 \\ Q & C \end{pmatrix} \cdot \begin{pmatrix} 0 & -I_{2m} \\ I_J & 0 \end{pmatrix}$$

Therefore, the boundary value problem (3.1.1), (3.1.2) is equivalent to the problem (3.1.40), (3.1.41).

Example 2. Let us consider the formally adjoint problems to the classical boundary value problem (3.1.12), (3.1.13) with a normal system of boundary operators B_k ($k = 1, \dots, m$) of orders less than $2m$. For this problem we have established two formally adjoint problems – the problem (3.1.20), (3.1.21)

$$L^+ v = f \quad \text{in } \Omega, \qquad Pv + Q^+ \underline{v} = \underline{g} \quad \text{on } \partial\Omega$$

which is formally adjoint to (3.1.12), (3.1.13) with respect to the Green formula (3.1.9) and the problem (3.1.18), (3.1.19)

$$L^+ v = f \quad \text{in } \Omega, \qquad B'_k v = h_k \quad \text{on } \partial\Omega$$

which is formally adjoint to (3.1.12), (3.1.13) with respect to the classical Green formula (3.1.17). We show that both problems are equivalent.

In this case we have $J = m$, $J' = 0$, $d = 2m$. The components P_k of the vector P are differential operators of order not greater than $\sigma_k \stackrel{def}{=} 2m - k$ ($k = 1, \dots, 2m$), and the orders of the operator $Q^+_{j,k}$ in (3.1.21) satisfy the inequality $\operatorname{ord} Q^+_{j,k} \le \sigma_k + \tau_j$, where $\tau_j = \mu_j + 1 - 2m$ ($j = 1, \dots, m$). Furthermore, the order of the operator B'_k ($k = 1, \dots, m$) in (3.1.19) is equal to $\mu'_k \stackrel{def}{=} 2m - 1 - \mu_{k+m}$. Since $\mu_1 + \dots + \mu_{2m} = m(2m-1)$, we have

$$\sum_{k=1}^{2m} \sigma_k + \sum_{j=1}^{m} \tau_j = \sum_{k=1}^{m} \mu'_k \,,$$

i.e., the condition (3.1.36) is satisfied. Let $\mu'_k \stackrel{def}{=} \operatorname{ord} B'_k = 2m - 1 - \mu_{k-m}$ for $k = m+1, \dots, 2m$, and let S be the matrix

$$S = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \cdot \Lambda^+ \,,$$

where the matrix Λ is defined by (3.1.15). The elements $S_{k,j}$ ($k, j = 1, \dots, 2m$) of the matrix S are tangential differential operators satisfying the condition $\operatorname{ord} S_{k,j} \le \mu'_k - \sigma_j$. From (3.1.16) it follows that

$$S \cdot P = \begin{pmatrix} B'_1 \\ \vdots \\ B'_{2m} \end{pmatrix}.$$

By (3.1.15), the matrix Q^+ consists of the first m columns of $(\Lambda^+)^{-1}$. Thus, we have

$$\Lambda^+ \cdot Q^+ = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$$

and therefore,

$$S \cdot Q^+ = \begin{pmatrix} 0 \\ I_m \end{pmatrix} \cdot (-I_m).$$

This proves the equivalence of the boundary value problems (3.1.18), (3.1.19) and (3.1.20), (3.1.21).

Example 3. The boundary value problems

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = g_1, \quad \frac{\partial u}{\partial \nu} = g_2 \quad \text{on } \partial\Omega \tag{3.1.42}$$

and

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} + u = h_1, \quad \frac{\partial u}{\partial \nu} - u = h_2 \quad \text{on } \partial\Omega \tag{3.1.43}$$

are not equivalent in the sense of Definition 3.1.5, although the vectors of the boundary operators in (3.1.42) and (3.1.43) are connected by the equality

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \partial/\partial\nu \end{pmatrix} = \begin{pmatrix} \partial/\partial\nu + 1 \\ \partial/\partial\nu - 1 \end{pmatrix}.$$

However, here the condition (3.1.36) on the orders of the boundary operators is not satisfied. Note that the first problem is elliptic, while the second one is not elliptic.

THEOREM 3.1.3. *Any two equivalent boundary value problems are elliptic or not elliptic simultaneously.*

Proof: Let Problem 1 and Problem 2 be equivalent boundary value problems such that the assumptions of Definition 3.1.5 are valid. Since the equivalence relation is symmetric, it suffices to show that the validity of Condition (ii) in Definition 3.1.2 for Problem 1 implies the validity of this condition for Problem 2.

Let $x^{(0)}$ be an arbitrary point on the boundary $\partial\Omega$ and $\xi' \neq 0$ an arbitrary vector tangential to $\partial\Omega$ at $x^{(0)}$. We assume that the problem

$$L^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})D_t\big)\, u(t) = 0 \quad \text{for } t > 0, \tag{3.1.44}$$

$$B^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})D_t\big)u(t)|_{t=0} + C^\circ(x^{(0)}, \xi')\underline{u} = 0 \tag{3.1.45}$$

has only the trivial solution in $\mathcal{M}^+ \times \mathbb{C}^J$ and prove that the problem

$$L^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})D_t\big)\, u(t) = 0 \quad \text{for } t > 0, \tag{3.1.46}$$

$$B'^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})D_t\big)u(t)|_{t=0} + C'^\circ(x^{(0)}, \xi')\underline{v} = 0 \tag{3.1.47}$$

has only the trivial solution in $\mathcal{M}^+ \times \mathbb{C}^{J'}$.

We denote by $S^\circ(x^{(0)}, D_x)$ the "principal part" of $S(x, D_x)$ with coefficients frozen in $x^{(0)}$, i.e., the elements $S^\circ_{k,j}(x^{(0)}, D_x)$ of the matrix $S^\circ(x^{(0)}, D_x)$ contain only the derivatives of order $\mu'_k - \mu_j$. Analogously, let $S'^\circ(x^{(0)}, D_x)$, be the "principal part" of $S'(x, D_x)$ with coefficients frozen in $x^{(0)}$. From the invertibility of $S(x, D_x)$ and $S'(x, D_x)$ and from the conditions (3.1.36), (3.1.37) it follows that $S^\circ(x^{(0)}, \xi')$ and $S'^\circ(x^{(0)}, \xi')$ are invertible matrices. Furthermore, (3.1.38) yields

$$S^\circ(x^{(0)}, \xi') \cdot \begin{pmatrix} B^\circ(x^{(0)}, \xi' + \nu(x^{(0)})\, D_t) \\ \mathcal{B}^\circ(x^{(0)}, \xi' + \nu(x^{(0)})D_t) \end{pmatrix} = \begin{pmatrix} B'^\circ(x^{(0)}, \xi' + \nu(x^{(0)})D_t) \\ \mathcal{B}'^\circ(x^{(0)}, \xi' + \nu(x^{(0)})D_t) \end{pmatrix} \tag{3.1.48}$$

and

$$S^\circ(x^{(0)},\xi')\cdot\begin{pmatrix} C^\circ(x^{(0)},\xi') & 0 \\ 0 & I_{d-m-J}\end{pmatrix} = \begin{pmatrix} C'^\circ(x^{(0)},\xi') & 0 \\ 0 & I_{d-m-J'}\end{pmatrix}\cdot S'^\circ(x^{(0)},\xi'). \tag{3.1.49}$$

Let $(u,\underline{v})\in\mathcal{M}^+(\eta)\times\mathbb{C}^{J'}$ be a solution of problem (3.1.46), (3.1.47). We set

$$v_j = -B'^\circ_{j+m}(x^{(0)},\xi'+\nu(x^{(0)})D_t)u(t)|_{t=0}$$

for $j=J'+1,\dots,d-m$ and

$$\underline{u}' = \begin{pmatrix} u_1 \\ \vdots \\ u_{d-m}\end{pmatrix} = S'^\circ(x^{(0)},\xi')^{-1}\cdot\begin{pmatrix} v_1 \\ \vdots \\ v_{d-m}\end{pmatrix}.$$

Then (3.1.47) yields

$$\begin{pmatrix} B'^\circ(x^{(0)},\xi'+\nu(x^{(0)})D_t)u \\ \mathcal{B}'^\circ(x^{(0)},\xi'+\nu(x^{(0)})D_t)u\end{pmatrix} + \begin{pmatrix} C'^\circ(x^{(0)},\xi') & 0 \\ 0 & I_{d-m-J'}\end{pmatrix} S'^\circ(x^{(0)},\xi')\,\underline{u}' = 0$$

for $t=0$. Multiplying this equation by $S^\circ(x^{(0)},\xi')^{-1}$ and using (3.1.48), (3.1.49), we get

$$B^\circ\big(x^{(0)},\xi'+\nu(x^{(0)})D_t\big)u(t)|_{t=0} + C^\circ(x^{(0)},\xi')\underline{u} = 0,$$
$$B^\circ_{j+m}\big(x^{(0)},\xi'+\nu(x^{(0)})D_t\big)u(t)|_{t=0} + u_j = 0, \qquad j=J+1,\dots,d-m,$$

where $\underline{u}$ denotes the vector $(u_1,\dots,u_J)$. Since problem (3.1.44), (3.1.45) has only the trivial solution, we can conclude that $u=0$, $u_j=0$ for $j=1,\dots,d-m$, and, consequently, $v_j=0$ for $j=1,\dots,d-m$. Hence, problem (3.1.46), (3.1.47) has only the trivial solution. This proves that the ellipticity of Problem 1 implies the ellipticity of Problem 2. ∎

The validity of condition (3.1.38) implies the following connection between the solutions of the equivalent Problems 1 and 2: Let $\underline{h}=(h_1,\dots,h_{m+J'})$ be a vector-function on $\partial\Omega$ and let $\underline{g}=(g_1,\dots,g_{m+J})$, $\underline{g}'=(g_{m+J+1},\dots,g_d)$ be determined by the relation

$$\begin{pmatrix}\underline{g}\\ \underline{g}'\end{pmatrix} = S^{-1}\begin{pmatrix}\underline{h}\\ 0\end{pmatrix}$$

Furthermore, let $(u,\underline{u})$ be a solution of Problem 1 and

$$\underline{u}' = \underline{g}' - \mathcal{B}u|_{\partial\Omega}\,.$$

We denote by $\underline{v}$ the vector-function which consists of the first J' components of the vector

$$S'\begin{pmatrix}\underline{u}\\ \underline{u}'\end{pmatrix}.$$

Then $(u,\underline{v})$ is a solution of Problem 2.

3.2. An a priori estimate for the solution

Now we study the operator of the boundary value problem (3.1.1), (3.1.2) in a domain with smooth boundary. First we extend this operator to Sobolev spaces of arbitrary integer order. Using the results of the foregoing chapter, we prove a regularity assertion and an a priori estimates for the solutions of elliptic problems. Furthermore, we prove that the ellipticity is necessary for the validity of such estimates.

3.2.1. Sobolev spaces. Let Ω be a bounded domain in $\mathbb{R}^n$. We define the Sobolev space $W_2^l(\Omega)$ for arbitrary integer $l \geq 0$ as the set of all functions $u \in L_2(\Omega)$ such that all generalized derivatives $D_x^\alpha u$ with $|\alpha| \leq l$ are elements of $L_2(\Omega)$. The Sobolev space $W_2^l(\Omega)$ is a separable Hilbert space with the norm

$$(3.2.1) \qquad \|u\|_{W_2^l(\Omega)} = \Big(\int\limits_\Omega \sum_{|\alpha|\leq l} |D_x^\alpha u(x)|^2 \, dx \Big)^{1/2}.$$

Note that the set $C_0^\infty(\overline{\Omega})$ of all infinitely differentiable functions in $\overline{\Omega}$ with compact support is dense in $W_2^l(\Omega)$ if the boundary $\partial\Omega$ is smooth. Therefore, in this case the space $W_2^l(\Omega)$ can be defined as the closure of $C_0^\infty(\overline{\Omega})$ with respect to the norm (3.2.1).

Furthermore, the space $\overset{\circ}{W}{}_2^l(\Omega)$ is defined as the closure of the set $C_0^\infty(\Omega)$ of all infinitely differentiable functions u having compact support in Ω with respect to the norm (3.2.1). By $W_2^l(\Omega)^*$ we denote the dual space to $W_2^l(\Omega)$ equipped with the norm

$$(3.2.2) \qquad \|u\|_{W_2^l(\Omega)^*} = \sup \Big\{ |\,(u,v)_\Omega\,| \, : \, v \in W_2^l(\Omega), \, \|v\|_{W_2^l(\Omega)} \leq 1 \Big\}.$$

Here $(\cdot,\cdot)_\Omega$ denotes the extension of the scalar product in $L_2(\Omega)$ to $W_2^l(\Omega)^* \times W_2^l(\Omega)$.

If $l \geq 1$ and the boundary $\partial\Omega$ is smooth, then the trace of every function $u \in W_2^l(\Omega)$ on $\partial\Omega$ exists. The space of traces of functions from $W_2^l(\Omega)$ $(l \geq 1)$ on $\partial\Omega$ is denoted by $W_2^{l-1/2}(\partial\Omega)$. The norm in this space is

$$\|u\|_{W_2^{l-1/2}(\partial\Omega)} = \inf \Big\{ \|v\|_{W_2^l(\Omega)} \, : \, v \in W_2^l(\Omega), \, v\big|_{\partial\Omega} = u \Big\}.$$

By means of suitable diffeomorphisms, the norm in $W_2^{l-1/2}(\partial\Omega)$ can be locally given by the norm in $W_{2,per}^{l-1/2}(\mathbb{R}^{n-1})$ (see Section 2.1). We denote the dual space to $W_2^{l-1/2}(\partial\Omega)$ by $W_2^{-l+1/2}(\partial\Omega)$ and define the norm in this space analogously to (3.2.2).

Finally, we define the space $\tilde{W}_2^{l,k}(\Omega)$ for arbitrary integer k, l, $k \geq 0$, as follows. In the case $l \geq 0$ the space $\tilde{W}_2^{l,k}(\Omega)$ consists of all pairs $(u, \underline{\phi})$ such that $u \in W_2^l(\Omega)$ and $\underline{\phi} = (\phi_1, \ldots, \phi_k)$ is a vector-functions with components $\phi_j \in W_2^{l-j+1/2}(\partial\Omega)$ satisfying the condition

$$D_\nu^{j-1} u\Big|_{\partial\Omega} = \phi_j \quad \text{for } j \leq \min(k,l).$$

For $l < 0$ we set

$$\tilde{W}_2^{l,k}(\Omega) = W_2^{-l}(\Omega)^* \times \prod_{j=1}^k W_2^{l-j+1/2}(\partial\Omega).$$

In particular, by this notation, the space $\tilde{W}_2^{l,0}(\Omega)$ coincides with $W_2^l(\Omega)$ for nonnegative l and with the space $W_2^{-l}(\Omega)^*$ for $l < 0$. In the case $l \geq k$ the space $\tilde{W}_2^{l,k}(\Omega)$ can be identified with $W_2^l(\Omega)$. The norm in $\tilde{W}_2^{l,k}(\Omega)$ is defined in a natural way as

$$\|(u, \underline{\phi})\|_{\tilde{W}_2^{l,k}(\Omega)} = \Big(\|u\|^2_{\tilde{W}_2^{l,0}(\Omega)} + \sum_{j=1}^{k} \|\phi_j\|^2_{W_2^{l-j+1/2}(\partial\Omega)}\Big)^{1/2}, \tag{3.2.3}$$

where by $\|\cdot\|_{\tilde{W}_2^{l,0}(\Omega)}$ we denote the norm in $W_2^l(\Omega)$ if $l \geq 0$ and in $W_2^{-l}(\Omega)^*$ if $l < 0$.

Obviously, there are the following imbeddings:

$$\begin{aligned} &W_2^{l+1}(\Omega) \subset W_2^l(\Omega) \quad \text{for } l \geq 0, \\ &W_2^{l+1/2}(\partial\Omega) \subset W_2^{l-1/2}(\partial\Omega), \\ &\tilde{W}_2^{l+1,k}(\Omega) \subset \tilde{W}_2^{l,k}(\Omega), \end{aligned}$$

where in each case the first space is dense in the second one (cf. Lemma 1.3.1). These imbeddings are continuous and for bounded Ω even compact. From the density of $\tilde{W}_2^{l_1,k}(\Omega)$ in $\tilde{W}_2^{l,k}(\Omega)$ for $l_1 > l$ it follows that $\tilde{W}_2^{l,k}(\Omega)$ is the closure of the set

$$\big\{(u, \underline{\phi}) \in C_0^\infty(\overline{\Omega}) \times C_0^\infty(\partial\Omega)^k \; : \; \underline{\phi} = \big(u|_{\partial\Omega}, D_\nu u|_{\partial\Omega}, \dots, D_\nu^{k-1} u|_{\partial\Omega}\big)\big\}$$

with respect to the norm (3.2.3).

3.2.2. The operator of the boundary value problem. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the boundary value problem (3.1.1), (3.1.2), where as before, $\operatorname{ord} B_k < 2m$ for $k = 1, \dots, m+J$ and the coefficients of L, B_k, and $C_{k,j}$ are assumed to be smooth. In order to simplify the notation, we set

$$W_2^{l+\underline{\tau}-1/2}(\partial\Omega) = \prod_{j=1}^{J} W_2^{l+\tau_j-1/2}(\partial\Omega) \quad \text{and} \quad W_2^{l-\underline{\mu}-1/2}(\partial\Omega) = \prod_{k=1}^{m+J} W_2^{l-\mu_k-1/2}(\partial\Omega).$$

Then the operator $\mathcal{A} : \; (u, \underline{u}) \to (f, \underline{g})$ of problem (3.1.1), (3.1.2) continuously maps

$$W_2^l(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \tag{3.2.4}$$

into the space

$$W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega), \tag{3.2.5}$$

where l is an arbitrary integer not less than $2m$.

Extension of the operator $\mathcal{A}$. In the following, we denote by $\mathcal{A}$ also the operator

$$\begin{aligned} &\tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \ni \big(u, \mathcal{D}u|_{\partial\Omega}, \underline{u}\big) \\ &\to \big(Lu,\, Bu|_{\partial\Omega} + C\underline{u}\big) \in W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega), \quad l \geq 2m. \end{aligned} \tag{3.2.6}$$

Our goal is to extend this operator to the space

$$\tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \tag{3.2.7}$$

with arbitrary integer $l < 2m$. We start with the extension of the operator L.

Let l be a fixed integer, $0 \le l < 2m$. We write the differential operator L in the form

$$L(x, D_x) = \sum_{|\alpha| \le 2m-l} D_x^\alpha \, L_\alpha(x, D_x)\,, \tag{3.2.8}$$

where L_α are differential operators of order $\le l$.

LEMMA 3.2.1. *Let L be the operator (3.2.8). Then the formula*

$$\begin{aligned}\int_\Omega Lu \cdot \overline{v}\, dx &= \sum_{|\alpha|\le 2m-l} \int_\Omega L_\alpha(x, D_x)u \cdot \overline{D_x^\alpha v}\, dx \\ &\quad + \sum_{j=l+1}^{2m} \int_{\partial\Omega} D_\nu^{j-1}u \cdot \overline{P_j v}\, d\sigma + \sum_{j=1}^{l} \int_{\partial\Omega} D_\nu^{j-1}u \cdot \overline{P_{l,j} v}\, d\sigma\end{aligned} \tag{3.2.9}$$

is valid for $u, v \in C_0^\infty(\overline{\Omega})$. Here P_j are the same operators as in (3.1.7) and $P_{l,j}$ are differential operators of order $\le 2m - j$ with smooth coefficients. Moreover, the functional

$$v \to \sum_{j=1}^{l} \int_{\partial\Omega} D_\nu^{j-1}u \cdot \overline{P_{l,j} v}\, d\sigma \tag{3.2.10}$$

is continuous on $W_2^{2m-l}(\Omega)$ for arbitrary $u \in W_2^l(\Omega)$.

Proof: Integrating by parts, we get

$$\int_\Omega Lu \cdot \overline{v}\, dx = \sum_{|\alpha|\le 2m-l} \int_\Omega L_\alpha(x, D_x)u \cdot \overline{D_x^\alpha v}\, dx + \sum_{j=1}^{2m} \int_{\partial\Omega} D_\nu^{j-1}u \cdot \overline{P_{l,j} v}\, d\sigma$$

with certain differential operators $P_{l,j}$ of order $\le 2m - j$. This and (3.1.7) imply

$$\begin{aligned}&\int_\Omega u \cdot \overline{L^+ v}\, dx - \sum_{|\alpha|\le 2m-l} \int_\Omega L_\alpha(x, D_x)u \cdot \overline{D_x^\alpha v}\, dx \\ &\quad + \sum_{j=1}^{l} \int_{\partial\Omega} D_\nu^{j-1}u \cdot \overline{(P_j - P_{l,j})v}\, d\sigma = \sum_{j=l+1}^{2m} \int_{\partial\Omega} D_\nu^{j-1}u \cdot \overline{(P_{l,j} - P_j)v}\, d\sigma.\end{aligned} \tag{3.2.11}$$

The left-hand side of (3.2.11) defines a linear and continuous functional on $W_2^l(\Omega)$ for arbitrary fixed $v \in W_2^{2m}(\Omega)$. Hence the right-hand side is also a continuous functional on $W_2^l(\Omega)$. This is only possible if $P_{l,j} = P_j$ for $j = l+1, \dots, 2m$.

Furthermore, the functional

$$v \to \sum_{|\alpha|\le 2m-l} \int_\Omega L_\alpha(x, D_x)u \cdot \overline{D_x^\alpha v}\, dx$$

is continuous on $W_2^{2m-l}(\Omega)$ for arbitrary $u \in W_2^l(\Omega)$. By (2.3.20), the same is true for the functional

$$v \to \int_\Omega Lu \cdot \overline{v}\, dx - \sum_{j=l+1}^{2m} \int_{\partial\Omega} D_\nu^{j-1}u \cdot \overline{P_j v}\, d\sigma.$$

Consequently, the functional (3.2.10) is also continuous on $W_2^{2m-l}(\Omega)$ for arbitrary $u \in W_2^l(\Omega)$. The lemma is proved. ■

Using formula (3.1.7) and Lemma 3.2.1, we can prove the following lemma.

LEMMA 3.2.2. *The operator*

$$\tilde{W}_2^{2m,2m}(\Omega) \ni \left(u, \mathcal{D}u|_{\partial\Omega}\right) \to Lu \in L_2(\Omega) \tag{3.2.12}$$

can be uniquely extended to a continuous operator

$$\tilde{W}_2^{l,2m}(\Omega) \ni (u, \underline{\phi}) \to f \in W_2^{2m-l}(\Omega)^*, \quad l < 2m. \tag{3.2.13}$$

The functional $f = L(u, \underline{\phi})$ in (3.2.13) is given by

$$(f, v)_\Omega = (u, L^+ v)_\Omega + \sum_{j=1}^{2m} \left(\phi_j, P_j v\right)_{\partial\Omega}, \quad v \in W_2^{2m-l}(\Omega), \tag{3.2.14}$$

if $l \leq 0$ and by

$$\begin{aligned}(f, v)_\Omega \;=\; & \sum_{|\alpha| \leq 2m-l} \int_\Omega L_\alpha(x, D_x) u \cdot \overline{D_x^\alpha v}\, dx + \sum_{j=l+1}^{2m} (\phi_j\,, P_j v)_{\partial\Omega} \\ & + \sum_{j=1}^{l} \left(D_\nu^{j-1} u\,, P_{l,j} v\right)_{\partial\Omega}, \quad v \in W_2^{2m-l}(\Omega),\end{aligned} \tag{3.2.15}$$

if $0 < l < 2m$.

Here $(\cdot,\cdot)_\Omega$ denotes the extension of the scalar product in $L_2(\Omega)$ to each of the products $W_2^{2m-l}(\Omega)^* \times W_2^{2m-l}(\Omega)$ and $W_2^{-l}(\Omega)^* \times W_2^{-l}(\Omega)$, while $(\cdot,\cdot)_{\partial\Omega}$ is the extension of the scalar product in $L_2(\partial\Omega)$ to $W_2^{l-j+1/2}(\partial\Omega) \times W_2^{-l+j-1/2}(\partial\Omega)$.

Proof: Obviously the mapping (3.2.13), where f is defined by (3.2.14) and (3.2.15), is continuous. If $(u, \underline{\phi}) \in \tilde{W}_2^{2m,2m}(\Omega)$, then by (3.1.7) and Lemma 3.2.1, we have $f = Lu$. Thus, the operator (3.2.13) is an extension of the operator (3.2.12). The uniqueness of the extension follows from the density of $\tilde{W}_2^{2m,2m}(\Omega)$ in $\tilde{W}_2^{l,2m}(\Omega)$ for $l < 2m$. ■

Furthermore, by means of (3.1.6), we can extend the operator

$$\tilde{W}_2^{l,2m}(\Omega) \ni \left(u, \mathcal{D}u|_{\partial\Omega}\right) \to Bu|_{\partial\Omega} \in W_2^{l-\underline{\mu}-1/2}(\partial\Omega), \quad l \geq 2m,$$

to a continuous operator

$$\tilde{W}_2^{l,2m}(\Omega) \ni (u, \underline{\phi}) \to \underline{g} \in W_2^{l-\underline{\mu}-1/2}(\partial\Omega), \quad l < 2m. \tag{3.2.16}$$

The vector-function $\underline{g}$ in (3.2.16) is given by the equality

$$\underline{g} = B(u, \underline{\phi}) \stackrel{def}{=} Q \cdot \underline{\phi} \tag{3.2.17}$$

Thus, we obtain the following theorem.

THEOREM 3.2.1. *The operator (3.2.6) can be uniquely extended to a linear and continuous operator*

(3.2.18)

$$\tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \ni (u, \underline{\phi}, \underline{u}) \to (f, \underline{g}) \in W_2^{2m-l}(\Omega)^* \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$$

with $l < 2m$. This extension is given by

$$f = L(u, \underline{\phi}), \qquad \underline{g} = Q\underline{\phi} + C\underline{u},$$

where L is the operator (3.2.13) and Q is determined by (3.1.6).

REMARK 3.2.1. By Theorem 3.2.1, $\mathcal{A}$ is a continuous operator

$$\tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \to \tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$$

for arbitrary integer l.

In the case $l \le 0$ the functional f and the vector-function $\underline{g}$ in (3.2.18) satisfy the equality

$$(f, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} = (u, L^+ v)_\Omega + (\underline{\phi}\,,\, Pv|_{\partial\Omega} + Q^+\underline{v}\,)_{\partial\Omega} + (\underline{u}\,,\, C^+\underline{v})_{\partial\Omega}\,, \tag{3.2.19}$$

where v is an arbitrary function from $W_2^{2m-l}(\Omega)$ and $\underline{v} \in W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$. Here $(\cdot,\cdot)_{\partial\Omega}$ denotes the scalar product in each of the spaces $L_2(\partial\Omega)^J$, $L_2(\partial\Omega)^{2m}$ and $L_2(\partial\Omega)^{m+J}$. Therefore, $\mathcal{A}$ is adjoint to the operator

$$\begin{aligned} W_2^{2m-l}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \ni (v, \underline{v}) \to \big(L^+ v,\ Pv|_{\partial\Omega} + Q^+\underline{v},\ C^+\underline{v}\big) \\ \in W_2^{-l}(\Omega) \times \Big(\prod_{j=1}^{2m} W_2^{-l+j-1/2}(\partial\Omega)\Big) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega) \end{aligned} \tag{3.2.20}$$

of the formally adjoint problem if $l \le 0$.

Extension of the Green formula to distribution spaces. The Green formula (3.1.9) is valid, for example, for functions $u, v \in W_2^{2m}(\Omega)$ and vector-functions $\underline{u}$, $\underline{v}$ from corresponding Sobolev spaces on the boundary. However, the extension of the operators L, L^+, B, and P to Sobolev spaces of arbitrary order allows us to extend this formula to functions in Sobolev spaces of lower orders.

THEOREM 3.2.2. *Let l be an arbitrary integer number. Then for all $(u, \underline{\phi}, \underline{u}) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$ and $(v, \underline{\psi}, \underline{v}) \in \tilde{W}_2^{2m-l,2m}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$ the Green formula*

$$\begin{aligned} &\big(L(u, \underline{\phi}), v\big)_\Omega + \big(B(u, \underline{\phi}) + C\underline{u},\ \underline{v}\big)_{\partial\Omega} \\ &= \big(u,\ L^+(v, \underline{\psi})\big)_\Omega + \big(\underline{\phi},\ P(v, \underline{\psi}) + Q^+\underline{v}\big)_{\partial\Omega} + \big(\underline{u},\ C^+\underline{v}\big)_{\partial\Omega} \end{aligned} \tag{3.2.21}$$

is valid.

Proof: Since $B(u, \underline{\phi}) = Q\underline{\phi}$ and $P(v, \underline{\psi}) = T\underline{\psi}$, formula (3.2.21) is equivalent to

$$\big(L(u, \underline{\phi}), v\big)_\Omega = \big(u,\ L^+(v, \underline{\psi})\big)_\Omega + \big(\underline{\phi},\ T\underline{\psi}\big)_{\partial\Omega}\,. \tag{3.2.22}$$

For $(u, \underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$, $(v, \underline{\psi}) \in \tilde{W}_2^{2m-l,2m}(\Omega)$, $l \le 0$, we have $T\underline{\psi} = Pv|_{\partial\Omega}$, and (3.2.22) follows from Lemma 3.2.2. Hence the functional

$$\tilde{W}_2^{2m-l,2m}(\Omega) \ni (v, \underline{\psi}) \to \big(L(u, \underline{\phi}), v\big)_\Omega - \big(u,\ L^+(v, \underline{\psi})\big)_\Omega - \big(\underline{\phi},\ T\underline{\psi}\big)_{\partial\Omega} \tag{3.2.23}$$

is equal to zero for fixed $(u, \underline{\phi}) \in \tilde{W}_2^{l_1,2m}(\Omega)$, $l_1 \ge l$, $l \le 0$. Since this functional is continuous on $\tilde{W}_2^{2m-l_1,2m}(\Omega)$ and $\tilde{W}_2^{2m-l,2m}(\Omega)$ is dense in $\tilde{W}_2^{2m-l_1,2m}(\Omega)$, we conclude that the functional (3.2.23) vanishes on $\tilde{W}_2^{2m-l_1,2m}(\Omega)$ for arbitrary $(u, \underline{\phi}) \in \tilde{W}_2^{l_1,2m}(\Omega)$. This proves the validity of (3.2.22) for $(u, \underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$, $(v, \underline{\psi}) \in \tilde{W}_2^{2m-l,2m}(\Omega)$ with arbitrary integer l. ■

3.2.3. A regularity assertion for the solutions. In Section 2.3.5 we have formulated a regularity assertion for solutions of elliptic boundary value problems in the half-space. Now we extend this result to elliptic boundary value problems in bounded domains.

First note that, by means of Lemma 2.3.2, the following assertion holds.

LEMMA 3.2.3. *Let ζ, η be functions from $C_0^\infty(\overline{\Omega})$ such that $\eta = 1$ in a neighbourhood of* supp ζ. *If L is elliptic in $\overline{\Omega}$, then for all $(u, \underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$ the inequality*

$$(3.2.24) \qquad \sum_{j=1}^{2m} \|\zeta\phi_j\|_{W_2^{l-j+1/2}(\partial\Omega)} \le c\left(\|\eta u\|_{\tilde{W}_2^{l,0}(\Omega)} + \|\eta L(u,\underline{\phi})\|_{\tilde{W}_2^{l-2m,0}(\Omega)}\right)$$

is satisfied with a constant c independent of $(u, \underline{\phi})$.

Naturally, the functions ζ, η can be identically equal to one in $\overline{\Omega}$. This means that the inequality (3.2.24) is also valid if we omit the factors ζ and η.

THEOREM 3.2.3. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. If $(u, \underline{\phi}, \underline{u}) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$ is a solution of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$ with*

$$(3.2.25) \qquad (f, \underline{g}) \in \tilde{W}_2^{l-2m+1,0}(\Omega) \times W_2^{l-\underline{\mu}+1/2}(\partial\Omega).$$

Then $(u, \phi, \underline{u})$ is an element of the space

$$(3.2.26) \qquad \tilde{W}_2^{l+1,2m}(\Omega) \times W_2^{l+\underline{\tau}+1/2}(\partial\Omega),$$

and the following inequality is valid with a constant c independent of $(u, \underline{\phi}, \underline{u})$:

$$(3.2.27) \quad \|(u, \underline{\phi}, \underline{u})\|_{l+1} \le c\left(\|f\|_{\tilde{W}_2^{l-2m+1,0}(\Omega)} + \|\underline{g}\|_{W_2^{l-\underline{\mu}+1/2}(\partial\Omega)} + \|(u, \underline{\phi}, \underline{u})\|_l\right).$$

Here $\|\cdot\|_l$ denotes the norm in (3.2.7).

Proof: Let $\{\mathcal{U}_\mu\}_{\mu=1}^N$ be a sufficiently fine covering of $\overline{\Omega}$ with open sets $\mathcal{U}_\mu$, and let $\{\zeta_\mu\}_{\mu=1}^N$ be a partition of unity subordinate to this covering which satisfies the conditions in the proof of Lemma 3.2.2.

We prove first that the vectors

$$(f^{(\mu)}, \underline{g}^{(\mu)}) \stackrel{def}{=} \mathcal{A}\zeta_\mu(u, \underline{\phi}, \underline{u}) - \zeta_\mu \mathcal{A}(u, \underline{\phi}, \underline{u}),$$

$\mu = 1, \dots, N$, belong to the space (3.2.25) for arbitrary $(u, \underline{\phi}, \underline{u})$ from the space (3.2.7) and that the norm of $(f^{(\mu)}, \underline{g}^{(\mu)})$ in (3.2.25) is not greater than $c\,\|(u, \underline{\phi}, \underline{u})\|_l$, where c is a constant independent of $(u, \underline{\phi}, \underline{u})$. For $l \ge 2m$ this assertion follows immediately from the Leibniz formula

$$D_x^\alpha(\zeta_\mu u) = \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} (D_x^{\alpha-\alpha'}\zeta_\mu)\, D_x^{\alpha'} u\,.$$

In the case $l \le 0$ the functional $(f^{(\mu)}, \underline{g}^{(\mu)})$ is defined by the equality

$$\begin{aligned}(3.2.28) \quad (f^{(\mu)}, v)_\Omega + (\underline{g}^{(\mu)}, \underline{v})_{\partial\Omega} \;&=\; \big(u,\, \overline{\zeta}_\mu L^+ v - L^+(\overline{\zeta}_\mu v)\big)_\Omega \\ &+ \big(\underline{\phi}\,,\, \overline{\zeta}_\mu(Pv + Q^+\underline{v}) - P(\overline{\zeta}_\mu v) - Q^+(\overline{\zeta}_\mu \underline{v})\big)_{\partial\Omega}\,. \\ &+ \big(\underline{u}\,,\, \overline{\zeta}_\mu C^+\underline{v} - C^+(\overline{\zeta}_\mu \underline{v})\big)_{\partial\Omega}\end{aligned}$$

Using the Leibniz formula, we get the inequality

$$|(f^{(\mu)},v)_\Omega + (\underline{g}^{(\mu)},\underline{v})_{\partial\Omega}| \le c\,\|(u,\underline{\phi},\underline{u})\|_l \left(\|v\|_{W_2^{2m-l-1}(\Omega)} + \|\underline{v}\|_{W_2^{-l+\underline{\mu}-1/2}(\partial\Omega)}\right)$$

for the case $l \le 0$. If $0 < l < 2m$, then this inequality can be proved analogously by means of the local representation (2.3.23) for the operator L.

Consequently, the term

$$\mathcal{A}\,\zeta_\mu\,(u,\underline{\phi},\underline{u}) = \zeta_\mu(f,\underline{g}) + (f^{(\mu)},\underline{g}^{(\mu)}) \tag{3.2.29}$$

belongs to the space (3.2.25). Let the support of ζ_μ be contained in the interior of Ω. Then $\zeta_\mu\underline{\phi} = 0$, $\zeta_\mu\underline{u} = 0$, and (3.2.29) yields

$$L\,(\zeta_\mu u) = \zeta_\mu f + f^{(\mu)}\,.$$

If we extend the functions $\zeta_\mu u$, $f^{(1)}$ and the coefficients of L outside the support of ζ_μ to periodic functions, we can consider this equation as an elliptic equation in $\mathbb{R}^n$. Then Theorem 2.1.2 yields $\zeta_\mu u \in W^{l+1}_{2,per}(\mathbb{R}^n)$. From this it follows that $\zeta_\mu(u,\underline{\phi},\underline{u})$ belongs to (3.2.26).

If the support of ζ_μ is contained in a sufficiently small neighbourhood of a point $x^{(0)} \in \partial\Omega$, then we can conclude this assertion in the same way from Theorem 2.3.5. Furthermore, an estimate analogous to (3.2.27) holds. Hence

$$(u,\underline{\phi},\underline{u}) = \sum_{\mu=1}^{N} \zeta_\mu(u,\underline{\phi},\underline{u})$$

is an element of the space (3.2.26) and satisfies the estimate (3.2.27). ■

REMARK 3.2.2. By Lemma 3.2.3, the term $\|(u,\underline{\phi},\underline{u})\|_l$ on the right of (3.2.27) can be replaced by

$$\|u\|_{\tilde{W}_2^{l,0}(\Omega)} + \|\underline{u}\|_{W_2^{l+\underline{\tau}-1/2}(\partial\Omega)}\,.$$

A local estimate for the solutions. As a consequence of Theorem 3.2.3, the following local regularity assertion holds.

LEMMA 3.2.4. *Let ζ, η be C^∞-functions on $\overline{\Omega}$ satisfying the equations $\zeta\eta = \zeta$ and $D_\nu^j\zeta|_{\partial\Omega} = D_\nu^j\eta|_{\partial\Omega} = 0$ for $j = 1,\dots,2m-1$. We assume that the boundary value problem (3.1.1), (3.1.2) is elliptic. If $(u,\underline{\phi},\underline{u}) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$ is a solution of problem (3.1.1), (3.1.2), where $\eta f \in \tilde{W}_2^{l-2m+1,0}(\Omega)$ and $\eta\underline{g} \in W_2^{l-\underline{\mu}+1/2}(\partial\Omega)$, then $\zeta(u,\underline{\phi},\underline{u})$ is an element of the space (3.2.26) and satisfies the estimate*

$$\|\zeta(u,\underline{\phi},\underline{u})\|_{l+1} \le c\left(\|\zeta f\|_{\tilde{W}_2^{l-2m+1,0}(\Omega)} + \|\zeta\underline{g}\|_{W_2^{l-\underline{\mu}+1/2}(\partial\Omega)} + \|\eta(u,\underline{\phi},\underline{u})\|_l\right),$$

where $\|\cdot\|_l$, $\|\cdot\|_{l+1}$ denote the norms in the spaces (3.2.7), (3.2.26), respectively.

Proof: Analogously to the proof of Theorem 3.2.3, it can be shown that the norm of

$$(f^{(1)},\underline{g}^{(1)}) \stackrel{def}{=} \mathcal{A}\zeta(u,\underline{\phi},\underline{u}) - \zeta\mathcal{A}(u,\underline{\phi},\underline{u})$$

in the space (3.2.25) is not greater than $c\,\|\eta(u,\underline{\phi},\underline{u})\|_l$ for arbitrary $(u,\underline{\phi},\underline{u})$ from the space (3.2.7). Since

$$\mathcal{A}\zeta(u,\underline{\phi},\underline{u}) = \zeta(f,\underline{g}) + (f^{(1)},\underline{g}^{(1)}),$$

now the assertion of the lemma follows immediately from Theorem 3.2.3. ■

COROLLARY 3.2.1. *Let ζ, η be C^∞-functions on $\overline{\Omega}$ such that $\eta = 1$ in a neighbourhood of* $\operatorname{supp}\zeta$. *If $(u, \underline{u}) \in W_2^{2m}(\Omega) \times W_2^{2m+\underline{\tau}-1/2}(\partial\Omega)$ is a solution of the elliptic boundary value problem (3.1.1), (3.1.2) and $\eta(f, \underline{g})$ belongs to the space (3.2.5) with $l \ge 2m$, then $\zeta(u, \underline{u})$ is an element of the space (3.2.4) and the inequality*

$$(3.2.30)\quad \|\zeta u\|_{W_2^l(\Omega)} + \|\zeta \underline{u}\|_{W_2^{l+\underline{\tau}-1/2}(\partial\Omega)} \le c\left(\|\eta f\|_{W_2^{l-2m}(\Omega)} + \|\eta \underline{g}\|_{W_2^{l-\underline{\mu}-1/2}(\partial\Omega)} + \|\eta(u, \underline{u})\|_{L_2(\Omega)\times L_2(\partial\Omega)^J}\right)$$

is satisfied with a constant c independent of $(u, \underline{u})$.

Proof: If additionally to our assumptions on ζ and η the condition

$$D_\nu^j \zeta|_{\partial\Omega} = D_\nu^j \eta|_{\partial\Omega} = 0 \quad \text{for } j = 1, \ldots, 2m-1$$

is satisfied, then the assertion of the corollary follows by induction from Lemma 3.2.3 and Lemma 3.2.4. Now let ζ and η be arbitrary C^∞-functions such that $\eta = 1$ in a neighbourhood of $\operatorname{supp}\zeta$. Then there exist smooth functions $\zeta^{(1)}$, $\eta^{(1)}$ such that $\eta^{(1)} = 1$ in a neighbourhood of $\operatorname{supp}\zeta^{(1)}$, $\zeta\zeta^{(1)} = \zeta$, $\eta\eta^{(1)} = \eta^{(1)}$, and $D_\nu^j \zeta^{(1)}|_{\partial\Omega} = D_\nu^j \eta^{(1)}|_{\partial\Omega} = 0$ for $j = 1, \ldots, 2m$. Hence the inequality (3.2.30) with $\zeta^{(1)}$, $\eta^{(1)}$ instead of ζ, η is valid. This implies (3.2.30) with our given functions ζ, η. ■

REMARK 3.2.3. The assertions of Lemma 3.2.4 and Corollary 3.2.1 are also true if Ω is an unbounded domain and the function ζ has compact support.

Necessity of the ellipticity for the a priori estimate. In Sections 2.1 and 2.3 we have shown that the ellipticity is a necessary condition for a priori estimates of the form (3.2.30) in $\mathbb{R}^n$ and in $\mathbb{R}_+^n$. We prove the same result for problems in a smooth bounded domain.

THEOREM 3.2.4. *Suppose that the estimate*

$$(3.2.31)\quad \|(u, \underline{\phi})\|_{\tilde{W}_2^{l,2m}(\Omega)} + \|\underline{u}\|_{W_2^{l+\underline{\tau}-1/2}(\partial\Omega)} \le c\Big(\|L(u, \underline{\phi})\|_{\tilde{W}_2^{l-2m,0}(\Omega)} + \|B(u, \underline{\phi}) + C\underline{u}\|_{W_2^{l-\underline{\mu}-1/2}(\partial\Omega)} + \|(u, \underline{\phi})\|_{\tilde{W}_2^{l-1,2m}(\Omega)} + \|\underline{u}\|_{W_2^{l+\underline{\tau}-3/2}(\partial\Omega)}\Big)$$

is satisfied for all $(u, \underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$, $\underline{u} \in W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$. Then the boundary value problem (3.1.1), (3.1.2) is elliptic.

Proof: 1) First we show that L is elliptic in Ω. Let $x^{(0)}$ be an arbitrary interior point of Ω and let $u \in \tilde{W}_2^{l,0}(\Omega)$ be a function with support in a sufficiently small neighbourhood $\mathcal{U} \subset \Omega$ of this point. We extend u and the coefficients of L outside $\mathcal{U}$ to periodic functions on $\mathbb{R}^n$. Then (3.2.31) yields

$$\|u\|_{W_{2,per}^l(\mathbb{R}^n)} \le c\left(\|Lu\|_{W_{2,per}^{l-2m}(\mathbb{R}^n)} + \|u\|_{W_{2,per}^{l-1}(\mathbb{R}^n)}\right),$$

and from Lemma 2.1.2 it follows that $L(x^{(0)}, D_x)$ is elliptic.

2) Now let $x^{(0)}$ be an arbitrary point on the boundary $\partial\Omega$. With no loss of generality, we may assume that $\partial\Omega$ coincides with the hyper-plane $x_n = 0$ in a neighbourhood of $x^{(0)}$. Furthermore, let u, $\underline{u}$ be functions with support in a sufficiently small neighbourhood of $x^{(0)}$. Extending $(u, \underline{\phi})$, $\underline{u}$, and the coefficients of L, B, and

C outside this neighbourhood to periodic functions with respect to $x_1, \dots, x_{n-1}$, we get the estimate

$$\|(u,\underline{\phi})\|_{\tilde{W}^{l,2m}_{2,per}(\mathbb{R}^n_+)} + \|\underline{u}\|_{W^{l+\underline{\tau}-1/2}_{2,per}(\partial\mathbb{R}^n_+)} \le c\Big(\|L(u,\underline{\phi})\|_{\tilde{W}^{l-2m,0}_{2,per}(\mathbb{R}^n_+)}$$
$$+\|B(u,\underline{\phi})+C\underline{u}\|_{W^{l-\underline{\mu}-1/2}_{2,per}(\partial\mathbb{R}^n_+)} + \|(u,\underline{\phi})\|_{\tilde{W}^{l-1,2m}_{2,per}(\mathbb{R}^n_+)} + \|\underline{u}\|_{W^{l+\underline{\tau}-3/2}_{2,per}(\partial\mathbb{R}^n_+)}\Big).$$

Hence by Lemma 2.3.3, the problem

$$L(x^{(0)},D_x)u = f \quad \text{for } x_n > 0,$$
$$B(x^{(0)},D_x)u|_{x_n=0} + C(x^{(0)},D_x)\underline{u} = \underline{g}$$

is elliptic. This proves the ellipticity of the boundary value problem (3.1.1), (3.1.2). ■

3.3. The adjoint operator

In the previous section we studied the operator $\mathcal{A}$ of an elliptic boundary value problem mapping a Cartesian product of Sobolev spaces into other Sobolev spaces. Now we explore the adjoint operator $\mathcal{A}^*$ acting in the corresponding dual spaces. In particular, we are interested in the relations between this operator and the operator $\mathcal{A}^+$ of the formally adjoint boundary value problem and in regularity assertions for solutions of the adjoint equation.

3.3.1. Relations between the adjoint operator and the operator of the formally adjoint problem. We suppose again that the orders of the differential operators B_k are less than $2m$. Then the operator $\mathcal{A}$ of the boundary value problem (3.1.1), (3.1.2) continuously maps (3.2.4) into (3.2.5) for arbitrary integer $l \ge 2m$. Hence the adjoint operator $\mathcal{A}^* : (v,\underline{v}) \to (F,\underline{h})$ of $\mathcal{A}$ is a linear and continuous mapping from the dual space of (3.2.5) into the dual space of (3.2.4), i.e.,

$$\mathcal{A}^* : W_2^{l-2m}(\Omega)^* \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \to W_2^l(\Omega)^* \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega). \tag{3.3.1}$$

The functional $(F,\underline{h}) = \mathcal{A}^*(v,\underline{v})$ is defined by the equality

$$(u,F)_\Omega + (\underline{u},\underline{h})_{\partial\Omega} = (Lu,v)_\Omega + (Bu|_{\partial\Omega} + C\underline{u}\,,\,\underline{v})_{\partial\Omega}\,, \tag{3.3.2}$$

where $(u,\underline{u})$ is an arbitrary element of the space (3.2.4).

Now we consider the operator $\mathcal{A}^+$ of the formally adjoint problem (3.1.10), (3.1.11). By Theorem 3.2.1, this operator can be extended to a continuous operator from

$$\tilde{W}_2^{2m-l,2m}(\Omega) \times \prod_{k=1}^{m+J} W_2^{-l+\mu_k+1/2}(\partial\Omega), \tag{3.3.3}$$

$l \le 0$, into

$$\tilde{W}_2^{-l,0}(\Omega) \times \prod_{j=1}^{2m} W_2^{-l+j-1/2}(\partial\Omega) \times \prod_{j=1}^{J} W_2^{-l-\tau_j+1/2}(\partial\Omega). \tag{3.3.4}$$

According to (3.1.7), (3.1.8), the mapping

$$\tilde{W}_2^{2m-l,2m}(\Omega) \ni (v,\underline{\psi}) \to f = L^+(v,\underline{\psi}) \in W_2^l(\Omega)^*$$

is defined for $l \ge 2m$ by the equality

$$(u,f)_\Omega = (Lu,v)_\Omega - \big(T^+\mathcal{D}u,\,\underline{\psi}\big)_{\partial\Omega}\,, \quad u \in W_2^l(\Omega),$$

while $P(v, \underline{\psi}) = T\underline{\psi}$. Hence the operator

$$\mathcal{A}^+ \, : \, (v, \underline{\psi}, \underline{v}) \to (f, \underline{g}, \underline{h}) = \big(L^+(v, \underline{\psi}),\, T\underline{\psi} + Q^+\underline{v},\, C^+\underline{v}\big)$$

which maps (3.3.3) into (3.3.4) is defined for $l \geq 2m$ by the equality

$$\text{(3.3.5)} \qquad \begin{aligned} &(u, f)_\Omega + (\underline{w}, \underline{g})_{\partial\Omega} + (\underline{u}, \underline{h})_{\partial\Omega} \\ &= (Lu, v)_\Omega + \big(\underline{w} - \mathcal{D}u|_{\partial\Omega}\,,\, T\underline{\psi}\big)_{\partial\Omega} + (Q\underline{w} + C\underline{u}\,,\, \underline{v})_{\partial\Omega}\,, \end{aligned}$$

where $(u, \underline{w}, \underline{u})$ is an arbitrary element of

$$\text{(3.3.6)} \qquad W_2^l(\Omega) \times \prod_{j=1}^{2m} W_2^{l-j+1/2}(\partial\Omega) \times \prod_{j=1}^{J} W_2^{l+\tau_j-1/2}(\partial\Omega)\,.$$

Comparing formulas (3.3.2) and (3.3.5), we obtain the following relation between the operators $\mathcal{A}^*$ and $\mathcal{A}^+$ (cf. Theorem 1.4.1).

LEMMA 3.3.1. *Let $(f, \underline{g}, \underline{h})$ be an arbitrary element of the space (3.3.4), where $l \geq 2m$. Furthermore, let the functional $F \in W_2^l(\Omega)^*$ be defined by*

$$\text{(3.3.7)} \qquad (u, F)_\Omega = (u, f)_\Omega + (\mathcal{D}u|_{\partial\Omega}\,, \underline{g})_{\partial\Omega}\,, \quad u \in W_2^l(\Omega).$$

Then $(v, \underline{\psi}, \underline{v})$ is a solution of the equation

$$\text{(3.3.8)} \qquad \mathcal{A}^+\,(v, \underline{\psi}, \underline{v}) = (f, \underline{g}, \underline{h})$$

in the space (3.3.3) if and only if

$$\text{(3.3.9)} \qquad \mathcal{A}^*\,(v, \underline{v}) = (F, \underline{h})$$

and $T\underline{\psi} + Q^+\underline{v} = \underline{g}$.

Proof: 1) Let $(v, \underline{\psi}, \underline{v})$ be a solution of the equation (3.3.8) in the space (3.3.3). Then in particular, $\underline{g} = P(v, \underline{\psi}) + Q^+\underline{v} = T\underline{\psi} + Q^+\underline{v}$. Furthermore, inserting $\underline{w} = \mathcal{D}u|_{\partial\Omega}$ into (3.3.5), we obtain

$$\text{(3.3.10)} \qquad (u, f)_\Omega + (\mathcal{D}u, \underline{g})_{\partial\Omega} + (\underline{u}, \underline{h})_{\partial\Omega} = (Lu, v)_\Omega + (Bu + C\underline{u}, \underline{v})_{\partial\Omega}$$

for arbitrary $u \in W_2^l(\Omega)$, $\underline{u} \in W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$. Hence the functionals F and $\underline{h}$ satisfy (3.3.3), i.e., $(v, \underline{v})$ is a solution of the equation (3.3.9).

2) Suppose that $(v, \underline{v}) \in W_2^{l-2m}(\Omega)^* \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$ is a solution of equation (3.3.9) with a functional F of the form (3.3.7) and $\underline{\psi} \in \prod W_2^{2m-l-j+1/2}(\partial\Omega)$ is a vector-function satisfying the equation $T\underline{\psi} + Q^*\underline{v} = \underline{g}$. Then (3.3.10) is satisfied for all $u \in W_2^l(\Omega)$, $\underline{u} \in W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$. This implies $C^+\underline{v} = \underline{h}$ and

$$(u, f)_\Omega + (\mathcal{D}u, \underline{g})_{\partial\Omega} = (Lu, v)_\Omega + (Bu, \underline{v})_{\partial\Omega}$$

for all $u \in W_2^l(\Omega)$. Inserting $\underline{g} = T\underline{\psi} + Q^+\underline{v}$ into the last equality and using the representation (3.1.6) for the vector Bu, we get

$$(u, f)_\Omega = (Lu, v)_\Omega - (\mathcal{D}u, T\underline{\psi})_{\partial\Omega}\,, \quad u \in W_2^l(\Omega),$$

i.e., $f = L^+(v, \underline{\psi})$. Consequently, $(v, \underline{\psi}, \underline{v})$ is a solution of the equation (3.3.8). The proof is complete. ■

Motivated by the representation (3.3.7) for the functional F in Lemma 3.3.1, we introduce the following subspace of $W_2^k(\Omega)^*$. For integer $k \geq 0$ and $l \geq -k$ let $D_2^{l,k}(\Omega)$ be the set of all functionals $F \in W_2^k(\Omega)^*$ which have the form

$$(3.3.11) \qquad (u, F)_\Omega = (u, f)_\Omega + \sum_{j=1}^{k} \big(D_\nu^{j-1} u|_{\partial\Omega}\, g_j\big)_{\partial\Omega}, \quad u \in W_2^k(\Omega),$$

where $g_j \in W_2^{l+j-1/2}(\partial\Omega)$, $f \in \tilde{W}_2^{l,0}(\Omega)$, i.e., $f \in W_2^l(\Omega)$ if $l \geq 0$ and $f \in W_2^{-l}(\Omega)^*$ if $l < 0$.

REMARK 3.3.1. If l is a negative integer, then the functional

$$u \to \sum_{j=1}^{-l} (D_\nu^{j-1} u|_{\partial\Omega}\,,\, g_j)_{\partial\Omega}$$

belongs to $W_2^{-l}(\Omega)^*$ for given $g_j \in W_2^{l+j-1/2}(\partial\Omega)$. Hence the space $D_2^{l,k}(\Omega)$ can be defined as the set of all functionals $F \in W_2^k(\Omega)^*$ which have the form

$$(u, F)_\Omega = (u, f)_\Omega + \sum_{j=-l+1}^{k} (D_\nu^{j-1} u|_{\partial\Omega}\,,\, g_j)_{\partial\Omega}\,, \quad u \in W_2^k(\Omega),$$

where $f \in W_2^{-l}(\Omega)^*$, $g_j \in W_2^{l+j-1/2}(\partial\Omega)$. This representation is unique.

The norm of the functional F in $D_2^{l,k}(\Omega)$ is defined as the infimum of the sum

$$\|f\|_{\tilde{W}_2^{l,0}(\Omega)} + \sum_{j=1}^{k} \|g_j\|_{W_2^{l+j-1/2}(\partial\Omega)}\,,$$

where f and g_j satisfy (3.3.11).

In the case $l \leq -k$, $k \geq 0$ we set $D_2^{l,k}(\Omega) = W_2^{-l}(\Omega)^*$. Then, in particular, we have

$$D_2^{l,0}(\Omega) = \tilde{W}_2^{l,0}(\Omega) = \begin{cases} W_2^l(\Omega) & \text{if} \quad l \geq 0\,, \\ W_2^{-l}(\Omega)^* & \text{if} \quad l < 0\,. \end{cases}$$

It can be easily seen that the space $D_2^{l_1,k}(\Omega)$ is continuously imbedded into $D_2^{l,k}(\Omega)$ if $l_1 > l$. Furthermore, $D_2^{l_1,k}(\Omega)$ is dense in $D_2^{l,k}(\Omega)$ if $l_1 > l$.

Since $W_2^{l-2m}(\Omega)^* = D_2^{-l+2m,0}(\Omega)$ and $W_2^l(\Omega)^* = D_2^{-l,2m}(\Omega)$ for $l \geq 2m$, the adjoint operator $\mathcal{A}^*$ continuously maps the space

$$(3.3.12) \qquad D_2^{-l+2m,0}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$$

for $l \geq 2m$ into the space

$$(3.3.13) \qquad D_2^{-l,2m}(\Omega) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega).$$

From Theorem 3.2.1 and Lemma 3.3.1 it follows that the restriction of $\mathcal{A}^*$ to the space

$$(3.3.14) \qquad W_2^{-l+2m}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$$

with $l < 2m$ continuously maps (3.3.14) into (3.3.13). Consequently, the adjoint operator $\mathcal{A}^*$ realizes a linear and continuous mapping

$$D_2^{-l+2m,0}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \to D_2^{-l,2m}(\Omega) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega)$$

for arbitrary integer l.

3.3.2. A regularity assertion for the solution of the adjoint problem. Using the relation between the adjoint operator $\mathcal{A}^*$ and the formally adjoint operator $\mathcal{A}^+$ given in Lemma 3.3.1 and the regularity assertion for solutions of elliptic boundary value problems in Section 3.2, we obtain the following assertion for the solution of the adjoint problem.

THEOREM 3.3.1. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. If* $(v, \underline{v})$ *is a solution of the adjoint equation* $\mathcal{A}^*(v, \underline{v}) = (F, \underline{h})$ *in the space (3.3.12) and*

$$(F, \underline{h}) \in D_2^{-l+1,2m}(\Omega) \times W_2^{-l-\underline{\tau}+3/2}(\partial\Omega),$$

then $(v, \underline{v})$ *is an element of the space*

$$D_2^{-l+1+2m,0}(\Omega) \times W_2^{-l+\underline{\mu}+3/2}(\partial\Omega) \tag{3.3.15}$$

and satisfies the estimate

$$\|(v, \underline{v})\|_{l+1} \le c\left(\|F\|_{D_2^{-l+1,2m}(\Omega)} + \|\underline{h}\|_{W_2^{-l-\underline{\tau}+3/2}(\partial\Omega)} + \|(v, \underline{v})\|_l\right), \tag{3.3.16}$$

where $\|\cdot\|_l$ *and* $\|\cdot\|_{l+1}$ *denote the norms in the spaces (3.3.12) and (3.3.15), respectively.*

Proof: By the assumptions of the lemma, the functional F has the form (3.3.7), where $f \in \tilde{W}_2^{-l+1,0}(\Omega)$ and $g_j \in W_2^{-l+j+1/2}(\partial\Omega)$. We choose f and g_j such that

$$\|f\|_{\tilde{W}_2^{-l+1,0}(\Omega)} + \sum_{j=1}^{2m} \|g_j\|_{W_2^{-l+j+1/2}(\partial\Omega)} \le 2\,\|F\|_{D_2^{-l+1,2m}(\Omega)}.$$

Since L is elliptic, the operator T is an isomorphism

$$\prod_{j=1}^{2m} W_2^{-l+2m-j+1/2}(\partial\Omega) \to \prod_{j=1}^{2m} W_2^{-l+j-1/2}(\partial\Omega)$$

(see Remark 3.1.2). Hence there exists a vector-function $\underline{\psi} \in \prod W_2^{-l+2m-j+1/2}(\partial\Omega)$ satisfying the equation $T\underline{\psi} + Q^+\underline{v} = \underline{g}$, where $\underline{g}$ denotes the vector $(g_1, \dots, g_{2m})$. From Lemma 3.3.1 it follows that $(v, \underline{\psi}, \underline{v})$ is a solution of the equation (3.3.8). Here $(v, \underline{\psi}, \underline{v})$ is an element of the space $\tilde{W}_2^{q,2m}(\Omega) \times W_2^{q-2m+\underline{\mu}+1/2}(\partial\Omega)$ with $q = \min(0, 2m-l)$. Applying Theorem 3.2.3, we obtain $(v, \underline{\psi}) \in \tilde{W}_2^{-l+1+2m,2m}(\Omega)$, $\underline{v} \in W_2^{-l+\underline{\mu}+3/2}(\partial\Omega)$ and

$$\|(v, \underline{\psi})\|_{\tilde{W}_2^{-l+1+2m,2m}(\Omega)} + \|\underline{v}\|_{W_2^{-l+\underline{\mu}+3/2}(\partial\Omega)} \le c\Big(\|f\|_{\tilde{W}_2^{-l+1,0}(\Omega)}$$
$$+ \sum_{j=1}^{2m} \|g_j\|_{W_2^{-l+j+1/2}(\partial\Omega)} + \|(v, \underline{\psi})\|_{\tilde{W}_2^{-l+2m,2m}(\Omega)} + \|\underline{v}\|_{W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)}\Big).$$

Since the norm of $\underline{\psi}$ can be estimated by the norms of $\underline{g}$ and $\underline{v}$, this implies (3.3.16). ∎

Note that the regularity assertion of Theorem 3.3.1 is true, in particular, for $F \in D_2^{-l+1,0}(\Omega)$ (i.e., $F \in W_2^{-l+1}(\Omega)$ if $l \le 1$, $F \in W_2^{l-1}(\Omega)^*$ if $l > 1$), since this space is contained in $D_2^{-l+1,2m}(\Omega)$. Furthermore, as a consequence of Theorem 3.3.1 and Lemma 3.3.1, we obtain the following assertion.

COROLLARY 3.3.1. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. If* $(v, \underline{v}) \in W_2^k(\Omega)^* \times W_2^{-k-2m+\underline{\mu}+1/2}(\partial\Omega)$, $k \geq 0$, *is a solution of the equation* $\mathcal{A}^*(v, \underline{v}) = (F, \underline{h})$, *where*

$$(F, \underline{h}) \in D_2^{l-2m,2m}(\Omega) \times W_2^{l-2m-\underline{\tau}+1/2}(\partial\Omega), \tag{3.3.17}$$

$l \geq -k$, *then*

$$(v, \underline{v}) \in D_2^{l,0}(\Omega) \times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega). \tag{3.3.18}$$

Moreover, in the case $l \geq 2m$, *the vector-function* $(v, \underline{v})$ *is a solution of the formally adjoint problem (3.1.10), (3.1.11), where the function* $f \in W_2^{l-2m}(\Omega)$ *and the vector-function*

$$\underline{g} = (g_1, \dots, g_{2m}) \in \prod W_2^{l-2m+j-1/2}(\partial\Omega)$$

are determined by the equality

$$(u, F)_\Omega = (u, f)_\Omega + \big(\mathcal{D}u|_{\partial\Omega}, \underline{g}\big)_{\partial\Omega}, \qquad u \in W_2^{2m}(\Omega).$$

3.4. Solvability of elliptic boundary value problems in smooth domains

Using the results of Sections 3.2 and 3.3, we prove now that the operator of an elliptic boundary value problem is Fredholm. Furthermore, we state conditions for the solvability of elliptic boundary value problems. Analogous assertions hold for the adjoint operator.

3.4.1. The Fredholm property. We start with an abstract definition of the Fredholm property for linear and continuous operators.

DEFINITION 3.4.1. The linear and continuous operator $\mathcal{A}$ from the Banach space $\mathcal{X}$ into the Banach space $\mathcal{Y}$ is said to be a *Fredholm operator,* if

(i) $\dim\ker\mathcal{A} < \infty$, $\mathcal{R}(\mathcal{A})$ is closed.
(ii) $\dim\operatorname{coker}\mathcal{A} = \dim(\mathcal{Y}/\mathcal{R}(\mathcal{A})) < \infty$.

Here $\ker\mathcal{A}$ denotes the kernel and $\mathcal{R}(\mathcal{A})$ denotes the range of the operator $\mathcal{A}$. The *index* of the operator $\mathcal{A}$ is defined as the difference $\dim\ker\mathcal{A} - \dim\operatorname{coker}\mathcal{A}$.

In connection with the notion of the Fredholm operator the following lemma (see J. Peetre [**192**]) plays an important role.

LEMMA 3.4.1. *Let* $\mathcal{X}$, $\mathcal{Y}$, *and* $\mathcal{Z}$ *be Banach spaces, where* $\mathcal{X}$ *is compactly imbedded into* $\mathcal{Z}$. *Furthermore, let* $\mathcal{A}$ *be a linear and continuous operator from* $\mathcal{X}$ *into* $\mathcal{Y}$. *Then the following assertions are equivalent:*

1) $\dim\ker\mathcal{A} < \infty$ *and* $\mathcal{R}(\mathcal{A})$ *is closed in* $\mathcal{Y}$.
2) *There exists a constant* $c > 0$ *such that for each* $x \in \mathcal{X}$ *the following estimate is satisfied:*

$$\|x\|_{\mathcal{X}} \leq c\,(\|\mathcal{A}x\|_{\mathcal{Y}} + \|x\|_{\mathcal{Z}}).$$

3.4.2. The Fredholm property for the operator of an elliptic boundary value problem. As in the previous sections, we assume that the orders of the operators B_k are less than $2m$. From Theorem 3.2.4 and Lemma 3.4.1 it follows that the ellipticity of the boundary value problem (3.1.1), (3.1.2) is necessary for the Fredholm property of the operator

$$\mathcal{A} : \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \to \tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega) \tag{3.4.1}$$

of the boundary value problem (3.1.1), (3.1.2). We prove that the ellipticity is also sufficient for the Fredholm property. In the following, we denote the operator (3.4.1) by $\mathcal{A}_l$. In the case $l \geq 2m$ the operator $\mathcal{A}_l$ can be identified with the operator

$$W_2^l(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \ni (u, \underline{u}) \quad \to \quad \big(Lu,\, Bu|_{\partial\Omega} + C\underline{u}\big) \in W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega). \tag{3.4.2}$$

Furthermore, let

$$\mathcal{A}_l^+ : \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega) \to \tilde{W}_2^{l-2m,0}(\Omega) \times \Big(\prod_{j=1}^{2m} W_2^{l-2m+j-1/2}(\partial\Omega)\Big) \times W_2^{l-2m-\underline{\tau}+1/2}(\partial\Omega)$$

be the operator of the formally adjoint problem (3.1.10), (3.1.11).

For $l \geq 2m$ we denote the adjoint operator to the operator (3.4.2) by $\mathcal{A}_l^*$, while for $l < 2m$ the operator $\mathcal{A}_l^*$ is defined as the restriction of $\mathcal{A}_{2m}^*$ to the space $D_2^{-l+2m,0}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$.

We introduce the following three sets. By $\mathcal{N}$ we denote the set of all $(u, \underline{\phi}, \underline{u}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^{2m} \times C^\infty(\partial\Omega)^J$ such that $(u, \underline{u})$ is a solution of the homogeneous boundary value problem (3.1.1), (3.1.2) and $\underline{\phi} = \mathcal{D}u|_{\partial\Omega}$. Analogously, $\mathcal{N}^+$ is the set of all $(v, \underline{\psi}, \underline{v}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^{2m} \times C^\infty(\partial\Omega)^{m+J}$ such that $(v, \underline{v})$ is a solution of the homogeneous formally adjoint problem (3.1.10), (3.1.11) and $\underline{\psi} = \mathcal{D}v|_{\partial\Omega}$. Finally, we put $\mathcal{N}^* = \{(v, \underline{v}) : (v, \mathcal{D}v|_{\partial\Omega}, \underline{v}) \in \mathcal{N}^+\}$.

Using Lemma 3.4.1 and the regularity assertion for the solutions of elliptic boundary value problems in Theorem 3.2.3, we get the following lemma.

LEMMA 3.4.2. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. Then the kernels of the operators $\mathcal{A}_l$, $\mathcal{A}_l^+$, and $\mathcal{A}_l^*$ are independent of l and contain only C^∞-functions. More precisely, we have*

$$\ker \mathcal{A}_l = \mathcal{N}, \quad \ker \mathcal{A}_l^+ = \mathcal{N}^+, \quad \textit{and} \quad \ker \mathcal{A}_l^* = \mathcal{N}^*.$$

Furthermore, the spaces $\mathcal{N}$, $\mathcal{N}^+$, $\mathcal{N}^*$ have finite dimensions and the ranges of the operators $\mathcal{A}_l$, $\mathcal{A}_l^+$, $\mathcal{A}_l^*$ are closed.

Proof: Since $\mathcal{A}_{l+1}$ is the restriction of $\mathcal{A}_l$, we have $\ker \mathcal{A}_{l+1} \subset \ker \mathcal{A}_l$. However, by Theorem 3.2.3, every $(u, \underline{\phi}, \underline{u}) \in \ker \mathcal{A}_l$ belongs also to the kernel of $\mathcal{A}_{l+1}$. Hence the kernel of $\mathcal{A}_l$ is independent of l. Since the intersection of all Sobolev spaces in Ω and $\partial\Omega$ coincides with $C^\infty(\overline{\Omega})$ and $C^\infty(\partial\Omega)$, respectively, we obtain $\ker \mathcal{A}_l = \mathcal{N}$. Furthermore, from (3.2.27) and from the compactness of the imbeddings $W_2^{l+1}(\Omega) \subset W_2^l(\Omega)$, $W_2^{l+1/2}(\partial\Omega) \subset W_2^{l-1/2}(\partial\Omega)$ it follows that $\mathcal{R}(\mathcal{A}_l)$ is closed and $\dim \mathcal{N} < \infty$.

Analogously, the assertions concerning the operators $\mathcal{A}_l^+$ and $\mathcal{A}_l^*$ can be proved by means of Theorem 3.1.2, Theorem 3.3.1 and Corollary 3.3.1. ■

THEOREM 3.4.1. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. Then the operator $\mathcal{A}_l$ is Fredholm for arbitrary integer l. The kernel of $\mathcal{A}_l$ is the space $\mathcal{N}$ defined above and the range of $\mathcal{A}_l$ consists of all elements $(f, \underline{g}) \in \tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$ such that*

$$(f, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} = 0 \tag{3.4.3}$$

for each $(v, \underline{v}) \in \mathcal{N}^*$.

Proof: Condition (i) in the definition of the Fredholm property has been already verified in Lemma 3.4.1. It remains to show that the cokernel of $\mathcal{A}_l$ has finite dimension.

First let l be not greater than zero. Then $\mathcal{A}_l$ is adjoint to the operator (3.2.20) (see Remark 3.2.1). Since $\mathcal{R}(\mathcal{A}_l)$ is closed, the equation

$$\mathcal{A}_l\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g}) \tag{3.4.4}$$

is solvable in $\tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$ if and only if the equality (3.4.3) is satisfied for all elements $(v, \underline{v})$ of the kernel of the operator (3.2.20). By Lemma 3.4.2, the kernel of the operator (3.2.20) coincides with $\mathcal{N}^*$. Thus, we get the condition of the theorem for the solvability of the equation (3.4.4).

We assume now that $l \geq 0$. Obviously,

$$\mathcal{R}(\mathcal{A}_l) \subset \mathcal{R}(\mathcal{A}_0) \cap \Big(\tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)\Big).$$

Let $(f, \underline{g})$ be an arbitrary element of $\mathcal{R}(\mathcal{A}_0) \cap \big(\tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)\big)$. Then there exists an element $(u, \underline{\phi}, \underline{u}) \in \tilde{W}_2^{0,2m}(\Omega) \times W_2^{\underline{\tau}-1/2}(\partial\Omega)$ such that $\mathcal{A}_0\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$. From the regularity assertion in Theorem 3.2.3 it follows that $(u, \underline{\phi}, \underline{u})$ is an element of the space $\tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$. Consequently, $(f, \underline{g})$ belongs to the range of the operator $\mathcal{A}_l$. Therefore, $\mathcal{R}(\mathcal{A}_l)$ is the intersection of $\mathcal{R}(\mathcal{A}_0)$ with the space $\tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$ and, by the first part of the proof, we get the assertion of the theorem. ■

COROLLARY 3.4.1. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. Then the operator $\mathcal{A}_l^+$ is Fredholm. The kernel of $\mathcal{A}_l^+$ is the set $\mathcal{N}^+$ and the range of $\mathcal{A}_l^+$ is the set of all*

$$(f, \underline{g}, \underline{h}) \in \tilde{W}_2^{l-2m,0}(\Omega) \times \Big(\prod_{j=1}^{2m} W_2^{l-2m+j-1/2}(\partial\Omega)\Big) \times W_2^{l-2m-\underline{\tau}+1/2}(\partial\Omega) \tag{3.4.5}$$

satisfying the condition

$$(f, u)_\Omega + (\underline{g}, \underline{\phi})_{\partial\Omega} + (\underline{h}, \underline{u})_{\partial\Omega} = 0$$

for all $(u, \underline{\phi}, \underline{u}) \in \mathcal{N}$.

Proof: The first part of the corollary is an immediate consequence of Theorems 3.1.2 and 3.4.1. We show the last assertion. By Theorem 3.4.1, the range of $\mathcal{A}_l^+$ consists of all $(f, \underline{g}, \underline{h})$ in the space of (3.4.5) such that

$$(f, u)_\Omega + (\underline{g}, \underline{w})_{\partial\Omega} + (\underline{h}, \underline{u})_{\partial\Omega} = 0$$

for all solutions of the homogeneous problem (3.1.40), (3.1.41) which is formally adjoint to the problem (3.1.10), (3.1.11). However, the set of the solutions of the homogeneous problem (3.1.40), (3.1.41) coincides with $\mathcal{N}$, since the operator T^+ is invertible (see Remark 3.1.2). This proves the last assertion of the corollary. ∎

Furthermore, the following assertion holds.

THEOREM 3.4.2. *If the boundary value problem (3.1.1), (3.1.2) is elliptic, then the operator*

$$\mathcal{A}_l^* \,:\, D_2^{-l+2m,0}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \to D_2^{-l,2m}(\Omega) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega)$$

is Fredholm for arbitrary integer l. The kernel of $\mathcal{A}_l^*$ is the space $\mathcal{N}^*$ and the range of $\mathcal{A}_l^*$ consists of all $(F, \underline{h}) \in D_2^{-l,2m}(\Omega) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega)$ satisfying the condition

$$(u, F)_\Omega + (\underline{u}, \underline{h})_{\partial\Omega} = 0$$

for all solutions $(u, \underline{u}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^J$ of the homogeneous problem (3.1.1), (3.1.2).

Proof: For $l \geq 2m$ the assertions of the theorem follow from Theorem 3.4.1. Furthermore, according to Lemma 3.4.2, for arbitrary integer l the kernel of $\mathcal{A}_l^*$ coincides with $\mathcal{N}^*$ and the range of $\mathcal{A}_l^*$ is closed. From Theorem 3.3.1 we conclude that

$$\mathcal{R}(\mathcal{A}_l^*) = \mathcal{R}(\mathcal{A}_{2m}^*) \cap \Big(D_2^{-l+2m,0}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)\Big)$$

for $l < 2m$. This proves the theorem for $l < 2m$. ∎

A solvability condition for classical boundary value problems. Now we consider the classical boundary value problem

$$Lu = f \quad \text{in } \Omega, \qquad B_k u = g_k \quad \text{on } \partial\Omega, \quad k = 1, \dots, m. \tag{3.4.6}$$

Suppose that this boundary value problem is elliptic,

$$\operatorname{ord} B_k = \mu_k < 2m \quad \text{for } k = 1, \dots, m,$$

and the system of the operators B_k is normal on $\partial\Omega$. Then the classical Green formula (3.1.17)

$$\int\limits_\Omega Lu \cdot \overline{v}\, dx + \sum_{k=1}^m \int\limits_{\partial\Omega} B_k u \cdot \overline{B'_{k+m} v}\, d\sigma = \int\limits_\Omega u \cdot \overline{L^+ v}\, dx + \sum_{k=1}^m \int\limits_{\partial\Omega} B_{k+m} u \cdot \overline{B'_k v}\, d\sigma$$

is satisfied for all $u, v \in W_2^{2m}(\Omega)$

Using the relations between the formally adjoint problems (3.1.18), (3.1.19) and (3.1.20), (3.1.21) given in Lemma 3.1.1, we get the following description of the range of the operator

$$W_2^l(\Omega) \ni u \to \big(Lu,\, Bu_{\partial\Omega}\big) \in W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega), \quad l \geq 2m. \tag{3.4.7}$$

COROLLARY 3.4.2. *Suppose that the boundary value problem (3.4.6) is elliptic, $\operatorname{ord} B_k = \mu_k < 2m$ and the system of the operators $B_1, \dots, B_m$ is normal on $\partial\Omega$. Then the kernel of the operator (3.4.7) is a finite-dimensional subspace of $C^\infty(\overline{\Omega})$, while the range of this operator consists of all elements $(f, \underline{g}) \in W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$ satisfying the condition*

$$(f, v)_\Omega + \sum_{k=1}^m (g_k\,,\, B'_{m+k}\, v)_{\partial\Omega} = 0 \tag{3.4.8}$$

for all solutions $v \in C^\infty(\overline{\Omega})$ *of the homogeneous formally adjoint problem*

$$L^+ v = 0 \quad \text{in } \Omega, \qquad B'_k v = 0 \quad \text{on } \partial\Omega, \quad k = 1, \dots, m.$$

Analogously, the formally adjoint problem

$$L^+ v = f \quad \text{in } \Omega, \qquad B'_k v = g_k \quad \text{on } \partial\Omega, \quad k = 1, \dots, m,$$

with any given $f \in W_2^{l-2m}(\Omega)$, $g_k \in W_2^{l-2m-\mu_{m+k}+1/2}(\partial\Omega)$, $\mu_{m+k} = \operatorname{ord} B_{m+k}$, *is solvable in* $W_2^l(\Omega)$ *if and only if*

$$(f, u)_\Omega + \sum_{k=1}^{m} (g_k\,,\, B_{m+k}\, u)_{\partial\Omega} = 0 \tag{3.4.9}$$

for all solutions u of the homogeneous boundary value problem (3.4.6).

Proof: The assertion concerning the kernel of the operator (3.4.7) follows from Lemma 3.4.2. Furthermore, by Theorem 3.4.1, the range of this operator consists of all $(f, \underline{g}) \in W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$ satisfying the condition

$$(f, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} = 0$$

for all solutions $(v, \underline{v})$ of the homogeneous formally adjoint problem

$$L^+ v = 0 \quad \text{in } \Omega, \qquad Pv + Q^+ \underline{v} = 0 \quad \text{on } \partial\Omega. \tag{3.4.10}$$

Due to Lemma 3.1.1, $(v, \underline{v})$ is a solution of problem (3.4.10) if and only if v satisfies the equations $L^+ v = 0$ in Ω, $B'_k v = 0$ on $\partial\Omega$, and the components v_k of the vector-function $\underline{v}$ coincide with the traces of $B'_{m+k} v$ on $\partial\Omega$. This proves the validity of the condition (3.4.8). Analogously, we get (3.4.9). ■

3.4.3. Existence of left and right regularizers. Another method to establish the Fredholm property for the operator $\mathcal{A}_l$ consists in the construction of left and right regularizers.

DEFINITION 3.4.2. Let $\mathcal{A}$ be a linear and continuous operator from the Banach space $\mathcal{X}$ into the Banach space $\mathcal{Y}$. A linear and continuous operator $\mathcal{R}_l : \mathcal{Y} \to \mathcal{X}$ is said to be a *left regularizer* for the operator $\mathcal{A}$ if $\mathcal{R}_l \mathcal{A} - I$ is compact in $\mathcal{X}$. Analogously, a linear and continuous operator $\mathcal{R}_r : \mathcal{Y} \to \mathcal{X}$ is said to be a *right regularizer* for the operator $\mathcal{A}$ if $\mathcal{A}\mathcal{R}_r - I$ is compact in $\mathcal{Y}$. Here I denotes the identity operator in $\mathcal{X}$ and $\mathcal{Y}$, respectively.

It is known (see e.g. [**256**, Ch.2, §12]) that the existence of left and right regularizers is necessary and sufficient for the Fredholm property of the operator $\mathcal{A}$. In the following let

$$\mathcal{X}_l = \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \quad \text{and} \quad \mathcal{Y}_l = \tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega).$$

THEOREM 3.4.3. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. Then there exist both left and right regularizers for the operator* $\mathcal{A}_l$.

Proof: Existence of a right regularizer. Let $\{\mathcal{U}_\nu\}_{1\le\nu\le N}$ be a sufficiently fine covering of $\overline{\Omega}$. Furthermore, let ζ_ν, η_ν, $\nu = 1, \dots, N$, be C^∞ functions satisfying the conditions

$$\operatorname{supp} \zeta_\nu \subset \operatorname{supp} \eta_\nu \subset \mathcal{U}_\nu\,, \quad \zeta_\nu\, \eta_\nu = \zeta_\nu\,, \quad \sum_{\nu=1}^{N} \zeta_\nu = 1 \text{ in } \Omega.$$

For arbitrary $(f, \underline{g}) \in \mathcal{Y}_l$ let $\eta_\nu(f, \underline{g})$ denote the function $\eta_\nu f$ if $\operatorname{supp} \eta_\nu \cap \partial\Omega = \emptyset$ and the pair $(\eta_\nu f, \eta_\nu \underline{g})$ in the contrary case. The same notation will be used for $\zeta_\nu(u, \underline{u})$ if $(u, \underline{u})$ is an arbitrary element of the space $\mathcal{X}_l$.

For every index $\nu = 1, \ldots, N$ we introduce linear operators R_ν such that $\zeta_\nu R_\nu \eta_\nu$ continuously map $\mathcal{Y}_l$ into $\mathcal{X}_l$ and

$$\mathcal{A}_l \, \zeta_\nu \, R_\nu \, \eta_\nu = \zeta_\nu \, (I + T_\nu) + K_\nu \, , \tag{3.4.11}$$

where T_ν are linear and continuous operators in $\mathcal{Y}_l$ with small norms and K_ν are linear and continuous operators from $\mathcal{Y}_l$ into $\mathcal{Y}_{l+1}$.

First let ν be an integer such that $\mathcal{U}_\nu \cap \partial\Omega = \emptyset$. By $L^\circ(x^{(\nu)}, \partial_x)$ we denote the principal part of the operator L with coefficients frozen in a fixed point $x^{(\nu)} \in \mathcal{U}_\nu$. Then by Theorem 2.1.1 (see also Remark 2.1.1), there exists the operator

$$R_\nu = \big(L^\circ(x^{(\nu)}, D_x + \tfrac{1}{2}\vec{1})\big)^{-1} \; : \; W^{l-2m}_{2,per}(\mathbb{R}^n) \to W^l_{2,per}(\mathbb{R}^n).$$

This operator realizes also a continuous mapping from $W^{l-2m+1}_{2,per}(\mathbb{R}^n)$ onto the space $W^{l+1}_{2,per}(\mathbb{R}^n)$. Since the restrictions of the spaces $W^l_{2,per}(\mathbb{R}^n)$ and $W^l_2(\mathbb{R}^n)$ to $\mathcal{U}_\nu$ coincide, the operator $\zeta_\nu \, R_\nu \, \eta_\nu$ can be considered as a continuous operator from $\mathcal{Y}_l$ into $\mathcal{X}_l$. We extend the coefficients a_α of L outside $\mathcal{U}_\nu$ to smooth 2π-periodic functions $a_{\nu,\alpha}$ on $\mathbb{R}^n$ satisfying the condition $|a_{\nu,\alpha}(x) - a_\alpha(x^{(\nu)})| < \varepsilon$ and denote the differential operator with the coefficients $a_{\nu,\alpha}$ by L_ν. Then

$$\begin{aligned} \mathcal{A}_l \, \zeta_\nu \, R_\nu \, \eta_\nu(f, \underline{g}) &= L_\nu \, \zeta_\nu \, R_\nu \, \eta_\nu f \\ &= \zeta_\nu f + \zeta_\nu \big(L_\nu - L^\circ(x^{(\nu)}, D_x + \tfrac{1}{2}\vec{1})\big) \, R_\nu \, \eta_\nu \, f + [L_\nu, \zeta_\nu] \, R_\nu \, \eta_\nu \, f, \end{aligned}$$

where $[L_\nu, \zeta_\nu] = L_\nu \zeta_\nu - \zeta_\nu L_\nu$ is the commutator of L_ν and ζ_ν. Here the operator $L_{\nu,1} \overset{def}{=} L_\nu - L^\circ(x^{(\nu)}, D_x + \tfrac{1}{2}\vec{1})$ can be written in the form

$$L_{\nu,1} = L_{\nu,1} \, S_\rho + L_{\nu,1} \, (I - S_\rho) \, ,$$

where S_ρ denotes the operator (2.1.8). If ρ is large, then by Lemma 2.1.1, the operator norm $W^l_{2,per}(\mathbb{R}^n) \to W^{l-2m}_{2,per}(\mathbb{R}^n)$ of $L_{\nu,1} \, S_\rho$ is small, while $L_{\nu,1} \, (I - S_\rho)$ continuously maps $W^l_{2,per}(\mathbb{R}^n)$ into $W^{l-2m+1}_{2,per}(\mathbb{R}^n)$. Thus the operator R_ν has the desired properties.

Analogously, the operator R_ν can be constructed in the case $\mathcal{U}_\nu \cap \partial\Omega \neq \emptyset$ by means of Theorem 2.2.1 (see also Remark 2.2.3). The representation (3.4.11) for $\mathcal{A} \, \zeta_\nu \, R_\nu \, \eta_\nu$ can be shown using the operators $S_{\rho,1}$, $S_{\rho,2}$ in the proof of Theorem 2.3.5.

We define the operator $\mathcal{R}_r \, : \, \mathcal{Y}_l \to \mathcal{X}_l$ as follows:

$$\mathcal{R}_r = \sum_{\nu=1}^{N} \zeta_\nu \, R_\nu \, \eta_\nu \, (I + T_\nu)^{-1} \, .$$

Then by (3.4.11), we have

$$\mathcal{A} \, \mathcal{R}_r = I + \sum_{\nu=1}^{N} K_\nu \, (I + T_\nu)^{-1} \, ,$$

i.e., $\mathcal{R}_r$ is a right regularizer for the operator $\mathcal{A}$.

Existence of a left regularizer. Let R_ν be the operators introduced in the first part of the proof. For these operators also the representation

$$\zeta_\nu \, R_\nu \, \eta_\nu \, \mathcal{A} = \zeta_\nu \, (I + T_\nu + K_\nu)$$

holds, where T_ν is a linear and continuous operator in $\mathcal{X}_l$ with small norm and K_ν is a compact operator in $\mathcal{X}_l$. For example, in the case $\operatorname{supp}\eta_\nu \cap \partial\Omega = \emptyset$ we have

$$\zeta_\nu\, R_\nu\, \eta_\nu\, \mathcal{A}_l\,(u,\underline{u}) = \zeta_\nu\, R_\nu\, \eta_\nu\, L_\nu u = \zeta_\nu\, R_\nu\, \big(L_\nu\,(\eta_\nu u) - [L_\nu, \eta_\nu]\, u\big)$$
$$= \zeta_\nu\, \Big(u + R_\nu\, L_{\nu,1} S_\rho\,(\eta_\nu u) + R_\nu\, L_{\nu,1}(I - S_\rho)\,(\eta_\nu u) - R_\nu\,[L_\nu, \eta_\nu]\, u\Big)$$

where $L_{\nu,1}$, S_ρ are the same operators as in the first part of the proof. Here the operator $R_\nu L_{\nu,1} S_\rho \eta_\nu$ has a small norm $\mathcal{X}_l$ and $R_\nu L_{\nu,1}(I - S_\rho)\eta_\nu$, $R_\nu[L_\nu, \eta_\nu]$ are continuous operators from $\mathcal{X}_l$ into $\mathcal{X}_{l+1}$. An analogous representation holds in the case $\operatorname{supp}\eta_\nu \cap \partial\Omega \neq \emptyset$. Thus,

$$\mathcal{R}_l = \sum_{\nu=0}^{N} (I + T_\nu)^{-1}\, \zeta_\nu\, R_\nu\, \eta_\nu$$

is a left regularizer for the operator $\mathcal{A}_l$. ■

REMARK 3.4.1. We have always assumed in this chapter that $\operatorname{ord} B_k < 2m$ for $k = 1, \dots, m + J$. In fact, for the validity of Theorems 3.4.1 - 3.4.3 it is sufficient that there is the representation (3.1.6) for the vector of the operators B_k.

3.5. The Green function of the boundary value problem

In this section we consider the solutions of elliptic boundary value problems with δ-distributions on the right-hand sides. These solutions are of special interest, since they occur in integral representations for solutions of the same problem with arbitrary right-hand sides. One calls these solutions *Green functions* of the given boundary value problem.

For the sake of simplicity, we start with the case when the kernel and the cokernel of the operator $\mathcal{A}$ of the boundary value problem are trivial. Then the Green functions are uniquely determined as solutions corresponding to δ-distributions in the right-hand sides of the differential equation or boundary conditions. In the case of nontrivial kernel and cokernel we introduce generalized Green functions.

3.5.1. A representation of the solution for uniquely solvable boundary value problems. We suppose that $\operatorname{ord} B_k < 2m$ for $k = 1, \dots, m + J$ and that the operator

$$\mathcal{A} \,:\, \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \to \tilde{W}_2^{l-2m,0}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$$

of the boundary value problem (3.1.1), (3.1.2) is an isomorphism for one and, consequently, for all integer l.

Let $f(x) = \delta(x - y)$, where y is a fixed point in $\overline{\Omega}$, and let $\underline{g} = 0$. Since $f \in W_2^{2m-l}(\Omega)^*$ for $2m - l > n/2$, there exists a unique solution

$$\text{(3.5.1)} \quad \big(G(\cdot, y),\, \underline{\Phi}(\cdot, y), \underline{G}(\cdot, y)\big) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega), \quad l < 2m - n/2,$$

of the boundary value problem (3.1.1), (3.1.2). We call this solution *Green function* of the boundary value problem (3.1.1), (3.1.2).

LEMMA 3.5.1. *The function $G(\cdot,\cdot)$ is smooth in $\overline{\Omega}\times\overline{\Omega}\backslash\operatorname{diag}\overline{\Omega}$, while the vector-functions $\underline{\Phi}(\cdot,\cdot)$ and $\underline{G}(\cdot,\cdot)$ are smooth in $\partial\Omega \times \overline{\Omega}\backslash\operatorname{diag}\partial\Omega$, where $\operatorname{diag}\overline{\Omega} = \{(x,x) : x \in \overline{\Omega}\}$, $\operatorname{diag}\partial\Omega = \{(x,x) : x \in \partial\Omega\}$. Furthermore, we have*

$$\text{(3.5.2)} \qquad \underline{\Phi}(x,y) = \mathcal{D}\, G(x,y)$$

for $x \in \partial\Omega$, $y \in \overline{\Omega}$, $x \neq y$.

Proof: Using local estimates (see Lemma 3.2.4), we obtain that $G(x,y)$, $\underline{\Phi}(x,y)$, and $\underline{G}(x,y)$ are smooth with respect to x and continuous with respect to y for $x \neq y$. Furthermore, (3.5.2) holds. Differentiating (3.1.1), (3.1.2) with respect to y and repeating the above consideration, we conclude that G, $\underline{\Phi}$, and $\underline{G}$ are smooth with respect to both variables outside the diagonals. ∎

By Corollary 3.4.1, the unique solvability of problem (3.1.1), (3.1.2) implies the unique solvability of the formally adjoint problem (3.1.10), (3.1.11). Hence for every fixed $y \in \overline{\Omega}$ there exists a unique solution

(3.5.3)
$$\big(G_*(\cdot,y),\, \underline{\Psi}(\cdot,y), \underline{\mathbb{G}}_*(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega), \quad l < 2m - n/2,$$

of the formally adjoint problem (3.1.10), (3.1.11) with the right-hand sides

$$f(x) = \delta(x-y), \quad \underline{g} = 0, \quad \underline{h} = 0.$$

Due to the Lemma 3.5.1, the functions $G_*(\cdot,\cdot)$, $\underline{\Psi}(\cdot,\cdot)$ and $\underline{\mathbb{G}}_*(\cdot,\cdot)$ are smooth with respect to both variables outside $\operatorname{diag}\overline{\Omega}$ and $\operatorname{diag}\partial\Omega$, respectively. Furthermore, $\underline{\Psi}(x,y) = \mathcal{D}\, G_*(x,y)$ for $x \in \partial\Omega$, $y \in \overline{\Omega}$, $x \neq y$.

LEMMA 3.5.2. *For $y, z \in \overline{\Omega}$, $y \neq z$ we have*

$$G_*(y,z) = \overline{G(z,y)} \tag{3.5.4}$$

Proof: 1) In the case $2m > n$ there exists an integer number l such that $n/2 < l < 2m - n/2$. Inserting

$$u = G(\cdot,y), \ \underline{\phi} = \underline{\Phi}(\cdot,y), \ \underline{u} = \underline{G}(\cdot,y),$$
$$\underline{v} = G_*(\cdot,z), \ \underline{\psi} = \underline{\Psi}(\cdot,z), \ \underline{v} = \underline{\mathbb{G}}_*(\cdot,z)$$

into the Green formula (3.2.21), we get

$$\big(\delta(x-y),\, G_*(x,z)\big)_\Omega = \big(G(x,y),\, \delta(x-z)\big)_\Omega,$$

i.e., (3.5.4) is true for $y \neq z$.

2) Let $2m \leq n$ and let ζ, η be smooth cut-off functions equal to one near y and z, respectively, such that $\operatorname{supp}\zeta \cap \operatorname{supp}\eta = \emptyset$. Since $(1-\zeta)\,\big(G(\cdot,y), \underline{\Phi}(\cdot,y), \underline{G}(\cdot,y)\big)$ and $(1-\eta)\,\big(G_*(\cdot,y), \underline{\Psi}(\cdot,y), \underline{\mathbb{G}}_*(\cdot,y)\big)$ are smooth, the Green formula (3.2.21) can be applied to each of the pairs

$$(u,\underline{\phi},\underline{u}) = \zeta\,(G,\underline{\Phi},\underline{G})(\cdot,y), \qquad (v,\underline{\psi},\underline{v}) = (1-\eta)\,\big(G_*,\underline{\Psi},\underline{\mathbb{G}}_*\big)(\cdot,z),$$
$$(u,\underline{\phi},\underline{u}) = (1-\zeta)\,(G,\underline{\Phi},\underline{G})(\cdot,y), \qquad (v,\underline{\psi},\underline{v}) = \eta\,\big(G_*,\underline{\Psi},\underline{\mathbb{G}}_*\big)(\cdot,z),$$
$$(u,\underline{\phi},\underline{u}) = (1-\zeta)\,\big(G,\underline{\Phi},\underline{G}\big)(\cdot,y), \qquad (v,\underline{\psi},\underline{v}) = (1-\eta)\,\big(G_*,\underline{\Psi},\underline{\mathbb{G}}_*\big)(\cdot,z).$$

Moreover, since $\operatorname{supp}\zeta \cap \operatorname{supp}\eta = \emptyset$, both sides of the equality (3.2.21) vanish for $(u,\underline{\phi},\underline{u}) = \zeta\,\big(G(\cdot,y),\underline{\Phi}(\cdot,y),\underline{G}(\cdot,y)\big)$, and $(v,\underline{\psi},\underline{v}) = \eta\,\big(G_*(\cdot,y),\underline{\Psi}(\cdot,y),\underline{\mathbb{G}}(\cdot,y)\big)$. Consequently, analogously to the first part of the proof, we obtain (3.5.4). ∎

Under our assumptions on the operator $\mathcal{A}$, there exist unique solutions

$$\big(\mathbb{G}_k(\cdot,y), \underline{\Phi}^{(k)}(\cdot,y), \underline{G}^{(k)}(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega), \tag{3.5.5}$$

$l < \mu_k + 1 - n/2$, of the boundary value problem (3.1.1), (3.1.2) with the right-hand sides

$$f(x) = 0, \quad \underline{g}(x) = \delta(x-y)\,\underline{e}_k\,,$$

where y is an arbitrary point on $\partial\Omega$, $k = 1,\dots,m+J$, and $\underline{e}_k$ denotes the k-th unit vector $(0,\dots,0,1,0,\dots,0)$ in $\mathbb{C}^{m+J}$. We call the solutions (3.5.5) *Poisson functions* for the boundary value problem (3.1.1), (3.1.2).

Furthermore, the formally adjoint problem (3.1.10), (3.1.11) with the right-hand sides

$$f(x) = 0, \quad \underline{g}(x) = 0, \quad \underline{h}(x) = \delta(x-y)\,\underline{e}_j\,,$$

where y is an arbitrary point on $\partial\Omega$ and $\underline{e}_j$ is the j-th unit vector in $\mathbb{C}^J$, has a unique solution

$$(3.5.6) \qquad \big(G_*^{(j)}(\cdot,y), \underline{\Psi}^{(j)}(\cdot,y), \underline{\mathcal{G}}_*^{(j)}(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega),$$

$l < 2m+\tau_j - n/2$. Analogously, to Lemma 3.5.1, it can be shown that the solutions (3.5.5) and (3.5.6) are smooth with respect to both variables outside the diagonal.

We denote the components of the vectors $\underline{G}$ in (3.5.1) by $G_j(\cdot,y)$, $j = 1,\dots,J$, and the components of the vector $\underline{\mathbb{G}}_*(\cdot,y)$ in (3.5.3) by $\mathbb{G}_*^{(k)}$. Moreover, let

$$\underline{\mathcal{G}}^{(k)} = \begin{pmatrix} \mathcal{G}_{1,k} \\ \vdots \\ \mathcal{G}_{J,k} \end{pmatrix} \quad \text{and} \quad \underline{\mathcal{G}}_*^{(j)} = \begin{pmatrix} \mathcal{G}_*^{(1,j)} \\ \vdots \\ \mathcal{G}_*^{(m+J,j)} \end{pmatrix}.$$

LEMMA 3.5.3. *There are the following relations between the functions (3.5.1), (3.5.3), (3.5.5) and (3.5.6):*

$$(3.5.7) \qquad G_*^{(j)}(y,z) = \overline{G_j(z,y)} \quad \textit{for } y \in \overline{\Omega},\ z \in \partial\Omega,\ y \neq z,$$

$$(3.5.8) \qquad \mathbb{G}_*^{(k)}(y,z) = \overline{\mathbb{G}_k(z,y)} \quad \textit{for } y \in \partial\Omega,\ z \in \overline{\Omega},\ y \neq z,$$

$$(3.5.9) \qquad \mathcal{G}_*^{(k,j)}(y,z) = \overline{\mathcal{G}_{j,k}(z,y)} \quad \textit{for } y,z \in \partial\Omega,\ y \neq z,$$

$k = 1,\dots,m+J$, $j = 1,\dots,J$.

Proof: The equalities (3.5.7) - (3.5.9) can be proved analogously to (3.5.4). If, for example, we insert

$$(u,\underline{\phi},\underline{u}) = \big(G(\cdot,y), \underline{\Phi}(\cdot,y), \underline{G}(\cdot,y)\big), \quad (v,\underline{\psi},\underline{v}) = \big(G_*^{(j)}(\cdot,z), \underline{\Psi}^{(j)}(\cdot,z), \mathcal{G}_*^{(j)}(\cdot,z)\big)$$

into the Green formula (3.2.21), we obtain (3.5.7). Here, as in the proof of Lemma 3.5.2, one has to use cut-off functions ζ and η equal to one in a neighbourhood of y and z, respectively, such that $\operatorname{supp}\zeta \cap \operatorname{supp}\eta = \emptyset$ if $2m \le n - \tau_j + 1$.

In the same way we obtain (3.5.8) and (3.5.9). ∎

As a consequence of the foregoing Lemma, the following theorem holds.

THEOREM 3.5.1. *Let f and $\underline{g}$ be smooth functions. Then the solution $(u,\underline{u})$ of the boundary value problem (3.1.1), (3.1.2) is given by the formulas*

$$(3.5.10) \qquad u(y) = \int_\Omega G(y,x)\cdot f(x)\,dx + \sum_{k=1}^{m+J} \int_{\partial\Omega} \mathbb{G}_k(y,x)\cdot g_k(x)\,dx,$$

$$(3.5.11) \qquad u_j(y) = \int_\Omega G_j(y,x)\cdot f(x)\,dx + \sum_{k=1}^{m+J} \int_{\partial\Omega} \mathcal{G}_{j,k}(y,x)\cdot g_k(x)\,dx\,.$$

Proof: To obtain (3.5.10), we insert $(v, \underline{\psi}, \underline{v}) = \big(G_*(\cdot,y), \underline{\Psi}(\cdot,y), \underline{\mathbb{G}}_*(\cdot,y)\big)$ into the Green formula (3.2.21). If we insert $(v, \underline{\psi}, \underline{v}) = \big(G_*^{(j)}(\cdot,y), \underline{\Psi}^{(j)}(\cdot,y), \underline{\mathcal{G}}_*^{(j)}(\cdot,y)\big)$ into (3.2.21), we get (3.5.11). ∎

3.5.2. Representation of the solution in the general case. Now we introduce the Green function for the boundary value problem (3.1.1), (3.1.2) without the assumption that this problem is uniquely solvable. We suppose only that the orders of the operators B_k are less than $2m$ and problem (3.1.1), (3.1.2) is elliptic.

Then by Theorem 3.4.1, problem (3.1.1), (3.1.2) is solvable if and only if $(f, \underline{g})$ is orthogonal to $\mathcal{N}^*$ with respect to the scalar product

$$\langle (f,\underline{g})\,,\,(v,\underline{v})\rangle_* = (f,v)_\Omega + (\underline{g},\underline{v})_{\partial\Omega}\,.$$

Here $\mathcal{N}^*$ is the set of all solutions $(v,\underline{v}) \in C^\infty(\overline{\Omega})\times C^\infty(\partial\Omega)^{m+J}$ of the homogeneous formally adjoint problem (3.1.10), (3.1.11).

Let $\{(v^{(1)},\underline{v}^{(1)}), \dots, (v^{(d)},\underline{v}^{(d)})\}$ be an orthonormal (with respect to the scalar product $\langle\cdot,\cdot\rangle_*$) basis of $\mathcal{N}^*$. Then $\sum\limits_{s=1}^{d} \overline{v^{(s)}(y)}\,\big(v^{(s)},\underline{v}^{(s)}\big)$ is the orthogonal projection of $(\delta(\cdot,y),0)$ onto $\mathcal{N}^*$. Consequently, for every $y \in \Omega$ there exists a solution $(u,\underline{\phi},\underline{u}) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$, $l < 2m - n/2$, of problem (3.1.1), (3.1.2) with the right-hand sides

$$f(x) = \delta(x-y) - \sum_{s=1}^{d} \overline{v^{(s)}(y)}\, v^{(s)}(x), \quad \underline{g}(x) = -\sum_{s=1}^{d} \overline{v^{(s)}(y)}\, \underline{v}^{(s)}(x)\,. \tag{3.5.12}$$

This solution is not uniquely determined if the kernel $\mathcal{N}$ of the operator (3.4.1) is not trivial. Let

$$\big(G(\cdot,y),\,\underline{\Phi}(\cdot,y),\underline{G}(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega), \quad l < 2m - n/2,$$

be the unique solution of problem (3.1.1), (3.1.2) with the right-hand sides (3.5.12) which is orthogonal to $\mathcal{N}$, i.e., satisfies the condition

$$\big(G(\cdot,y),\,u\big)_\Omega + \big(\underline{\Phi}(\cdot,y),\,\underline{\phi}\big)_{\partial\Omega} + \big(\underline{G}(\cdot,y),\,\underline{u}\big)_{\partial\Omega} = 0$$

for all $(u,\underline{\phi},\underline{u}) \in \mathcal{N}$. This solution is called *generalized Green function* of the boundary value problem (3.1.1), (3.1.2).

Let $v_k^{(s)}$ denote the k-th component of the vector-function $\underline{v}^{(s)}$. It can be easily seen that $\sum\limits_{s=1}^{d} \overline{v_k^{(s)}(y)}\,\big(v^{(s)},\underline{v}^{(s)}\big)$ is the orthogonal projection of $(f,\underline{g}) = (0, \delta(\cdot - y)\,\underline{e}_k)$ onto $\mathcal{N}^*$. Hence for every $y \in \partial\Omega$, $k = 1, \dots, m+J$ there exists a unique solution

$$\big(\mathbb{G}_k(\cdot,y),\,\underline{\Phi}^{(k)}(\cdot,y),\underline{\mathcal{G}}^{(k)}(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega), \quad l < \mu_k + 1 - n/2,$$

of problem (3.1.1), (3.1.2) with the right-hand sides

$$f(x) = -\sum_{s=1}^{d} \overline{v_k^{(s)}(y)}\, v^{(s)}(x), \quad \underline{g}(x) = \delta(x-y)\,\underline{e}_k - \sum_{s=1}^{d} \overline{v_k^{(s)}(y)}\, \underline{v}^{(s)}(x)$$

which is orthogonal to $\mathcal{N}$. This solution is called *generalized Poisson function* for the boundary value problem (3.1.1), (3.1.2).

Analogously to Lemma 3.5.1, the smoothness of $G(\cdot,\cdot)$, $\underline{\Phi}(\cdot,\cdot)$, $\underline{G}(\cdot,\cdot)$, $\mathbb{G}^{(k)}(\cdot,\cdot)$, $\underline{\Phi}^{(k)}(\cdot,\cdot)$, and $\underline{\mathcal{G}}^{(k)}(\cdot,\cdot)$ with respect to both variables outside $\operatorname{diag}\overline{\Omega}$ and $\operatorname{diag}\partial\Omega$, respectively, holds.

In the same way as above, we introduce the generalized Green and Poisson functions for the formally adjoint problem (3.1.10), (3.1.11).

As in Section 3.4, we denote the kernel of the operator $\mathcal{A}^+$ by $\mathcal{N}^+$, i.e., $\mathcal{N}^+ = \{(v,\underline{\psi},\underline{v}) \,:\, (v,\underline{v}) \in \mathcal{N}^*,\ \underline{\psi} = \mathcal{D}v|_{\partial\Omega}\}$. Obviously,

$$\langle (v^{(1)},\underline{\psi}^{(1)},\underline{v}^{(1)})\,,\,(v^{(2)},\underline{\psi}^{(2)},\underline{v}^{(2)})\rangle_+ = (v^{(1)},v^{(2)})_\Omega + (\underline{v}^{(1)},\underline{v}^{(2)})_{\partial\Omega}$$

is a scalar product in $\mathcal{N}^+$.

Let $\{(u^{(s)},\underline{\phi}^{(s)},\underline{u}^{(s)})\}_{1\le s\le d'}$ be a basis in the kernel $\mathcal{N}$ of the operator $\mathcal{A}$ such that

$$(u^{(j)},u^{(s)})_\Omega + (\mathcal{D}u^{(j)},\mathcal{D}u^{(s)})_{\partial\Omega} + (\underline{u}^{(j)},\underline{u}^{(s)})_{\partial\Omega} = \delta_{j,s} \quad \text{for } j,s = 1,\dots,d'.$$

Then

$$\sum_{s=1}^{d'} \overline{u^{(s)}(y)}\,\big(u^{(s)},\underline{\phi}^{(s)},\underline{u}^{(s)}\big)$$

is the orthogonal projection of $(\delta(\cdot-y),0,0)$ onto $\mathcal{N}$ with respect to the scalar product

$$\langle (f,\underline{g},\underline{h})\,,\,(u,\underline{\phi},\underline{u})\rangle = (f,u)_\Omega + (\underline{g},\underline{\phi})_{\partial\Omega} + (\underline{h},\underline{u})_{\partial\Omega}\,.$$

Consequently, by Corollary 3.4.1, there exists a unique solution

$$\big(G_*(\cdot,y),\underline{\Psi}(\cdot,y),\underline{\mathbb{G}}_*(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega)\times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega), \quad l < 2m - n/2,$$

of the formally adjoint problem (3.1.10), (3.1.11) with the right-hand sides

$$f(x) = \delta(x-y) - \sum_{s=1}^{d'} \overline{u^{(s)}(y)}u^{(s)}(x)\,, \quad \underline{g}(x) = -\sum_{s=1}^{d'}\overline{u^{(s)}(y)}\underline{\phi}^{(s)}(x)\,,$$

$$\underline{h}(x) = -\sum_{s=1}^{d'}\overline{u^{(s)}(y)}\underline{u}^{(s)}(x)$$

which is orthogonal (with respect to the scalar product $\langle\cdot,\cdot\rangle_+$) to $\mathcal{N}^+$.

Analogously, for every $y\in\partial\Omega$, $j=1,\dots,J$ there exists a unique solution

$$\big(G_*^{(j)}(\cdot,y),\underline{\Psi}^{(j)}(\cdot,y),\underline{\mathcal{G}}_*^{(j)}(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega)\times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega)$$

$(l < 2m + \tau_j - n/2)$ of the boundary value problem (3.1.10), (3.1.11) with the right-hand sides

$$f(x) = -\sum_{s=1}^{d'}\overline{u_j^{(s)}(y)}\,u^{(s)}(x)\,, \quad \underline{g}(x) = -\sum_{s=1}^{d'}\overline{u_j^{(s)}(y)}\,\underline{\phi}^{(s)}(x)\,,$$

$$\underline{h}(x) = \delta(x-y)\,\underline{e}_j - \sum_{s=1}^{d'}\overline{u_j^{(s)}(y)}\,\underline{u}^{(s)}(x)$$

which is orthogonal to $\mathcal{N}^+$.

LEMMA 3.5.4. *There are the following relations between the Green and Poisson functions introduced above:*

$$G_*(y,z) = \overline{G(z,y)} \quad \textit{for } y,z\in\overline{\Omega},\ y\neq z, \tag{3.5.13}$$

$$G_*^{(j)}(y,z) = \overline{G_j(z,y)} \quad \textit{for } y\in\overline{\Omega},\ z\in\partial\Omega,\ y\neq z, \tag{3.5.14}$$

$$\mathbb{G}_*^{(k)}(y,z) = \overline{\mathbb{G}_k(z,y)} \quad \textit{for } y\in\partial\Omega,\ z\in\overline{\Omega},\ y\neq z, \tag{3.5.15}$$

$$\mathcal{G}_*^{(k,j)}(y,z) = \overline{\mathcal{G}_{j,k}(z,y)} \quad \textit{for } y,z\in\partial\Omega,\ y\neq z, \tag{3.5.16}$$

$j=1,\dots,J$, $k=1,\dots,m+J$.

Proof: The proof proceeds analogously to Lemmas 3.5.2, 3.5.3. If we insert

$$(u,\underline{\phi},\underline{u}) = \big(G(\cdot,y),\,\underline{\Phi}(\cdot,y),\underline{G}(\cdot,y)\big) \quad\text{and}\quad (v,\underline{\psi},\underline{v}) = \big(G_*(\cdot,z),\underline{\Psi}(\cdot,z),\underline{\mathbb{G}}_*(\cdot,z)\big)$$

into the Green formula (3.2.21), we obtain

$$\overline{G_*(y,z)} - \sum_{s=1}^{d} \overline{v^{(s)}(y)}\,\Big(\big(v^{(s)},G_*(\cdot,z)\big)_\Omega + \big(\underline{v}^{(s)},\underline{\mathbb{G}}_*(\cdot,z)\big)_{\partial\Omega}\Big)$$

$$= G(z,y) - \sum_{s=1}^{d'} u^{(s)}(y)\,\Big(\big(G(\cdot,y),u^{(s)}\big)_\Omega + \big(\underline{\Phi}(\cdot,y),\underline{\phi}^{(s)}\big)_{\partial\Omega} + \big(\underline{G}(\cdot,y),\underline{u}^{(s)}\big)_{\partial\Omega}\Big).$$

Since $\big(G(\cdot,y),\,\underline{\Phi}(\cdot,y),\underline{G}(\cdot,y)\big)$ is orthogonal to $\mathcal{N}$ and $\big(G_*(\cdot,z),\underline{\Psi}(\cdot,z),\underline{\mathbb{G}}_*(\cdot,z)\big)$ is orthogonal to $\mathcal{N}^+$, this implies (3.5.13). In the same way, the relations (3.5.14) - (3.5.16) hold. ∎

As a consequence of Lemma 3.5.4, we obtain the following representations for the solutions of problem (3.1.1), (3.1.2).

THEOREM 3.5.2. *Let f and $\underline{g}$ be smooth functions satisfying the condition (3.4.3). Then every solution $(u,\underline{u})$ of the boundary value problem (3.1.1), (3.1.2) is given by the formulas*

$$u(y) = \int_\Omega G(y,x)\cdot f(x)\,dx + \sum_{k=1}^{m+J}\int_{\partial\Omega} \mathbb{G}_k(y,x)\cdot g_k(x)\,dx + \sum_{s=1}^{d'} c_s\,u^{(s)}(y)\,, \tag{3.5.17}$$

(3.5.18)

$$u_j(y) = \int_\Omega G_j(y,x)\cdot f(x)\,dx + \sum_{k=1}^{m+J}\int_{\partial\Omega} \mathcal{G}_{j,k}(y,x)\cdot g_k(x)\,dx + \sum_{s=1}^{d'} c_s\,u_j^{(s)}(y)\,,$$

where $c_s = (u,u^{(s)})_\Omega + \big(\mathcal{D}u|_{\partial\Omega}\,,\underline{\phi}^{(s)}\big)_{\partial\Omega} + (\underline{u},\underline{u}^{(s)})_{\partial\Omega}\,.$

Proof: Formulas (3.5.17), (3.5.18) follow from the Green formula (3.2.21) setting $(v,\underline{\psi},\underline{v}) = \big(G_*(\cdot,y),\underline{\Psi}(\cdot,y),\underline{\mathbb{G}}_*(\cdot,y)\big)$, $(v,\underline{\psi},\underline{v}) = \big(G_*^{(j)}(\cdot,y),\underline{\Psi}^{(j)}(\cdot,y),\underline{\mathcal{G}}_*^{(j)}(\cdot,y)\big)$. ∎

3.5.3. The Green function for classical boundary value problems. Now we consider the classical boundary value problem

$$Lu = f \quad\text{in } \Omega, \qquad Bu = \underline{g} \quad\text{on } \partial\Omega, \tag{3.5.19}$$

where B is a vector of differential operators $B_1,\dots,B_m$, $\operatorname{ord} B_k = \mu_k \le 2m-1$, which form a normal system on $\partial\Omega$. Let $\mathcal{B}$ be a vector of differential operators

$B_{m+1}, \dots, B_{2m}$, $\operatorname{ord} B_{m+k} = \mu_{m+k}$, such that the operators $B_1, \dots, B_{2m}$ form a Dirichlet system of order $2m$ on $\partial\Omega$. Then there exist $2m \times m$-matrices Λ_1, Λ_2 of tangential differential operators on $\partial\Omega$ such that

$$\mathcal{D}u = \Lambda_1 \, Bu + \Lambda_2 \, \mathcal{B}u \qquad \text{on } \partial\Omega \tag{3.5.20}$$

for $u \in W_2^{2m}(\Omega)$ (cf. (3.1.15)). With the notation $B' = \Lambda_2^+ \, P$ and $\mathcal{B}' = -\Lambda_1^+ \, P$ we get the classical Green formula (see (3.1.17))

$$(Lu, v)_\Omega + (Bu, \mathcal{B}'v)_{\partial\Omega} = (u, L^+v)_\Omega + (\mathcal{B}u, B'v)_{\partial\Omega}$$

which is valid for arbitrary $u, v \in W_2^{2m}(\Omega)$. According to Lemma 3.2.2, Theorem 3.2.1, the operators in this formula can be continuously extended to the space $\tilde{W}_2^{l,2m}(\Omega)$ with $l < 2m$. Then analogously to Theorem 3.2.2, the following assertion holds.

LEMMA 3.5.5. *The Green formula*

$$\big(L(u,\underline{\phi}), v\big)_\Omega + \big(B(u,\underline{\phi}), \mathcal{B}'(v,\underline{\psi})\big)_{\partial\Omega} = \big(u, L^+(v,\underline{\psi})\big)_\Omega + \big(\mathcal{B}(u,\underline{\phi}), B'(v,\underline{\psi})\big)_{\partial\Omega} \tag{3.5.21}$$

is valid for all $(u,\underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$, $(v,\underline{\psi}) \in \tilde{W}_2^{2m-l,2m}(\Omega)$, *where* l *is an arbitrary integer number.*

Proof: By (3.2.22), we have

$$\big(L(u,\underline{\phi}), v\big)_\Omega = \big(u, L^+(v,\underline{\psi})\big)_\Omega + \big(\underline{\phi}, \, P(v,\underline{\psi})\big)_{\partial\Omega}. \tag{3.5.22}$$

Furthermore, from (3.5.20) it follows that $\Lambda_1 \, B(u,\underline{\phi}) + \Lambda_2 \, \mathcal{B}(u,\underline{\phi}) = \underline{\phi}$ for all $(u,\underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$. Consequently,

$$\begin{aligned} &\big(B(u,\underline{\phi}), \mathcal{B}'(v,\underline{\psi})\big)_{\partial\Omega} - \big(\mathcal{B}(u,\underline{\phi}), B'(v,\underline{\psi})\big)_{\partial\Omega} \\ &= -\big(B(u,\underline{\phi}), \, \Lambda_1^+ \, P(v,\underline{\psi})\big)_{\partial\Omega} - \big(\mathcal{B}(u,\underline{\phi}), \, \Lambda_2^+ \, P(v,\underline{\psi})\big)_{\partial\Omega} = -\big(\underline{\phi}, \, P(v,\underline{\psi})\big)_{\partial\Omega}. \end{aligned}$$

This together with (3.5.20) implies (3.5.21). ■

The boundary value problem

$$L^+v = f \ \text{ in } \Omega, \quad B'v = \underline{g} \ \text{ on } \partial\Omega \tag{3.5.23}$$

is formally adjoint to problem (3.5.19) with respect to the classical Green formula given above, while the boundary value problem

$$L^+v = 0 \ \text{ in } \Omega, \quad Pv + Q^+\underline{v} = 0 \ \text{ on } \partial\Omega. \tag{3.5.24}$$

is formally adjoint to problem (3.5.19) with respect to the Green formula (3.1.9).

We suppose that the boundary value problem (3.5.19) is elliptic. As in the foregoing subsection, let $\{(v^{(s)}, \underline{v}^{(s)})\}_{1 \le s \le d}$ be an orthonormal (with respect to the scalar product $(\langle \cdot, \cdot \rangle_*)$ basis in the set $\mathcal{N}^*$ of the solutions $(v, \underline{v}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^m$ of the homogeneous formally adjoint problem (3.5.24). Then by Lemma 3.1.1, the vector-function $\underline{v}^{(s)}$ is equal to $\mathcal{B}'v^{(s)}|_{\partial\Omega}$ for $s = 1, \dots, d$ and $\{v^{(s)}\}_{1 \le s \le d}$ is a basis in the set $\mathcal{N}_0^+$ of the solutions $v \in C^\infty(\overline{\Omega})$ of the homogeneous boundary value problem (3.5.23) satisfying the condition

$$\big(v^{(j)}, v^{(s)}\big)_\Omega + \big(\mathcal{B}'v^{(j)}, \, \mathcal{B}'v^{(s)}\big)_{\partial\Omega} = \delta_{j,s} \qquad \text{for } j, s = 1, \dots, d.$$

In the previous subsection we showed that problem (3.5.19) with the right-hand sides

$$(3.5.25)\qquad \begin{cases} f(x) = \delta(x-y) - \sum_{s=1}^{d} \overline{v^{(s)}(y)}\, v^{(s)}(x), \\ \underline{g}(x) = -\sum_{s=1}^{d} \overline{v^{(s)}(y)}\, \mathcal{B}' v^{(s)}(x), \quad y \in \overline{\Omega}, \end{cases}$$

is solvable in $\tilde{W}_2^{l,2m}(\Omega)$, $l < 2m - n/2$. Now, in contrast to the previous subsection, we define $\big(G(\cdot,y), \underline{\Phi}(\cdot,y)\big)$ as the (uniquely determined) solution of problem (3.5.19) with the right-hand sides (3.5.25) satisfying the condition

$$\big(G(\cdot,y), u\big)_\Omega + \big(\mathcal{B}(G(\cdot,y), \underline{\Phi}(\cdot,y)),\, \mathcal{B}u\big)_{\partial\Omega} = 0 \quad \text{for all } u \in \mathcal{N}_0\,,$$

where $\mathcal{N}_0 = \{u \in C^\infty(\overline{\Omega}) \,:\, Lu = 0,\, Bu|_{\partial\Omega} = 0\}$. The function $G(\cdot,\cdot)$ is called *Green function* for the boundary value problem (3.5.19).

Analogously, if $\{u^{(s)}\}_{s=1,\dots,d'}$ is a basis in $\mathcal{N}_0$ satisfying the condition

$$\big(u^{(j)}, u^{(s)}\big)_\Omega + \big(\mathcal{B}u^{(j)},\, \mathcal{B}u^{(s)}\big)_{\partial\Omega} = \delta_{j,s} \quad \text{for } j,s = 1,\dots,d',$$

then there exists a uniquely determined solution $\big(G_*(\cdot,y), \underline{\Psi}(\cdot,y)\big) \in \tilde{W}_2^{l,2m}(\Omega)$, $l < 2m - n/2$, of problem (3.5.23) with the right-hand sides

$$f(x) = \delta(x-y) - \sum_{s=1}^{d'} \overline{u^{(s)}(y)}\, u^{(s)}(x), \quad \underline{g}(x) = -\sum_{s=1}^{d} \overline{u^{(s)}(y)}\, \mathcal{B}u^{(s)}(x),\ y \in \overline{\Omega},$$

satisfying the condition

$$\big(G_*(\cdot,y), v\big)_\Omega + \big(\mathcal{B}'(G_*(\cdot,y), \underline{\Psi}(\cdot,y)),\, \mathcal{B}'v\big)_{\partial\Omega} = 0 \quad \text{for all } v \in \mathcal{N}_0^+\,.$$

LEMMA 3.5.6. *The functions $G(\cdot,\cdot)$ and $G_*(\cdot,\cdot)$ are infinitely differentiable in $\overline{\Omega} \times \overline{\Omega}\backslash \operatorname{diag}\overline{\Omega}$. Furthermore, $\underline{\Phi}(x,y) = \mathcal{D}\,G(x,y)$, $\underline{\Psi}(x,y) = G_*(x,y)$ for $x \in \partial\Omega$, $y \in \overline{\Omega}$, $x \neq y$ and*

$$(3.5.26)\qquad G_*(y,z) = \overline{G(z,y)} \quad \textit{for } x,y \in \overline{\Omega}, x \neq y.$$

Proof: The smoothness of G and G_* can be proved in the same way as in Lemma 3.5.1, while (3.5.26) follows from (3.5.21) setting $(u,\underline{\phi}) = \big(G(\cdot,y), \underline{\Phi}(\cdot,y)\big)$ and $(v,\underline{\psi}) = \big(G_*(\cdot,z), \underline{\Psi}(\cdot,z)\big)$. Here in the case $2m \le n$ one has to use smooth cut-off functions ζ and η having the same properties as in the proof of Lemma 3.5.2. ■

We denote the components of the vector B' by $B'_1,\dots,B'_m$ and the components of the vector $\mathcal{B}'$ by $B'_{m+1},\dots,B'_{m+k}$. Inserting $(v,\underline{\psi}) = \big(G_*(\cdot,y), \underline{\Psi}(\cdot,y)\big)$ into (3.5.21) and using (3.5.26), we obtain the following theorem.

THEOREM 3.5.3. *Let f and $\underline{g}$ be smooth functions satisfying condition (3.4.8). Then every solution u of the boundary value problem (3.5.19) is given by the formula*

$$(3.5.27)\qquad u(y) = \int_\Omega G(y,x)\cdot f(x)\,dx + \sum_{k=1}^{m+J} \int_{\partial\Omega} \overline{B'_{m+k}(x,D_x)G(y,x)}\cdot g_k(x)\,dx + \sum_{s=1}^{d'} c_s\, u^{(s)}(y)\,,$$

where $c_s = (u, u^{(s)})_\Omega + \big(\mathcal{B}u\,, \mathcal{B}\,u^{(s)}\big)_{\partial\Omega}$.

3.6. Elliptic boundary value problems with parameter

This section is dedicated to elliptic boundary value problems polynomially depending on a complex parameter λ. Such problems arise, e.g., from model problems in an infinite cylinder or an infinite cone (see Sections 5.2, 6.1) applying the Fourier and the Mellin transformation, respectively. The main result in this section is the unique solvability of such problems for complex λ with sufficiently great modulus lying near the imaginary axis. Furthermore, we prove an a priori estimate for the solutions containing the parameter λ.

3.6.1. Ellipticity with parameter. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We consider the boundary value problem

$$L(\lambda)\, u = f \qquad \text{in } \Omega, \tag{3.6.1}$$

$$B(\lambda)\, u + C(\lambda)\, \underline{u} = \underline{g} \quad \text{on } \partial\Omega, \tag{3.6.2}$$

where

$$L(\lambda) = L(x, D_x, \lambda) = \sum_{|\alpha|+j\le 2m} a_{\alpha,j}(x)\, D_x^\alpha\, \lambda^j \tag{3.6.3}$$

is a differential operator of order $2m$ with coefficients $a_{\alpha,j} \in C^\infty(\overline{\Omega})$, $B(\lambda)$ is a vector of differential operators

$$B_k(x, D_x, \lambda) = \sum_{|\alpha|+j\le \mu_k} b_{k;\alpha,j}(x)\, D_x^\alpha\, \lambda^j\,, \quad k = 1, \dots, m+J, \tag{3.6.4}$$

with coefficients $b_{k;\alpha,j} \in C^\infty(\overline{\Omega})$, and $C(\lambda) = C(x, D_x, \lambda)$ is a matrix of tangential differential operators

(3.6.5)
$$C_{k,j}(x, D_x, \lambda) = \sum_{|\alpha|+s\le \mu_k+\tau_j} c_{k,j;\alpha,s}(x)\, D_x^\alpha\, \lambda^j\,, \quad k = 1, \dots, m+J,\ j = 1, \dots, J,$$

with smooth coefficients.

DEFINITION 3.6.1. The problem (3.6.1), (3.6.2) is said to be *elliptic with parameter* if the boundary value problem

$$L(x, D_x, iD_t)\, u(x,t) = f(x,t), \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{3.6.6}$$

$$B(x, D_x, iD_t)\, u(x,t) + C(x, D_x, iD_t)\, \underline{u}(x,t) = \underline{g}(x,t), \quad x \in \partial\Omega,\ t \in \mathbb{R} \tag{3.6.7}$$

is elliptic (see Definition 3.1.2) in the infinite cylinder $\mathcal{C} = \Omega \times \mathbb{R}$.

REMARK 3.6.1. If the problem (3.6.1), (3.6.2) is elliptic with parameter, then this problem is elliptic for every fixed λ.

Indeed, it is evident that the ellipticity of the operator $L(x, D_x, iD_t)$ implies the ellipticity of the operator $L(x, D_x, \lambda)$ for every fixed λ. Furthermore, condition (ii) in Definition 3.1.2 is satisfied for the boundary value problem (3.6.1), (3.6.2) if it is satisfied for the problem (3.6.6), (3.6.7).

3.6.2. The Green formula for parameter-depending operators. Suppose that the orders of the differential operators $B_k(x, D_x, \lambda)$ are less than $2m$. Then the vector $B(x, D_x, \lambda)u$ admits the representation

$$B(x, D_x, \lambda)u\Big|_{\partial\Omega} = Q(x, D_x, \lambda) \cdot \mathcal{D}u\Big|_{\partial\Omega}, \tag{3.6.8}$$

where $Q(x, D_x, \lambda)$ is a matrix of tangential differential operators $Q_{k,j}(x, D_x, \lambda)$, $k = 1, \dots, m+J$, $j = 1, \dots, J$, polynomially depending on λ, $\operatorname{ord} Q_{k,j} \le \mu_k + 1 - j$, $Q_{k,j} \equiv 0$ if $j > \mu_k + 1$.

Let $L^+(\lambda) = L^+(x, D_x, \lambda)$ be the formally adjoint operator to $L(x, D_x, \overline{\lambda})$, i.e.,

$$L^+(x, D_x, \lambda)\, u = \sum_{|\alpha|+j\le 2m} D_x^\alpha(\overline{a_{\alpha,j}(x)}\, u) \cdot \lambda^j\, .$$

Analogously, let $C^+(\lambda) = C^+(x, D_x, \lambda)$ and $Q^+(\lambda) = Q^+(x, D_x, \lambda)$ be the formally adjoint operators to $C(x, D_x, \overline{\lambda})$ and $Q(x, D_x, \overline{\lambda})$, respectively. Then by Theorem 3.1.1, the following Green formula is satisfied for all functions $\hat{u}, \hat{v} \in C^\infty(\overline{\Omega})$ and all vector-functions $\underline{\hat{u}} \in C^\infty(\partial\Omega)^J$, $\underline{\hat{v}} \in C^\infty(\partial\Omega)^{m+J}$:

$$\int_\Omega L(\lambda)\hat{u} \cdot \bar{\hat{v}}\, dx + \int_{\partial\Omega} \big(B(\lambda)\hat{u} + C(\lambda)\underline{\hat{u}},\, \underline{\hat{v}}\big)_{\mathbb{C}^{m+J}}\, d\sigma \tag{3.6.9}$$
$$= \int_\Omega \hat{u} \cdot \overline{L^+(\overline{\lambda})\hat{v}}\, dx + \int_{\partial\Omega} \big(\mathcal{D}\hat{u},\, P(\overline{\lambda})\hat{v} + Q^+(\overline{\lambda})\underline{\hat{v}}\big)_{\mathbb{C}^{2m}}\, d\sigma + \int_{\partial\Omega} \big(\underline{\hat{u}}, C^+(\overline{\lambda})\underline{\hat{v}}\big)_{\mathbb{C}^J}\, d\sigma,$$

where $P(\lambda) = P(x, D_x, \lambda)$ is a vector of differential operators

$$P_j(x, D_x, \lambda) = \sum_{|\alpha|+s\le 2m-j} p_{j;\alpha,s}(x)\, D_x^\alpha\, \lambda^s \tag{3.6.10}$$

with coefficients $p_{j;\alpha,s} \in C^\infty(\overline{\Omega})$.

Now let u, v be arbitrary functions from $C_0^\infty(\overline{\mathcal{C}})$ on the cylinder $\mathcal{C} = \Omega \times \mathbb{R}$, and let $\underline{u}, \underline{v}$ be arbitrary vector-functions from $C_0^\infty(\partial\mathcal{C})^J$ and $C_0^\infty(\partial\mathcal{C})^{m+J}$, respectively. By $\hat{u}$, $\hat{v}$, $\underline{\hat{u}}$, $\underline{\hat{v}}$ we denote the Fourier transforms of u, v, $\underline{u}$, $\underline{v}$ with respect to the last variable t (see (1.2.2)). Setting $\lambda = i\tau$ in (3.6.9) and integrating (3.6.9) relative to τ over the real axis, then by means of Parseval's equality, it holds

$$\int_{\mathcal{C}} L(iD_t)u \cdot \bar{v}\, dx\, dt + \int_{\partial\mathcal{C}} \big(B(iD_t)u + C(iD_t)\underline{u},\, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma\, dt \tag{3.6.11}$$
$$= \int_{\mathcal{C}} u \cdot \overline{L^+(-iD_t)v}\, dx\, dt + \int_{\partial\mathcal{C}} \big(\mathcal{D}u,\, P(-iD_t)v + Q^+(-iD_t)\underline{v}\big)_{\mathbb{C}^{2m}}\, d\sigma\, dt$$
$$+ \int_{\partial\mathcal{C}} \big(\underline{u}, C^+(-iD_t)\underline{v}\big)_{\mathbb{C}^J}\, d\sigma\, dt.$$

Hence the boundary value problem

$$L^+(-iD_t)\, v = f \qquad \text{in } \mathcal{C},$$
$$P(-iD_t)\, v + Q^+(-iD_t)\, \underline{v} = \underline{g}, \quad C^+(-iD_t)\, \underline{v} = \underline{h} \quad \text{on } \partial\mathcal{C}$$

is formally adjoint to the problem (3.6.6), (3.6.7), and as a consequence of Theorem 3.1.2, we get the following assertion.

LEMMA 3.6.1. *The boundary value problem*

$$L^+(-\lambda)\, v = f \qquad \text{in } \Omega, \tag{3.6.12}$$

$$P(-\lambda)\, v + Q^+(-\lambda)\, \underline{v} = \underline{g}, \quad C^+(-\lambda)\, \underline{v} = \underline{h} \quad \text{on } \partial\Omega \tag{3.6.13}$$

is elliptic with parameter if and only if the problem (3.6.1), (3.6.2) is elliptic with parameter.

3.6.3. An estimate for the solutions of parameter-depending elliptic boundary value problems. From Theorem 3.2.3 it follows that the solution of the boundary value problem (3.6.1), (3.6.2) satisfies an estimate of the form (3.2.27) for every fixed λ. However, the constant c in this estimate depends on the parameter λ. Using a local estimate for solutions of the elliptic boundary value problem (3.6.6), (3.6.7) in the cylinder $\mathcal{C} = \Omega \times \mathbb{R}$, we prove an estimate for solutions of the corresponding parameter-depending boundary value problem (3.6.1), (3.6.2), where the parameter λ appears explicitly. Beforehand, we prove the following two lemmas.

LEMMA 3.6.2. *Let ζ be a smooth real-valued function on $\mathbb{R}^1$ with compact support and let c be an arbitrary real constant. Then*

$$\int\limits_{-\infty}^{+\infty} (1+c\tau)^l \big|(\mathcal{F}_{t\to\tau}\zeta)(\tau)\big|^2\, d\tau = \sum_{j=0}^{[l/2]} \binom{l}{2j} c^{2j}\, \|D_t^j\zeta\|^2_{L_2(\mathbb{R})}$$

for each integer $l \ge 0$. Here $\mathcal{F}_{t\to\tau}$ denotes the Fourier transformation (1.2.2) and $[l/2]$ is integral part of $l/2$.

Proof: By the Parseval equality, we have

$$\int\limits_{-\infty}^{+\infty} (1+c\tau)^l\, (\mathcal{F}_{t\to\tau}\zeta)(\tau)\cdot\overline{(\mathcal{F}_{t\to\tau}\zeta)(\tau)}\, d\tau \;=\; \int\limits_{-\infty}^{+\infty} (1+cD_t)^l\, \zeta(t)\cdot\overline{\zeta(t)}\, dt$$

$$= \sum_{j=0}^{l} \binom{l}{j} \int\limits_{-\infty}^{+\infty} \big((cD_t)^j\zeta(t)\big)\cdot\zeta(t)\, dt.$$

Integrating by parts, we get

$$\int\limits_{-\infty}^{+\infty} \big(D_t^j\zeta(t)\big)\cdot\zeta(t)\, dt = \int\limits_{-\infty}^{+\infty} |D_t^{j/2}\zeta(t)|^2\, dt$$

if j is an even number, and

$$\int\limits_{-\infty}^{+\infty} \big(D_t^j\zeta(t)\big)\cdot\zeta(t)\, dt = \frac{1}{2}\int\limits_{-\infty}^{+\infty} D_t\, \big(D_t^{(j-1)/2}\,\zeta(t)\big)^2\, dt = 0$$

if j is an odd number. This proves the lemma. ■

LEMMA 3.6.3. *Let ζ be a smooth real-valued function on $\mathbb{R}^1$ with compact support, $\zeta \not\equiv 0$, and let $\lambda = is$ be an arbitrary purely imaginary number.*

1) *There exist constants c_1, c_2 independent of u and λ such that for each $u \in W_2^l(\Omega)$, $l \geq 0$, the following inequalities are satisfied:*

(3.6.14)

$$c_1 \|v\|^2_{W_2^l(C)} \leq \|u\|^2_{W_2^l(\Omega)} + |\lambda|^{2l} \|u\|^2_{L_2(\Omega)} \leq \sum_{j=0}^{l} |\lambda|^{2j} \|u\|^2_{W_2^{l-j}(\Omega)} \leq c_2 \|v\|^2_{W_2^l(C)},$$

where the function v on C is defined as

$$v(x,t) = e^{\lambda t} \zeta(t) u(x). \tag{3.6.15}$$

2) *Analogously, for each $u \in W_2^{l-1/2}(\partial\Omega)$, $l \geq 1$, and v given by (3.6.15) the inequalities*

$$\begin{aligned} c_1 \|v\|^2_{W_2^{l-1/2}(\partial C)} &\leq \|u\|^2_{W_2^{l-1/2}(\partial\Omega)} + |\lambda|^{2l-1} \|u\|^2_{L_2(\partial\Omega)} \\ &\leq \sum_{j=0}^{l-1} |\lambda|^{2j} \|u\|^2_{W_2^{l-j-1/2}(\partial\Omega)} + |\lambda|^{2l-1} \|u\|^2_{L_2(\partial\Omega)} \leq c_2 \|v\|^2_{W_2^{l-1/2}(\partial C)} \end{aligned} \tag{3.6.16}$$

with constants c_1, c_2 independent of u are satisfied.

Proof: The first assertion can be easily proved estimating the W_2^l-norm (3.2.1) of the function v. Therefore, we only check the more complicated inequality (3.6.16).

First let $u \in W_2^{l-1/2}(\partial\Omega)$ be a function with support in a sufficiently small neighbourhood $\mathcal{U}$ of any point $x_0 \in \partial\Omega$. For the sake of simplicity, we assume that $\mathcal{U}$ is a subset of the hyper-plane $x_n = 0$ which can be identified with $\mathbb{R}^{n-1}$. (Otherwise, we make use of a diffeomorphism $x' = \kappa(x)$ which transforms $\mathcal{U}$ into a subset of the hyper-plane $x'_n = 0$.) Then the function u can be extended to a 2π-periodic function, and the Sobolev norms of u in (3.6.16) can be replaced by the norms in the spaces $W_{2,per}^{l-j-1/2}(\mathbb{R}^{n-1})$, while the norm of v can be replaced by the norm

(3.6.17)

$$\|v\|_{W_{2,per}^{l-1/2}(\mathbb{R}^{n-1}\times\mathbb{R})} = \Big(\sum_{q\in\mathbb{Z}^{n-1}} \langle q\rangle^{2l-2} \int_{\mathbb{R}} (1+\tau^2)^{l-1/2} |(\mathcal{F}_{t\to\tau} V)(q,\tau)|^2 \, d\tau \Big)^{1/2}$$

(cf. (2.2.6)), where

$$V(q,t) = \dot{v}(q, \langle q\rangle^{-1} t) = e^{\lambda t/\langle q\rangle} \zeta(\langle q\rangle^{-1} t) \, \dot{u}(q)$$

and $\dot{u}(q)$ are the Fourier coefficients of u. It can be easily verified that the expression (3.6.17) is equal to

$$\Big(\sum_{q\in\mathbb{Z}^{n-1}} |\dot{u}(q)|^2 \int_{\mathbb{R}} (\langle q\rangle^2 + |\tau - i\lambda|^2)^{l-1/2} |(\mathcal{F}_{t\to\tau}\zeta)(\tau)|^2 \, d\tau \Big)^{1/2}.$$

Using the inequality

$$(\langle q\rangle^2 + |\tau - i\lambda|^2)^{l-1/2} \leq c\,(\langle q\rangle^{2l-1} + |\tau|^{2l-1} + |\lambda|^{2l-1}),$$

we get the left part of the estimate (3.6.16). Furthermore, the inequality

$$(\langle q\rangle^2 + |\tau - i\lambda|^2)^{l-1/2} \geq c\Big(\sum_{j=0}^{l-1} \langle q\rangle^{2l-2j-1} |\tau - i\lambda|^{2j} + |\tau - i\lambda|^{2l-1} \Big)$$

yields

$$\|v\|^2_{W^{l-1/2}_{2,per}(\mathbb{R}^{n-1})} \geq c \sum_{q\in\mathbb{Z}^{n-1}} |\dot{u}(q)|^2 \Big(\sum_{j=0}^{l-1} \langle q\rangle^{2l-2j-1} \, |\lambda|^{2j} \, A_j + |\lambda|^{2l-1} \, A_{l-1/2}\Big),$$

where

$$A_j = A_j(\zeta,\lambda) \overset{def}{=} \int_{\mathbb{R}} |1 + i\lambda^{-1}\tau|^{2j} \, |(\mathcal{F}_{t\to\tau}\zeta)(\tau)|^2 \, d\tau \geq \|\zeta\|^2_{L_2(\mathbb{R})}$$

for $j = 0, 1, \dots, l-1$ and $j = l - 1/2$ (see Lemma 3.6.2). This proves (3.6.16) for functions u with sufficiently small support. If u has arbitrary support, then this inequality can be easily proved by means of a sufficiently fine partition of unity on $\partial\Omega$. ■

Before we give an estimate for the solution of the parameter-depending problem, we introduce the following norms which are equivalent to the norms in $W_2^l(\Omega)$ and $W_2^{l-1/2}(\partial\Omega)$, respectively, for arbitrary fixed λ :

$$(3.6.18) \qquad \|u\|_{W_2^l(\Omega,\lambda)} = \|u\|_{W_2^l(\Omega)} + |\lambda|^l \, \|u\|_{L_2(\Omega)} \, ,$$

$$(3.6.19) \qquad \|u\|_{W_2^{l-1/2}(\partial\Omega,\lambda)} = \|u\|_{W_2^{l-1/2}(\partial\Omega)} + |\lambda|^{l-1/2} \, \|u\|_{L_2(\partial\Omega)} \, .$$

THEOREM 3.6.1. *Suppose that the boundary value problem (3.6.1), (3.6.2) is elliptic with parameter. Then there exist positive real constants ρ and δ such that for all $\lambda \in \mathbb{C}$ satisfying the conditions*

$$(3.6.20) \qquad |\lambda| > \rho \quad \textit{and} \quad |\mathrm{Re}\,\lambda| < \delta\,|\mathrm{Im}\,\lambda|,$$

the boundary value problem (3.6.1), (3.6.2) has a unique solution

$$(3.6.21) \qquad (u, \underline{u}) \in W_2^l(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$$

for arbitrary given $f \in W_2^{l-2m}(\Omega)$, $\underline{g} = (g_1, \dots, g_{m+J}) \in W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$, $l \geq 2m$. This solution satisfies the estimate

$$(3.6.22) \qquad \|u\|_{W_2^l(\Omega,\lambda)} + \sum_{j=1}^{J} \|u_j\|_{W_2^{l+\tau_j-1/2}(\partial\Omega,\lambda)} \leq c\Big(\|f\|_{W_2^{l-2m}(\Omega,\lambda)} + \sum_{k=1}^{m+J} \|g_k\|_{W_2^{l-\mu_k-1/2}(\partial\Omega,\lambda)}\Big) ,$$

where the constant c is independent of u, $\underline{u}$ and λ.

Proof: First we prove the inequality (3.6.22) for purely imaginary λ, $|\lambda| > \rho$. Let $(u, \underline{u})$ be an arbitrary element of (3.6.21), $\lambda \in \mathbb{R}$, and let ζ, η be smooth functions of the variable t with compact support such that $\zeta \not\equiv 0$, $\zeta\,\eta = \zeta$. Then the function $v(x,t) = e^{\lambda t}\, u(x)$ and the vector-function $\underline{v}(x,t) = e^{\lambda t}\, \underline{u}(x)$ satisfy the estimate

$$\|\zeta v\|_{W_2^l(\mathbb{C})} + \|\zeta \underline{v}\|_{W_2^{l+\underline{\tau}-1/2}(\partial\mathcal{C})} \leq c \Big(\|\eta L(iD_t) v\|_{W_2^{l-2m}(\mathcal{C})} + \|\eta\big(B(iD_t)v + C(iD_t)\underline{v}\big)\|_{W_2^{l-\underline{\mu}-1/2}(\partial\mathcal{C})} + \|\eta v\|_{L_2(\mathcal{C})} + \|\eta \underline{v}\|_{L_2(\partial\mathcal{C})^J}\Big)$$

(see (3.2.30)). Using the equations $L(iD_t)v = e^{\lambda t}L(\lambda)u$, $B(iD_t)v + C(iD_t)\underline{v} = e^{\lambda t}(B(\lambda)u + C(\lambda)\underline{u})$ and the estimates in Lemma 3.6.3, we get (3.6.22) with the additional term

$$\|u\|_{L_2(\Omega)} + \|\underline{u}\|_{L_2(\partial\Omega)}$$

on the right-hand side. If $|\lambda| > \rho$ and ρ is sufficiently large, this implies (3.6.22) without this additional term.

Now let $\lambda = \pm i|\lambda|\, e^{i\theta}$ be a complex number, where $\theta \in (-\pi/2\,,\pi/2]$. Since

$$L(\lambda) - L(\pm i|\lambda|) = \sum_{|\alpha+j|\le 2m} a_{\alpha,j}(x)\,(\pm|i\lambda|)^j\,(e^{ij\theta} - 1)\,D_x^\alpha\,,$$

there exists a number δ for every $\varepsilon > 0$ such that

$$\|(L(\lambda) - L(\pm i|\lambda|))u\|_{W_2^{l-2m}(\Omega,\lambda)} \le \varepsilon \|u\|_{W_2^l(\Omega,\lambda)}$$

if $|\theta| < \delta$. Analogous estimates are valid for the norms of $(B_k(\lambda) - B_k(\pm i|\lambda|))u|_{\partial\Omega}$ and $(C_{k,j}(\lambda) - C_{k,j}(\pm i|\lambda|))u_j$ in $W_2^{l-\mu_k-1/2}(\partial\Omega,\lambda)$. Hence from the validity of (3.6.22) for purely imaginary λ, $|\lambda| > \rho$, it follows that this inequality is satisfied for complex λ with $|\lambda| > \rho$ and $\min(|\arg i\lambda|, |\arg(-i\lambda)|) < \delta$ if δ is sufficiently small. In particular, (3.6.22) implies the uniqueness of the solution (3.6.21) for these λ. Since the formally adjoint problem of (3.6.1), (3.6.2) is also elliptic with parameter, it follows that the problem (3.6.1), (3.6.2) is solvable in (3.6.21) for every given f, $\underline{g}$ as in the formulation of the theorem. The proof is complete. ■

CHAPTER 4

Variants and extensions

In this chapter the results of Chapter 3 are generalized to elliptic boundary value problems for a $2m$ order differential equation without any restrictions on the orders of the boundary operators and to elliptic boundary value problems for systems of differential equations. In particular, we derive a modification of the Green formula which was used in the previous chapters. This allows to introduce a formally adjoint boundary value problem for arbitrary boundary value problems. Furthermore, we extend the operator of the boundary value problem to Sobolev spaces of negative order. Analogously to Chapter 3, it can be proved that this operator is Fredholm if and only if the boundary value problem is elliptic.

Section 4.3 is concerned with boundary value problems in the variational form. Here additionally to the generalized solutions of the boundary value problem, so-called *variational solutions* appear. We study the relations between these solutions.

4.1. Elliptic problems with boundary operators of higher order in a smooth bounded domain

We consider the boundary value problem (3.1.1), (3.1.2) for the $2m$ order differential operator L in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. In Chapter 3 we have supposed that the orders of the boundary operators B_k are less than $2m$. Now we consider the boundary value problem (3.1.1), (3.1.2) without this restriction.

4.1.1. A modification of the Green formula. Let κ be an integer number, $\kappa \geq 2m$, $\kappa > \max \mu_k$, and let

$$\mathcal{A}^* : W_2^{\kappa-2m}(\Omega)^* \times W_2^{-\kappa+\underline{\mu}+1/2}(\partial\Omega) \to W_2^{\kappa}(\Omega)^* \times W_2^{-\kappa-\underline{\tau}+1/2}(\partial\Omega) \tag{4.1.1}$$

be the adjoint operator to the operator

$$W_2^{\kappa}(\Omega) \times W_2^{\kappa+\underline{\tau}-1/2}(\partial\Omega) \ni (u, \underline{u}) \quad \to \quad \big(Lu,\, Bu|_{\partial\Omega} + C\underline{u}\big) \in W_2^{\kappa-2m}(\Omega) \times W_2^{\kappa-\underline{\mu}-1/2}(\partial\Omega)$$

of the elliptic boundary value problem (3.1.1), (3.1.2). In Section 3.3 we have shown that for $\mu_k < 2m$ the restriction of the operator (4.1.1) to the space

$$D_2^{l,0}(\Omega) \times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega), \quad l \geq 2m - \kappa, \tag{4.1.2}$$

continuously maps the space (4.1.2) into $D_2^{l-2m,2m}(\Omega) \times W_2^{l-2m-\underline{\tau}+1/2}(\partial\Omega)$. By Theorem 3.4.2, this mapping is Fredholm if the boundary value problem (3.1.1), (3.1.2) is elliptic.

We consider the restriction of the operator (4.1.1) to the subspace

$$D_2^{l,\kappa-2m}(\Omega) \times W_2^{l-2m+\underline{\mu}+1/2}(\partial\Omega), \quad l \geq 2m - \kappa, \tag{4.1.3}$$

of $W_2^{\kappa-2m}(\Omega)^* \times W_2^{-\kappa+\underline{\mu}+1/2}(\partial\Omega)$. Let $(V, \underline{v})$ be an arbitrary element of the space (4.1.3). Then by the definition of the space $D_2^{l,\kappa-2m}(\Omega)$, there exist a function (or a functional) $v \in D_2^{l,0}(\Omega)$ and a vector $\underline{w} = (w_1, \dots, w_{l-2m})$ with $w_j \in W_2^{l+j-1/2}(\partial\Omega)$ such that

$$(\varphi, V)_\Omega = (\varphi, v)_\Omega + (\mathcal{D}^{(\kappa-2m)}\varphi|_{\partial\Omega}\,,\, \underline{w})_{\partial\Omega} \quad \text{for all } \varphi \in W_2^{\kappa-2m}(\Omega).$$

Here $\mathcal{D}^{(\kappa-2m)}$ denotes the vector

$$\mathcal{D}^{(\kappa-2m)} = \begin{pmatrix} 1 \\ D_\nu \\ \vdots \\ D_\nu^{\kappa-2m-1} \end{pmatrix}.$$

Hence the functional $(F, \underline{h}) = \mathcal{A}^*(V, \underline{v})$ satisfies the equation

$$\begin{aligned} (4.1.4) \qquad (u, F)_\Omega + (\underline{u}, \underline{h})_{\partial\Omega} &= (Lu, V)_\Omega + \big(Bu|_{\partial\Omega} + C\underline{u},\, \underline{v}\big)_{\partial\Omega} \\ &= (Lu, v)_\Omega + (\mathcal{D}^{(\kappa-2m)}Lu|_{\partial\Omega}\,,\, \underline{w})_{\partial\Omega} + \big(Bu|_{\partial\Omega} + C\underline{u}\,,\, \underline{v}\big)_{\partial\Omega}\,, \end{aligned}$$

for all $u \in W_2^\kappa(\Omega)$, $\underline{u} \in W_2^{\kappa+\underline{\tau}-1/2}(\partial\Omega)$.

We rewrite the right side of (4.1.4) for smooth functions u, v, $\underline{u}$, $\underline{v}$, $\underline{w}$. By (3.1.7), we have

$$\begin{aligned} (4.1.5) \qquad (Lu, v)_\Omega &= (u, L^+v)_\Omega + (\mathcal{D}^{(2m)}u|_{\partial\Omega}\,,\, Pv|_{\partial\Omega})_{\partial\Omega} \\ &= (u, L^+v)_\Omega + (\mathcal{D}^{(\kappa)}u|_{\partial\Omega}\,,\, P^{(\kappa)}v|_{\partial\Omega})_{\partial\Omega}\,, \end{aligned}$$

where $P^{(\kappa)}$ is the vector with κ components $P_1(x, D_x), \dots, P_{2m}(x, D_x), 0, \dots, 0$ and $P_j(x, D_x)$, $j = 1, \dots, 2m$, are the differential operators of order $2m - j$ which occur in formula (3.1.7).

Since $\operatorname{ord} B_k < \kappa$, there exists a matrix $Q^{(\kappa)}$ of tangential differential operators $Q_{k,j}$ $(k = 1, \dots, m+J,\ j = 1, \dots, \kappa)$ such that

$$(4.1.6) \qquad Bu|_{\partial\Omega} = Q^{(\kappa)} \cdot \mathcal{D}^{(\kappa)}u|_{\partial\Omega}\,.$$

Hence

$$(4.1.7) \qquad (Bu|_{\partial\Omega}\,,\, \underline{v})_{\partial\Omega} = \big(\mathcal{D}^{(\kappa)}u|_{\partial\Omega}\,,\, (Q^{(\kappa)})^+\underline{v}\big)_{\partial\Omega}\,.$$

Furthermore, we can write the vector $\mathcal{D}^{(\kappa-2m)}Lu$ on $\partial\Omega$ in the form

$$(4.1.8) \qquad \mathcal{D}^{(\kappa-2m)}Lu|_{\partial\Omega} = R^{(\kappa)}\mathcal{D}^{(\kappa)}u|_{\partial\Omega}\,,$$

where

$$R^{(\kappa)} = \Big(R_{s,j}(x, D_x)\Big)_{1\le s\le\kappa-2m,\ 1\le j\le\kappa}$$

is a matrix of tangential differential operators $R_{s,j}$ on $\partial\Omega$ of order not greater than $2m + s - j$, $R_{s,j} = 0$ if $j > 2m + s$. In particular, $R_{s,2m+s}$ are functions on $\partial\Omega$ which do not depend on s, i.e., we have $R_{s,2m+s} = R_{1,2m+1}$ for $s = 1, \dots, l - 2m$. If L is elliptic on $\overline{\Omega}$, then $R_{1,2m+1} \neq 0$ on $\partial\Omega$. Using (4.1.8), we obtain

$$(4.1.9) \qquad (\mathcal{D}^{(\kappa-2m)}Lu|_{\partial\Omega}\,,\, \underline{w})_{\partial\Omega} = (\mathcal{D}^{(\kappa)}u|_{\partial\Omega}\,,\, (R^{(\kappa)})^+\underline{w})_{\partial\Omega}\,.$$

The relations (4.1.5), (4.1.7), (4.1.9) yield the following modificated Green formula:

$$\begin{aligned} (4.1.10) \qquad &\big(Lu, v\big)_\Omega + \big(\mathcal{D}^{(\kappa-2m)}Lu|_{\partial\Omega}\,,\, \underline{w}\big)_{\partial\Omega} + \big(Bu|_{\partial\Omega} + C\underline{u}\,,\, \underline{v}\big)_{\partial\Omega} \\ &= \big(u, L^+v\big)_\Omega + \big(\mathcal{D}^{(\kappa)}u|_{\partial\Omega}\,,\, P^{(\kappa)}v|_{\partial\Omega} + (Q^{(\kappa)})^+\underline{v} + (R^{(\kappa)})^+\underline{w}\big)_{\partial\Omega} + \big(\underline{u}, C^+\underline{v}\big)_{\partial\Omega} \end{aligned}$$

This formula is valid for all $u, v \in C^\infty(\overline{\Omega})$, $\underline{u} \in C^\infty(\partial\Omega)^J$, $\underline{w} \in C^\infty(\partial\Omega)^{\kappa-2m}$, $\underline{v} \in C^\infty(\partial\Omega)^{m+J}$ if the vector B admits the representation (4.1.6) with $\kappa \geq 2m$.

DEFINITION 4.1.1. Let the Green formula (2.1.10) be valid for each $u, v \in C^\infty(\overline{\Omega})$, $\underline{u} \in C^\infty(\partial\Omega)^J$, $\underline{w} \in C^\infty(\partial\Omega)^{\kappa-2m}$, $\underline{v} \in C^\infty(\partial\Omega)^{m+J}$. Then the problem

$$L^+ v = f \quad \text{in } \Omega, \tag{4.1.11}$$

$$P^{(\kappa)} v + (R^{(\kappa)})^+ \underline{w} + (Q^{(\kappa)})^+ \underline{v} = \underline{g} \quad \text{on } \partial\Omega, \tag{4.1.12}$$

$$C^+ \underline{v} = \underline{h} \quad \text{on } \partial\Omega \tag{4.1.13}$$

is said to be *formally adjoint* to the boundary value problem (3.1.1), (3.1.2) with respect to the Green formula (4.1.10).

REMARK 4.1.1. The formally adjoint problem (4.1.11)–(4.1.13) depends both on the operators of the starting problem (3.1.1), (3.1.2) and on the choice of the number κ. Let κ_0 be the smallest integer number not less than $2m$ such that a representation of the form (4.1.6) is valid for the vector B. If $\kappa_0 < \kappa$, then the last $\kappa - \kappa_0$ rows of the matrix $(Q^{(\kappa)})^+$ are equal to zero and the system of the last $\kappa - \kappa_0$ equations in (4.1.12) has the triangular form

$$\begin{array}{ccccccc} \overline{R}_1 w_{\kappa_0-2m+1} & + & \overline{R}_{\kappa_0-2m+2,\kappa_0+1} w_{\kappa_0-2m+2} & + & \dots & = & g_{\kappa_0+1} \\ & & \overline{R}_1 w_{\kappa_0-2m+2} & + & \dots & = & g_{\kappa_0+2} \\ & & & \ddots & & & \\ & & & & \overline{R}_1 w_{\kappa-2m} & = & g_\kappa \end{array}$$

with the function $\overline{R}_1 \overset{def}{=} \overline{R}_{1,2m+1}$ in the diagonal. If the differential operator L is elliptic, then the elements in the diagonal do not vanish on $\partial\Omega$. Hence the formally adjoint problems with different values of κ, $\kappa \geq \kappa_0$, are equivalent. In particular, for $\kappa_0 = 2m$ the problem (4.1.11)–(4.1.13) is equivalent to the boundary value problem (3.1.10), (3.1.11).

We observe further that the order of the derivatives of v in the boundary conditions (4.1.12) is less than $2m$. Thus, the boundary value problem (4.1.11)–(4.1.13) belongs to the class of problems which has been studied in Chapter 3.

Analogously to Theorem 3.1.2, the following assertion holds.

THEOREM 4.1.1. *The boundary value problem (3.1.1), (3.1.2) is elliptic if and only if the formally adjoint problem (4.1.11)–(4.1.13) is elliptic.*

4.1.2. Extension of the operator of the boundary value problem. Let κ be an arbitrary integer, $\kappa \geq 2m$. We consider the operator

$$\tilde{W}_2^{l,\kappa}(\Omega) \ni \left(u, \mathcal{D}^{(\kappa)} u|_{\partial\Omega}\right) \to \left(Lu,\, \mathcal{D}^{(\kappa-2m)} Lu|_{\partial\Omega}\right) \in \tilde{W}_2^{l-2m,\kappa-2m}(\Omega), \quad l \geq \kappa, \tag{4.1.14}$$

which can be identified with the differential operator L. The operator (4.1.14) can be extended to a continuous operator

$$\tilde{W}_2^{l,\kappa}(\Omega) \ni (u, \underline{\phi}) \to (f, \underline{\Phi}) \in \tilde{W}_2^{l-2m,\kappa-2m}(\Omega), \quad l < \kappa,$$

as follows. Let $(u, \underline{\phi}) = (u, \phi_1, \dots, \phi_\kappa)$ be an arbitrary element of the space $\tilde{W}_2^{l,\kappa}(\Omega)$, $l < \kappa$. Then the first component $f \in \tilde{W}_2^{l-2m,0}(\Omega)$ of the pair $(f, \underline{\Phi}) \in \tilde{W}_2^{l-2m,\kappa-2m}(\Omega)$ is defined as

$$f = L(u, \underline{\phi}'),$$

where $\underline{\phi}' = (\phi_1, \dots, \phi_{2m})$ and L is the extension of the operator

$$\tilde{W}_2^{\kappa,2m}(\Omega) \ni \big(u, \mathcal{D}^{(2m)}u|_{\partial\Omega}\big) \to Lu \in \tilde{W}_2^{\kappa-2m,0}(\Omega)$$

to the space $\tilde{W}_2^{l,2m}(\Omega)$ (see Lemma 3.2.2), while

$$\underline{\Phi} = R^{(\kappa)}\,\underline{\phi}\,,$$

where $R^{(\kappa)}$ is given by (4.1.8). Obviously, the so defined mapping $(u, \underline{\phi}) \to (f, \underline{\Phi})$ is a continuous extension of the operator (4.1.14).

In the following, we denote the operator (4.1.14) and its extension to the space $\tilde{W}_2^{l,\kappa}(\Omega)$ with $l < \kappa$ also by L.

We suppose that $\operatorname{ord} B_k < \kappa$ for $k = 1, \dots, m + J$. According to (4.1.6), the operator

$$\tilde{W}_2^{l,\kappa}(\Omega) \ni (u, \underline{\phi}) \to Q^{(\kappa)}\,\underline{\phi} \in W_2^{l-\underline{\mu}-1/2}(\partial\Omega) \tag{4.1.15}$$

coincides with the operator

$$\tilde{W}_2^{l,\kappa}(\Omega) \ni \big(u, \mathcal{D}^{(\kappa)}u|_{\partial\Omega}\big) \to Bu|_{\partial\Omega} \in W_2^{l-\underline{\mu}-1/2}(\partial\Omega) \tag{4.1.16}$$

for $l \geq \kappa$. Hence in the case $l < \kappa$ the operator (4.1.15) is the continuous extension of the operator (4.1.16).

Thus, we have proved the following theorem.

THEOREM 4.1.2. *Let κ be an integer, $\kappa \geq 2m$, $\kappa > \max \operatorname{ord} B_k$. Then the operator*

$$\begin{aligned} &\tilde{W}_2^{l,\kappa}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \ni \big(u, \mathcal{D}^{(\kappa)}u|_{\partial\Omega}, \underline{u}\big) \\ &\quad \to \big(Lu,\, \mathcal{D}^{(\kappa-2m)}Lu|_{\partial\Omega}\,,\, Bu|_{\partial\Omega} + C\underline{u}\big) \in \tilde{W}_2^{l-2m,\kappa-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega) \end{aligned} \tag{4.1.17}$$

with $l \geq \kappa$ can be uniquely extended to a linear and continuous operator

$$\mathcal{A} : \tilde{W}_2^{l,\kappa}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \to \tilde{W}_2^{l-2m,\kappa-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega) \tag{4.1.18}$$

with $l < \kappa$. This extension has the form

$$(u, \underline{\phi}, \underline{u}) \to \big(f, R^{(\kappa)}\underline{\phi},\, Q^{(\kappa)}\underline{\phi} + C\underline{u}\big)\,,$$

where $f = Lu$ if $2m \leq l < \kappa$, while in the cases $l \leq 0$ and $0 < l < 2m$ the functional $f \in W_2^{2m-l}(\Omega)^*$ is given by the equalities (3.2.14) and (3.2.15), respectively.

In particular, in the case $l \leq 0$ the triple $(f, \underline{\Phi}, \underline{g}) = \mathcal{A}(u, \underline{\phi}, \underline{u})$ satisfies the equality

$$\begin{aligned} &(f, v)_\Omega + (\underline{\Phi}, \underline{w})_{\partial\Omega} + (\underline{g}, \underline{v})_{\partial\Omega} \\ &\quad = (u, L^+v)_\Omega + \big(\underline{\phi}\,,\, P^{(\kappa)}v|_{\partial\Omega} + (R^{(\kappa)})^+\underline{w} + (Q^{(\kappa)})^+\underline{v}\big)_{\partial\Omega} + (\underline{u}\,,\, C^+\underline{v})_{\partial\Omega}\,, \end{aligned} \tag{4.1.19}$$

for all $v \in W_2^{2m-l}(\Omega)$, $\underline{w} \in \prod_{s=1}^{\kappa-2m} W_2^{2m-l+s-1/2}(\partial\Omega)$, $\underline{v} \in W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$.

By (4.1.19), the operator $\mathcal{A}$ is adjoint to the operator

$$\begin{aligned} &W_2^{2m-l}(\Omega) \times \prod_{s=1}^{\kappa-2m} W_2^{2m-l+s-1/2}(\partial\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \ni (v, \underline{w}, \underline{v}) \\ &\quad \to \big(L^+v,\, P^{(\kappa)}v|_{\partial\Omega} + (R^{(\kappa)})^+\underline{w} + (Q^{(\kappa)})^+\underline{v},\, C^+\underline{v}\big) \\ &\quad\quad \in W_2^{-l}(\Omega) \times \prod_{j=1}^{\kappa} W_2^{-l+j-1/2}(\partial\Omega) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega) \end{aligned}$$

of the formally adjoint problem (4.1.11) - (4.1.13) if $l \le 0$.

Repeating the proofs in Chapters 1–3, we get the following theorems.

THEOREM 4.1.3. *Let κ be an integer number, $\kappa \ge 2m$, $\kappa > \max \operatorname{ord} B_k$, and let $(u, \underline{\phi}, \underline{u}) \in \tilde{W}_2^{l,\kappa}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$ be a solution of the elliptic boundary value (3.1.1), (3.1.2) with the right-hand sides*

$$(f, \underline{\Phi}) = L(u, \underline{\phi}) \in \tilde{W}_2^{l+1-2m,\kappa-2m}(\Omega), \qquad \underline{g} \in W_2^{l-\underline{\mu}+1/2}(\partial\Omega).$$

Then $(u, \underline{\phi}) \in \tilde{W}_2^{l+1,\kappa}(\Omega)$ and $\underline{u} \in W_2^{l+\underline{\tau}+1/2}(\partial\Omega)$. Furthermore, $(u, \underline{\phi}, \underline{u})$ satisfies an estimate analogous to (3.2.27).

THEOREM 4.1.4. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic and $\operatorname{ord} B_k < \kappa$ for $k = 1, \dots, m + J$, where κ is an integer number not less than $2m$. Then the operator (4.1.17) (or its extension (4.1.18)) is Fredholm for arbitrary integer l.*

The kernel of the operator $\mathcal{A}$ consists of all elements

$$(u, \underline{\phi}, \underline{u}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^\kappa \times C^\infty(\partial\Omega)^J$$

such that $\underline{\phi} = \mathcal{D}^{(\kappa)} u|_{\partial\Omega}$ and $(u, \underline{u})$ is a solution of the homogeneous boundary value problem (3.1.1), (3.1.2).

The element $(f, \underline{\Phi}, \underline{g}) \in \tilde{W}_2^{l-2m,\kappa-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$ belongs to the range of the operator $\mathcal{A}$ if and only if

$$(f, v)_\Omega + (\underline{\Phi}, \underline{w})_{\partial\Omega} + (\underline{g}, \underline{v})_{\partial\Omega} = 0$$

for all solutions $(v, \underline{w}, \underline{v}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^{\kappa-2m} \times C^\infty(\partial\Omega)^{m+J}$ of the homogeneous formally adjoint problem.

For $l \ge \kappa$ the operator $\mathcal{A}$ can be identified with the operator

$$W_2^l(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \ni (u, \underline{u}) \to \big(Lu,\ Bu|_{\partial\Omega} + C\underline{u}\big) \in W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega). \tag{4.1.20}$$

Then as a consequence of Theorem 4.1.4, the following assertion holds.

COROLLARY 4.1.1. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. Then the operator (4.1.20) is Fredholm for $l \ge 2m$, $l > \max \operatorname{ord} B_k$. The kernel of this operator consists only of C^∞-functions. The element*

$$(f, \underline{g}) \in W_2^{l-2m}(\Omega) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$$

belongs to the range of the operator (4.1.20) if and only if

$$(f, v)_\Omega + \big(\mathcal{D}^{(\kappa-2m)} f|_{\partial\Omega}\,,\ \underline{w}\big)_{\partial\Omega} + (\underline{g}, \underline{v})_{\partial\Omega} = 0$$

for all solutions $(v, \underline{w}, \underline{v}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^{\kappa-2m} \times C^\infty(\partial\Omega)^{m+J}$ of the homogeneous formally adjoint problem.

4.1.3. The adjoint operator. Let $l \geq \kappa$, where $\kappa \geq 2m$, $\kappa > \max \operatorname{ord} B_k$, and let

$$\mathcal{A}^* : W_2^{l-2m}(\partial\Omega)^* \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \to W_2^l(\Omega)^* \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega)$$

be the adjoint operator to (4.1.20). The functional $(F, \underline{h}) = \mathcal{A}^*(V, \underline{v})$ is given for arbitrary $V \in W_2^{l-2m}(\partial\Omega)^*$, $\underline{v} \in W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$ by the equality

$$(u, F)_\Omega + (\underline{u}, \underline{h})_{\partial\Omega} = (Lu, V)_\Omega + \big(Bu|_{\partial\Omega} + C\underline{u},\, \underline{v}\big)_{\partial\Omega}, \tag{4.1.21}$$

where u is an arbitrary function from $W_2^l(\Omega)$ and $\underline{u} \in W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$.

We consider the operator of the formally adjoint problem (4.1.11), (4.1.13) which maps the space

$$\tilde{W}_2^{2m-l,\kappa}(\Omega) \times \Big(\prod_{s=1}^{\kappa-2m} W_2^{2m-l+s-1/2}(\partial\Omega)\Big) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \tag{4.1.22}$$

into

$$\tilde{W}_2^{-l,\kappa-2m}(\Omega) \times \Big(\prod_{j=1}^{\kappa} W_2^{-l+j-1/2}(\partial\Omega)\Big) \times W_2^{-l-\underline{\tau}-1/2}(\partial\Omega) \tag{4.1.23}$$

Let $(v, \underline{\psi})$ be an arbitrary element of the space $\tilde{W}_2^{2m-l,\kappa}(\Omega)$, $l \geq \kappa$. Furthermore, let

$$\underline{w} \in \prod_{s=1}^{\kappa-2m} W_2^{2m-l+s-1/2}(\partial\Omega) \quad \text{and} \quad \underline{v} \in W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$$

be given. Then $(f, \underline{\Phi}) = L^+(v, \underline{\psi})$ satisfies the equalities

$$(u, f)_\Omega = (Lu, v)_\Omega - \big(\mathcal{D}^{(\kappa)}u,\, T^{(\kappa)}\underline{\psi}\big)_{\partial\Omega}, \quad u \in W_2^l(\Omega), \tag{4.1.24}$$

and

$$\underline{\Phi} = S^{(\kappa)}\,\underline{\psi}, \tag{4.1.25}$$

where $T^{(\kappa)}$ and $S^{(\kappa)}$ are matrices of tangential differential operators defined by

$$P^{(\kappa)}u|_{\partial\Omega} = T^{(\kappa)} \cdot \mathcal{D}^{(\kappa)}u|_{\partial\Omega}, \quad \mathcal{D}^{(\kappa-2m)}L^+u|_{\partial\Omega} = S^{(\kappa)}\,\mathcal{D}^{(\kappa)}u|_{\partial\Omega}, \tag{4.1.26}$$

$u \in C^\infty(\overline{\Omega})$. The boundary conditions (4.1.12), (4.1.13) for $(v, \underline{\psi})$, $\underline{w}$, and $\underline{v}$ can be written in the form

$$T^{(\kappa)}\underline{\psi} + (R^{(\kappa)})^+\underline{w} + (Q^{(\kappa)})^+\underline{v} = \underline{g}, \tag{4.1.27}$$

$$C^+\underline{v} = \underline{h}. \tag{4.1.28}$$

Comparing (4.1.21) with (4.1.24)–(4.1.28), we obtain the following relations between the operators $\mathcal{A}^+$ and $\mathcal{A}^*$ (cf. Lemma 3.3.1).

LEMMA 4.1.1. *Let $(v, \underline{\psi}, \underline{w}, \underline{v})$ and $(f, \underline{\Phi}, \underline{g}, \underline{h})$ be elements of the spaces (4.1.22) and (4.1.23), respectively, where $l \geq \kappa$. Furthermore, let the functionals $V \in W_2^{l-2m}(\Omega)^*$ and $F \in W_2^l(\Omega)^*$ be defined as follows*

$$(V, \varphi)_\Omega = (v, \varphi)_\Omega + \big(\underline{w},\, \mathcal{D}^{(\kappa-2m)}\varphi\big)_{\partial\Omega}, \qquad \varphi \in W_2^{l-2m}(\Omega), \tag{4.1.29}$$

$$(F, u)_\Omega = (f, u)_\Omega + \big(\underline{g},\, \mathcal{D}^{(\kappa)}u\big)_{\partial\Omega}, \qquad u \in W_2^l(\Omega). \tag{4.1.30}$$

Then $(v, \underline{\psi}, \underline{w}, \underline{v})$ is a solution of the equation

$$\mathcal{A}^+(v, \underline{\psi}, \underline{w}, \underline{v}) = (f, \underline{\Phi}, \underline{g}, \underline{h}) \tag{4.1.31}$$

if and only if $(V, \underline{v})$ *is a solution of the equation*

$$\mathcal{A}^* (V, \underline{v}) = (F, \underline{h}) \tag{4.1.32}$$

and $\underline{\psi}$ *satisfies the equations (4.1.25), (4.1.27).*

Proof: Let $(v, \underline{\psi}, \underline{w}, \underline{v})$ be a solution of (4.1.31). Then (4.1.25), (4.1.27) are satisfied and, according to (4.1.24), (4.1.30), we have

$$\begin{aligned} (u, F) &= (Lu, v)_\Omega + \left(\mathcal{D}^{(\kappa)} u,\ \underline{g} - T^{(\kappa)} \underline{\psi}\right)_{\partial\Omega} \\ &= (Lu, v)_\Omega + \left(\mathcal{D}^{(\kappa)} u,\ (R^{(\kappa)})^+ \underline{w} + (Q^{(\kappa)})^+ \underline{v}\right)_{\partial\Omega} \\ &= (Lu, v)_\Omega + \left(\mathcal{D}^{(\kappa-2m)} Lu,\ \underline{w}\right)_{\partial\Omega} + (Bu, \underline{v})_{\partial\Omega} = (Lu, V)_\Omega + (Bu, \underline{v})_{\partial\Omega} \end{aligned}$$

This together with (4.1.28) implies (4.1.21). Hence $(V, \underline{v})$ is a solution of the equation (4.1.32).

Analogously, it can be shown that (4.1.31) follows from (4.1.25), (4.1.27), and (4.1.32). ∎

Note that the equations (4.1.25), (4.1.27) have a unique solution $\underline{\psi}$ for arbitrary given $\underline{\Phi}$, $\underline{g}$, $\underline{v}$, $\underline{w}$ if the operator L is elliptic. This follows from the structure of the matrices $S^{(\kappa)}$ and $T^{(\kappa)}$ (cf. Remarks 3.1.2, 4.1.1).

Using Theorems 4.1.3, 4.1.4 and Lemma 4.1.1, we obtain the following result (cf. Theorem 3.4.2).

THEOREM 4.1.5. *Suppose that the boundary value problem (3.1.1), (3.1.2) is elliptic. Then the operator* $\mathcal{A}^*$ *(or its restriction) is Fredholm as a mapping from*

$$D_2^{-l+2m, \kappa-2m}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega) \tag{4.1.33}$$

into

$$D_2^{-l, \kappa}(\Omega) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega) \tag{4.1.34}$$

for arbitrary integer l*. The kernel of* $\mathcal{A}^*$ *consists of all pairs* $(V, \underline{v}) \in W_2^{\kappa-2m}(\Omega)^* \times C^\infty(\partial\Omega)^{m+J}$ *such that the functional* $V \in W_2^{\kappa-2m}(\Omega)^*$ *has the form*

$$(V, \varphi)_\Omega = (v, \varphi)_\Omega + \left(\underline{w},\ \mathcal{D}^{(\kappa-2m)} \varphi|_{\partial\Omega}\right)_{\partial\Omega}, \qquad \varphi \in W_2^\kappa(\Omega),$$

where $v \in C^\infty(\overline{\Omega})$, $\underline{w} \in C^\infty(\partial\Omega)^{\kappa-2m}$*, and* $(v, \underline{w}, \underline{v})$ *is a solution of the homogeneous formally adjoint boundary value problem (4.1.11)–(4.1.13). The equation*

$$\mathcal{A}^* (V, \underline{v}) = (F, \underline{h})$$

is solvable in (4.1.33) for a given element $(F, \underline{h})$ *of the space (4.1.34) if and only if*

$$(u, F)_\Omega + (\underline{u}, \underline{h})_{\partial\Omega} = 0$$

for all solutions $(u, \underline{u}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^J$ *of the homogeneous boundary value problem (3.1.1), (3.1.2).*

4.2. Boundary value problems for elliptic systems of differential equations

In this section boundary value problems for systems of N differential equations in a smooth bounded domain Ω are considered. First we introduce the notion of ellipticity for such problems which is necessary and sufficient for the Fredholm property of the operator of the boundary value problem. Furthermore, we derive a

Green formula and extend the operator of the boundary value problem to Sobolev spaces of arbitrary integer order.

4.2.1. Ellipticity of the boundary value problem. Let

$$L(x,D_x) = \big(L_{i,j}(x,D_x)\big)_{1\le i,j\le N}$$

be a matrix of linear differential operators with infinitely differentiable coefficients on $\overline{\Omega}$. We assume that the determinant of the matrix $\big(L_{i,j}(x,\xi)\big)_{1\le i,j\le N}$ is a polynomial of even order $2m \ge 2$ and there exist integer numbers $s_i \le 0$ and $t_j \ge 0$ $(i,j = 1,\dots,N)$ with $\max(s_1,\dots,s_N) = 0$ such that

$$\operatorname{ord} L_{i,j} \le s_i + t_j\,, \quad L_{i,j} \equiv 0 \text{ if } s_i + t_j < 0, \tag{4.2.1}$$

$$s_1 + s_2 + \cdots + s_N + t_1 + t_2 + \cdots + t_N = 2m\,. \tag{4.2.2}$$

Note that under these assumptions,

$$\min_{1\le i\le N} s_i + \max_{1\le j\le N} t_j \ge 0 \quad\text{and}\quad \max_{1\le i\le N} s_i + \min_{1\le j\le N} t_j \ge 0\,.$$

Otherwise, from (4.2.1) it follows that at least one row or one column of the matrix $(L_{i,j}(\xi))$ is identically equal to zero, i.e., the determinant of this matrix vanishes.

Furthermore, let

$$B(x,D_x) = \Big(B_{k,j}(x,D_x)\Big)_{1\le k\le m+J,\ 1\le j\le N}$$

be a $(m+J)\times N$-matrix of linear differential operators with smooth coefficients satisfying the condition

$$\operatorname{ord} B_{k,j} \le \sigma_k + t_j\,, \quad B_{k,j} \equiv 0 \text{ if } \sigma_k + t_j < 0, \tag{4.2.3}$$

where σ_k are given integer numbers. Additionally, let

$$C(x,D_x) = \Big(C_{k,\nu}(x,D_x)\Big)_{1\le k\le m+J,\ 1\le \nu\le J}$$

be a $(m+J)\times J$-matrix of tangential differential operators on $\partial\Omega$,

$$\operatorname{ord} C_{k,\nu} \le \sigma_k + \tau_\nu\,, \quad C_{k,\nu} \equiv 0 \text{ if } \sigma_k + \tau_\nu < 0. \tag{4.2.4}$$

Here $\tau_1,\dots,\tau_J$ are also integer numbers.

We consider the boundary value problem

$$L(x,D_x)\underline{\mathfrak{u}} = \underline{\mathfrak{f}} \quad \text{in } \Omega, \tag{4.2.5}$$

$$B(x,D_x)\underline{\mathfrak{u}} + C\underline{u} = \underline{g} \quad \text{on } \partial\Omega. \tag{4.2.6}$$

This means, for a given vector-function $\underline{\mathfrak{f}} = (\mathfrak{f}_1,\dots,\mathfrak{f}_N)$ on Ω and a given vector-function $\underline{g} = (g_1,\dots,g_{m+J})$ on $\partial\Omega$ one has to find a vector-function $\underline{\mathfrak{u}} = (\mathfrak{u}_1,\dots \mathfrak{u}_N)$ on Ω and a vector-function $\underline{u} = (u_1,\dots,u_J)$ on $\partial\Omega$ satisfying the equations (4.2.5), (4.2.6).

Again we denote by $L^\circ_{i,j}$ the principal part of the operator $L_{i,j}$, i.e., if $L_{i,j}(x,D_x)$ is a sum of terms $a^{i,j}_\alpha(x)\,D^\alpha_x$, then

$$L^\circ_{i,j}(x,D_x) = \sum_{|\alpha|=s_i+t_j} a^{i,j}_\alpha(x)\,D^\alpha_x.$$

In the cases $s_i + t_j < 0$ and $\operatorname{ord} L_{i,j} < s_i + t_j$ we set $L^\circ_{i,j}(x,D_x) = 0$. Analogously, the principal parts $B^\circ_{k,j}$ and $C^\circ_{k,\nu}$ of the operators $B_{k,j}$ and $C_{k,\nu}$ are defined. The corresponding matrix operators are denoted by L°, B°, and C°.

Under our assumptions on the orders of the differential operators $L_{i,j}$, $B_{k,j}$, and $C_{k,\nu}$, the operator $\mathcal{A}$ of problem (4.2.5), (4.2.6) can be considered as a continuous mapping

$$W_2^{l+\underline{t}}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \longrightarrow W_2^{l-\underline{s}}(\Omega) \times W_2^{l-\underline{\sigma}-1/2}(\partial\Omega) \tag{4.2.7}$$

for arbitrary arbitrary nonnegative integer $l > \max(\sigma_1, \dots, \sigma_{m+J})$, where

$$W_2^{l+\underline{t}}(\Omega) = \prod_{j=1}^{N} W_2^{l+t_j}(\Omega), \qquad W_2^{l+\underline{\tau}-1/2}(\partial\Omega) = \prod_{j=1}^{J} W_2^{l+\tau_j-1/2}(\partial\Omega),$$

and analogous notation is used on the right side of (4.2.7).

DEFINITION 4.2.1. The matrix operator $L(x, D_x)$ is said to be *elliptic* in $\overline{\Omega}$ if

$$\det L^\circ(x,\xi) \neq 0 \qquad \text{for all } x \in \overline{\Omega},\ \xi \in \mathbb{R}^n\backslash\{0\}.$$

If moreover the polynomial $\tau \to \det L^\circ(x, \xi + \tau\zeta)$ has exactly m zeros (counting multiplicity) in the upper half-plane $\operatorname{Im}\tau > 0$ for arbitrary linearly independent vectors ξ, $\zeta \in \mathbb{R}^n$, then the operator $L(x, D_x)$ is said to be *properly elliptic.*

For arbitrary $x^{(0)} \in \partial\Omega$ and arbitrary vectors ξ' tangential to $\partial\Omega$ in $x^{(0)}$ let $\mathcal{M}^+(\xi')$ be the linear m-dimensional space of the stable solutions of the equation

$$L^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})\, D_t\big)\, \underline{\mathfrak{u}}(t) = 0, \qquad t > 0,$$

which tend to zero as $t \to \infty$.

DEFINITION 4.2.2. The boundary value problem (4.2.5), (4.2.6) is said to be *elliptic* if

(i) the operator L is properly elliptic in $\overline{\Omega}$,
(ii) for every $x^{(0)} \in \partial\Omega$, every vector ξ' tangential to $\partial\Omega$ in $x^{(0)}$, and every $\underline{g} \in \mathbb{C}^{m+J}$ there exist exactly one function $\underline{\mathfrak{u}} \in \mathcal{M}^+(\xi')$ and one vector $\underline{u} \in \mathbb{C}^J$ satisfying the equation

$$B^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})\, D_t\big)\, \underline{\mathfrak{u}}(t)\big|_{t=0} + C^\circ(x^{(0)}, \xi')\, \underline{u} = \underline{g}.$$

4.2.2. The formally adjoint boundary value problem. In order to construct a formally adjoint boundary value problem to the given problem (4.2.5), (4.2.6), we generalize the Green formula (4.1.10) to the case $N > 1$. Let

$$\sigma_0 \stackrel{def}{=} \max(0, \sigma_1 + 1, \dots, \sigma_{m+J} + 1). \tag{4.2.8}$$

We consider the expression

$$\int_\Omega \big(L\underline{\mathfrak{u}}, \underline{\mathfrak{v}}\big)_{\mathbb{C}^N}\, dx \; + \int_{\partial\Omega} \sum_{i=1}^{N}\sum_{j=1}^{N} \big(\mathcal{D}^{(\sigma_0-s_i)} L_{i,j}\mathfrak{u}_j\, , \underline{w}^{(i)}\big)_{\mathbb{C}^{\sigma_0-s_i}}\, d\sigma \; + \int_{\partial\Omega} \big((B\underline{\mathfrak{u}} + C\underline{u}), \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma \tag{4.2.9}$$

for arbitrary vector-functions $\underline{\mathfrak{u}} = (\mathfrak{u}_1 \dots, \mathfrak{u}_N)$, $\underline{\mathfrak{v}} = (\mathfrak{v}_1, \dots, \mathfrak{v}_N) \in C^\infty(\overline{\Omega})^N$, $\underline{u} \in C^\infty(\partial\Omega)^J$, $\underline{v} \in C^\infty(\partial\Omega)^{m+J}$, and $\underline{w}^{(i)} \in C^\infty(\partial\Omega)^{\sigma_0-s_i}$, $i = 1, \dots, N$. Here $\mathcal{D}^{(\sigma_0-s_i)}$ denotes the vector with the components $1, D_\nu, \dots, D_\nu^{\sigma_0-s_i-1}$ if $\sigma_0 - s_i > 0$. In the case $\sigma_0 - s_i = 0$ we set $\mathcal{D}^{(\sigma_0-s_i)} = 0$, i.e., the corresponding term in (4.2.9) can be omitted.

Applying formula (4.1.5) to the differential operator $L_{i,j}(x, D_x)$, we get

$$\int_\Omega L_{i,j}\mathfrak{u}_j \cdot \overline{\mathfrak{v}_i}\, dx = \int_\Omega \mathfrak{u}_j \cdot \overline{L^+_{i,j}\mathfrak{v}_i}\, d\sigma + \int_{\partial\Omega} \big(\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\, , P^{i,j}\mathfrak{v}_i\big)_{\mathbb{C}^{\sigma_0+t_j}}\, d\sigma\, ,$$

where $L^+_{i,j}$ is the formally adjoint operator to $L_{i,j}$, and $P^{i,j} = (P^{i,j}_1, \dots , P^{i,j}_{\sigma_0+t_j})$ are vectors of linear differential operators $P^{i,j}_\mu(x, D_x)$, $\operatorname{ord} P^{i,j}_\mu \le s_i + t_j - \mu$.

If we denote by $\mathfrak{P}^{(j)}$ the $(\sigma_0 + t_j) \times N$-matrix with the columns $P^{1,j}, \dots , P^{N,j}$, we have

(4.2.10)

$$\int_\Omega \big(L\underline{\mathfrak{u}}, \underline{\mathfrak{v}}\big)_{\mathbb{C}^N}\, dx = \int_\Omega \big(\underline{\mathfrak{u}}, L^+\underline{\mathfrak{v}}\big)_{\mathbb{C}^N}\, dx + \sum_{j=1}^N \int_{\partial\Omega} \big(\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\, , \mathfrak{P}^{(j)}\underline{\mathfrak{v}}\big)_{\mathbb{C}^{\sigma_0+t_j}}\, d\sigma.$$

Furthermore, analogously to (4.1.9), for arbitrary $\mathfrak{u}_j \in C^\infty(\overline{\Omega})$ and arbitrary vector-functions $\underline{w}^{(i)} = (w^{(i)}_1, \dots , w^{(i)}_{\sigma_0-s_i}) \in C^\infty(\partial\Omega)^{\sigma_0-s_i}$ we get

$$\int_{\partial\Omega} \big(\mathcal{D}^{(\sigma_0-s_i)}L_{i,j}\mathfrak{u}_j\, , \underline{w}^{(i)}\big)_{\mathbb{C}^N}\, d\sigma = \int_{\partial\Omega} \big(\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j, (R^{i,j})^+\underline{w}^{(i)}\big)_{\mathbb{C}^{\sigma_0+t_j}}\, d\sigma, \tag{4.2.11}$$

where $R^{i,j}$ is a $(\sigma_0 - s_i) \times (\sigma_0 + t_j)$-matrix of tangential differential operators on $\partial\Omega$ and $(R^{i,j})^+$ denotes the formally adjoint matrix operator to $R^{i,j}$.

It remains to rewrite the last term in (4.2.9). To this end, we write the vector Bu in the form

$$B\underline{\mathfrak{u}}\,\big|_{\partial\Omega} = \sum_{j=1}^N Q^{(j)}\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\,\big|_{\partial\Omega}\, . \tag{4.2.12}$$

Here $Q^{(j)}$ are $(m + J) \times (\sigma_0 + t_j)$-matrices of tangential differential operators on $\partial\Omega$. Then

$$\int_{\partial\Omega} \big(B\underline{\mathfrak{u}}, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma = \sum_{j=1}^N \int_{\partial\Omega} \big(\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\, , (Q^{(j)})^+\underline{v}\big)_{\mathbb{C}^{\sigma_0+t_j}}\, d\sigma. \tag{4.2.13}$$

Thus, we have proved the following theorem.

THEOREM 4.2.1. *There exist matrices $\mathfrak{P}^{(j)}$, $Q^{(j)}$ and $R^{i,j}$ $(i, j = 1, \dots , N)$ such that the following Green formula is satisfied for arbitrary $\underline{\mathfrak{u}}, \underline{\mathfrak{v}} \in C^\infty(\overline{\Omega})^N$, $\underline{u} \in C^\infty(\partial\Omega)^J$, $\underline{v} \in C^\infty(\partial\Omega)^{m+J}$, $\underline{w}^{(i)} \in C^\infty(\partial\Omega)^{\sigma_0-s_i}$:*

$$\begin{aligned}
&\int_\Omega \big(L\underline{\mathfrak{u}}, \underline{\mathfrak{v}}\big)_{\mathbb{C}^N}\, dx + \int_{\partial\Omega} \sum_{i=1}^N\sum_{j=1}^N \big(\mathcal{D}^{(\sigma_0-s_i)}L_{i,j}\mathfrak{u}_j\, , \underline{w}^{(i)}\big)_{\mathbb{C}^{\sigma_0-s_i}}\, d\sigma \\
&\quad + \int_{\partial\Omega} \big((B\underline{\mathfrak{u}} + C\underline{u}), \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma = \int_\Omega \big(\underline{\mathfrak{u}}, L^+\underline{\mathfrak{v}}\big)_{\mathbb{C}^N}\, dx + \int_{\partial\Omega} \big(\underline{u}, C^+\underline{v}\big)_{\mathbb{C}^J}\, d\sigma \\
&\quad + \sum_{j=1}^N \int_{\partial\Omega} \Big(\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\, , \big(\mathfrak{P}^{(j)}\underline{\mathfrak{v}} + (Q^{(j)})^+\underline{v} + \sum_{i=1}^N (R^{i,j})^+\underline{w}^{(i)}\big)\Big)_{\mathbb{C}^{\sigma_0+t_j}}\, d\sigma.
\end{aligned} \tag{4.2.14}$$

Note that the operators $\underline{\mathfrak{v}} \to \mathfrak{P}_j\underline{\mathfrak{v}}|_{\partial\Omega}$, and the matrices $R^{i,j}$ in the Green formula (4.2.14) are uniquely determined by the operator L, while the matrices $Q^{(j)}$ are uniquely determined by the matrix B.

Analogously to the case $N = 1$, we define the formally adjoint boundary value problem of the problem (4.2.5), (4.2.6) by the operators on the right-hand side of the Green formula.

DEFINITION 4.2.3. Suppose that the Green formula (4.2.14) is valid for all $\underline{\mathfrak{u}}, \underline{\mathfrak{v}} \in C_0^\infty(\overline{\Omega})^N$, $\underline{u} \in C^\infty(\partial\Omega)^J$, $\underline{v} \in C^\infty(\partial\Omega)^{m+J}$, and $\underline{w}^{(i)} \in C^\infty(\partial\Omega)^{\sigma_0 - s_i}$, $(i = 1, \dots, N)$. Then the problem

$$L^+ \underline{\mathfrak{v}} = \underline{\mathfrak{f}} \ \text{ in } \Omega, \tag{4.2.15}$$

$$\mathfrak{P}^{(j)} \underline{\mathfrak{v}} + (Q^{(j)})^+ \underline{v} + \sum_{i=1}^{N} (R^{i,j})^+ \underline{w}^{(i)} = \underline{g}^{(j)} \quad \text{on } \partial\Omega, \ j = 1, \dots, N, \tag{4.2.16}$$

$$C^+ \underline{v} = \underline{h} \quad \text{on } \partial\Omega \tag{4.2.17}$$

is said to be *formally adjoint* to the boundary value problem (4.2.5), (4.2.6).

The following assertion can be proved analogously to Theorem 3.1.2.

THEOREM 4.2.2. *The boundary value problem (4.2.5), (4.2.6) is elliptic if and only if the formally adjoint problem (4.2.15)–(4.2.17) is elliptic.*

4.2.3. Extension of the operator of the boundary value problem. The operator

$$\begin{aligned} &\tilde{W}_2^{l+t_j, \sigma_0 + t_j}(\Omega) \ni \big(\mathfrak{u}_j, \mathcal{D}^{(\sigma_0 + t_j)} \mathfrak{u}_j|_{\partial\Omega}\big) \\ &\quad \to \big(L_{i,j}\mathfrak{u}_j, \mathcal{D}^{(\sigma_0 - s_i)} L_{i,j}\mathfrak{u}_j|_{\partial\Omega}\big) \in \tilde{W}_2^{l - s_i, \sigma_0 - s_i}(\Omega), \quad l \ge 0, \end{aligned} \tag{4.2.18}$$

can be extended in a unique way to the space $\tilde{W}_2^{l+t_j,\sigma_0+t_j}(\Omega)$ with $l < 0$. This extension was described in Section 4.1. We denote the operator (4.2.18) and its extension also by $L_{i,j}$.

Let $\tilde{W}_2^{l+\underline{t},\sigma_0+\underline{t}}(\Omega)$ be the product of the spaces $\tilde{W}_2^{l+t_j,\sigma_0+t_j}(\Omega)$, $j = 1, \dots, N$. By (4.2.12), the mapping

$$\tilde{W}_2^{l+\underline{t},\sigma_0+\underline{t}}(\Omega) \ni \{(\mathfrak{u}_j, \underline{\phi}^{(j)})\}_{1 \le j \le N} \to \sum_{j=1}^{N} Q^{(j)} \cdot \underline{\phi}^{(j)} \in W_2^{l - \underline{\sigma} - 1/2}(\partial\Omega),$$

$l < \sigma_0$, is the continuous extension of the operator

$$\big\{\big(\mathfrak{u}_j, \mathcal{D}^{(\sigma_0 + t_j)} \mathfrak{u}_j|_{\partial\Omega}\big)\big\}_{1 \le j \le N} \to B\mathfrak{u}\big|_{\partial\Omega}$$

which is defined on the space $\tilde{W}_2^{l+\underline{t},\sigma_0+\underline{t}}(\Omega)$ with $l \ge \sigma_0$.

Thus, analogously to Theorem 4.1.2, the following assertions hold.

THEOREM 4.2.3. *The operator*

$$\begin{aligned} &\tilde{W}_2^{l+\underline{t},\sigma_0+\underline{t}}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \ni \Big(\big\{\big(\mathfrak{u}_j, \mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j|_{\partial\Omega}\big)\big\}_{1\le j\le N}, \underline{u}\Big) \\ &\to \Big(\big\{\big(\mathfrak{f}_i, \mathcal{D}^{(\sigma_0 - s_i)}\mathfrak{f}_i|_{\partial\Omega}\big)\big\}_{1 \le i \le N}, \underline{g}\Big) \in \tilde{W}_2^{l-\underline{s},\sigma_0-\underline{s}}(\Omega) \times W_2^{l-\underline{\sigma}-1/2}(\partial\Omega), \quad l \ge \sigma_0, \end{aligned} \tag{4.2.19}$$

where

$$\mathfrak{f}_i = \sum_{j=1}^{N} L_{i,j}\, \mathfrak{u}_j\,, \qquad \underline{g} = B\underline{\mathfrak{u}}|_{\partial\Omega} + C\underline{u},$$

can be uniquely extended to a linear and continuous operator

$$\tag{4.2.20} \begin{aligned} &\tilde{W}_2^{l+\underline{t},\sigma_0+\underline{t}}(\Omega) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \ni \big(\{(\mathfrak{u}_j, \underline{\phi}^{(j)})\}_{1\le j\le N}\,,\, \underline{u}\big) \\ &\qquad \to \big(\{(\mathfrak{f}_i, \underline{\Phi}^{(i)})\}_{1\le i\le N}\,,\, \underline{g}\big) \in \tilde{W}_2^{l-\underline{s},\sigma_0-\underline{s}}(\Omega) \times W_2^{l-\underline{\sigma}-1/2}(\partial\Omega) \end{aligned}$$

with $l < \sigma_0$. *Here*

$$\underline{g} = \sum_{j=1}^{n} Q^{(j)} \underline{\phi}^{(j)} + C\underline{u} \quad \text{and} \qquad (\mathfrak{f}_i, \underline{\Phi}^{(i)}) = \sum_{j=1}^{N} L_{i,j}\,(\mathfrak{u}_j, \underline{\phi}^{(j)}), \quad i = 1, \dots, N.$$

The last equality means that $\mathfrak{f}_i$ *are functionals which have representations analogous to (3.2.14), (3.2.15) and*

$$\underline{\Phi}^{(i)} = \sum_{j=1}^{N} R^{i,j} \underline{\phi}^{(j)}.$$

In particular, in the case $l \le -\max(t_1, \dots, t_N)$ *we have*

$$\tag{4.2.21} \begin{aligned} &\sum_{i=1}^{N} \Big((\mathfrak{f}_i, \mathfrak{v}_i)_\Omega + (\underline{\Phi}^{(i)}, \underline{w}^{(i)})_{\partial\Omega} \Big) + (\underline{g}, \underline{v})_{\partial\Omega} \\ &= (\underline{\mathfrak{u}}, L^+ \underline{\mathfrak{v}})_\Omega + \sum_{j=1}^{N} \Big(\underline{\phi}^{(j)},\, \mathfrak{P}^{(j)} \underline{\mathfrak{v}} + (Q^{(j)})^+ \underline{v} + \sum_{i=1}^{N} (R^{i,j})^+ \underline{w}^{(i)} \Big)_{\partial\Omega} + (\underline{u}, C^+ \underline{v})_{\partial\Omega} \end{aligned}$$

for all $\underline{\mathfrak{v}} \in W_2^{-l+\underline{s}}(\Omega)$, $\underline{w}^{(i)} \in \prod_{\nu=1}^{\sigma_0-s_i} W_2^{-l+s_i+\nu-1/2}(\partial\Omega)$, *and* $\underline{v} \in W_2^{-l+\underline{\sigma}+1/2}(\partial\Omega)$.

We denote the operator (4.2.19) and its extension (4.2.20) also by $\mathcal{A}$. By (4.2.21), in the case $l \le -\max(t_1, \dots, t_N)$ the operator $\mathcal{A}$ is adjoint to the operator of the formally adjoint problem mapping the space

$$W_2^{-l+\underline{s}}(\Omega) \times \prod_{i=1}^{N} \Big(\prod_{\nu=1}^{\sigma_0-s_i} W_2^{-l+s_i+\nu-1/2}(\partial\Omega) \Big) \times W_2^{-l+\underline{\sigma}+1/2}(\partial\Omega)$$

into

$$\tag{4.2.22} W_2^{-l-\underline{t}}(\Omega) \times \prod_{i=1}^{N} \Big(\prod_{\nu=1}^{\sigma_0+t_j} W_2^{-l-t_j+\nu-1/2}(\partial\Omega) \Big) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega).$$

4.2.4. Solvability of elliptic boundary value problems for systems of differential equations. In the same way as in the case $N = 1$ (cf. Theorem 3.2.3), we can prove regularity assertions and a priori estimates for solutions of the boundary value problem (4.2.5), (4.2.6). Using the relations between the operator $\mathcal{A}^+$ of the formally adjoint problem (4.2.15)–(4.2.17) and the adjoint operator $\mathcal{A}^*$ (cf. Lemmas 3.3.1, 4.1.1), we obtain also regularity assertions for solutions of the equation

$$\mathcal{A}^*(\underline{V}, \underline{v}) = (\underline{F}, \underline{h}).$$

Here the operator $\mathcal{A}^*$ can be considered for arbitrary integer l as a continuous mapping

$$D_2^{-l+\underline{s},\sigma_0-\underline{s}}(\Omega) \times W_2^{-l+\underline{\sigma}+1/2}(\partial\Omega) \to D_2^{-l-\underline{t},\sigma_0+\underline{t}}(\Omega) \times W_2^{-l-\underline{\tau}+1/2}(\partial\Omega).$$

This leads to the following theorem.

THEOREM 4.2.4. *Suppose that the boundary value problem (4.2.5), (4.2.6) is elliptic. Then the operator (4.2.19) (or its extension (4.2.20)) is Fredholm for arbitrary integer l. The kernel of the operator $\mathcal{A}$ is independent of l and consists of all C^∞ vector-functions $\big(\{(\mathfrak{u}_j, \underline{\phi}^{(j)})\}_{1\le j\le N}, \underline{u}\big)$ such that $(\underline{\mathfrak{u}}, \underline{u})$ is a solution of the homogeneous problem (4.2.5), (4.2.6) and $\underline{\phi}^{(j)} = \mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j|_{\partial\Omega}$.*

The element

$$\big(\{(\mathfrak{f}_i, \underline{\Phi}^{(i)})\}_{1\le i\le N}\, \underline{g}\big) \in \tilde{W}_2^{l-\underline{s},\sigma_0-\underline{s}}(\Omega) \times W_2^{l-\underline{\sigma}-1/2}(\partial\Omega)$$

belongs to the range of $\mathcal{A}$ if and only if

$$\sum_{i=1}^{N}\Big(\big(\mathfrak{f}_i, \mathfrak{v}_i\big)_\Omega + \big(\underline{\Phi}^{(i)}, \underline{w}^{(i)}\big)_{\partial\Omega}\Big) + \big(\underline{g}, \underline{v}\big)_{\partial\Omega} = 0 \tag{4.2.23}$$

for all solutions $\big(\underline{\mathfrak{v}}, \{\underline{w}^{(i)}\}_{1\le i\le N}, \underline{v}\big) \in C^\infty(\overline{\Omega})^N \times \prod C^\infty(\partial\Omega)^{\sigma_0-s_i} \times C^\infty(\partial\Omega)^{m+J}$ of the homogeneous formally adjoint problem (4.2.15)–(4.2.17).

4.2.5. Examples.

Example 1. We consider the *Lamé system* of linear elasticity

$$L(D_x)\,\underline{\mathfrak{u}} = \Delta\underline{\mathfrak{u}} + \gamma\,\mathrm{grad}\,\mathrm{div}\,\underline{\mathfrak{u}} = \underline{\mathfrak{f}} \quad \text{in } \Omega, \tag{4.2.24}$$

where $\underline{\mathfrak{u}} = (\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$ denotes the displacement vector and Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

If we set $s_1 = s_2 = s_3 = 0$, $t_1 = t_2 = t_3 = 2$, we have

$$\det L^\circ(\xi) = \begin{vmatrix} -|\xi|^2 - \gamma\xi_1^2 & -\gamma\xi_1\xi_2 & -\gamma\xi_1\xi_3 \\ -\gamma\xi_1\xi_2 & -|\xi|^2 - \gamma\xi_2^2 & -\gamma\xi_2\xi_3 \\ -\gamma\xi_1\xi_3 & -\gamma\xi_2\xi_3 & -|\xi|^2 - \gamma\xi_3^2 \end{vmatrix} = -(1+\gamma)\,|\xi|^6 .$$

Hence the equation (4.2.24) is elliptic if $\gamma \ne -1$. Moreover, the polynomial (in τ)

$$\det L^\circ(\xi + \tau\zeta) = -(1+\gamma)\,\big(|\xi|^2 + 2\tau\,\xi\cdot\zeta + \tau^2\,|\zeta|^2\big)^3$$

has exactly two different zeros

$$\tau_{1,2} = -|\zeta|^{-2}\,\big(\xi\cdot\zeta \pm i\,\sqrt{|\xi|^2\,|\zeta|^2 - (\xi\cdot\zeta)^2}\big)$$

if ξ and ζ are linear independent vectors in $\mathbb{R}^3$, $\gamma \ne -1$. One of them lies in the upper the other in the lower half-plane. Both zeros have the multiplicity 3. Thus, the operator L is properly elliptic.

We prove the ellipticity of the Dirichlet problem for the Lamé system. Note that every rotation $y = Ax$ of the coordinate system, where A is an orthogonal matrix, transforms the equation

$$\Delta_x\underline{\mathfrak{u}} + \gamma\,\mathrm{grad}_x\,\mathrm{div}_x\,\underline{\mathfrak{u}} = \underline{\mathfrak{f}}$$

into the equation

$$\Delta_y A\underline{\mathfrak{u}} + \gamma\,\mathrm{grad}_y\,\mathrm{div}_y\,A\underline{\mathfrak{u}} = A\underline{\mathfrak{f}}\,.$$

Therefore, it suffices to verify condition (ii) of Definition 4.2.2 for the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. In our example the equation $L^\circ(\eta, D_{x_3})\,\underline{\mathfrak{u}}(x_3) = 0$ has

the form
(4.2.25)

$$\begin{pmatrix} -|\eta|^2 - \gamma\eta_1^2 + \partial_{x_3}^2 & -\gamma\eta_1\eta_2 & i\gamma\eta_1\partial_{x_3} \\ -\gamma\eta_1\eta_2 & -|\eta|^2 - \gamma\eta_2^2 + \partial_{x_3}^2 & i\gamma\eta_2\partial_{x_3} \\ i\gamma\eta_1\partial_{x_3} & i\gamma\eta_2\partial_{x_3} & -|\eta|^2 + (1+\gamma)\partial_{x_3}^2 \end{pmatrix} \underline{\mathfrak{u}}(x_3) = 0.$$

The set $\mathcal{M}^+(\eta)$ of the stable solutions is spanned by the vector-functions $\underline{\mathfrak{u}}^{(j)} = \underline{\mathfrak{v}}^{(j)}\, e^{-|\eta|\, x_3}$, $j = 1, 2, 3$, where

$$\underline{\mathfrak{v}}^{(1)} = \begin{pmatrix} \eta_2 \\ -\eta_1 \\ 0 \end{pmatrix}, \ \underline{\mathfrak{v}}^{(2)} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ i|\eta| \end{pmatrix}, \ \underline{\mathfrak{v}}^{(3)} = \begin{pmatrix} \eta_1\, x_3 \\ \eta_2\, x_3 \\ i(2\gamma^{-1} + 1 + |\eta|\, x_3) \end{pmatrix}.$$

Here the vectors $\underline{\mathfrak{v}}^{(1)}$, $\underline{\mathfrak{v}}^{(2)}$, and $\underline{\mathfrak{v}}^{(3)}|_{x_3=0}$ are linear independent for $\gamma \neq -2$, $\eta \neq 0$. Consequently, the Dirichlet problem

$$\Delta\underline{\mathfrak{u}} + \gamma\, \mathrm{grad}\, \mathrm{div}\, \underline{\mathfrak{u}} = \underline{\mathfrak{f}} \ \text{ in } \Omega, \qquad \underline{\mathfrak{u}} = \underline{g} \text{ on } \partial\Omega$$

is elliptic for $\gamma \neq -1$, $\gamma \neq -2$.

In the same way, the ellipticity of the Neumann problem

$$\Delta\underline{\mathfrak{u}} + \gamma\, \mathrm{grad}\, \mathrm{div}\, \underline{\mathfrak{u}} = \underline{\mathfrak{f}} \ \text{ in } \Omega,$$

$$\sum_{j=1}^{3} \Big(\frac{\partial \mathfrak{u}_i}{\partial x_j} + \frac{\partial \mathfrak{u}_j}{\partial x_i}\Big)\, \nu_j + \nu_i\, (\gamma - 1)\, \mathrm{div}\, \underline{\mathfrak{u}} = g_i \text{ on } \partial\Omega, \ i = 1, 2, 3,$$

for $\gamma \neq -1$ can be verified.

By Theorem 4.2.4, both the operator

$$\mathcal{A}_D : W_2^l(\Omega)^3 \to W_2^{l-2}(\Omega)^3 \times W_2^{l-1/2}(\partial\Omega)^3$$

of the Dirichlet problem and the operator

$$\mathcal{A}_N : W_2^l(\Omega)^3 \to W_2^{l-2}(\Omega)^3 \times W_2^{l-3/2}(\partial\Omega)^3$$

of the Neumann problem are Fredholm for $l \geq 2$.

Example 2. We consider the *Stokes system* of hydrodynamics

$$L(D_x) \binom{\underline{\mathfrak{u}}}{\mathfrak{p}} = \begin{pmatrix} -\Delta\underline{\mathfrak{u}} + \mathrm{grad}\, \mathfrak{p} \\ \mathrm{div}\, \underline{\mathfrak{u}} \end{pmatrix} = \underline{\mathfrak{f}} \ \text{ in } \Omega, \tag{4.2.26}$$

where $\underline{\mathfrak{u}} = (\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$ denotes the velocity, $\mathfrak{p}$ the pressure, and Ω is a smooth bounded domain in $\mathbb{R}^3$.

For $s_1 = s_2 = s_3 = 0$, $s_4 = -1$, $t_1 = t_2 = t_3 = 2$, $t_4 = 1$ we have

$$L^\circ(\xi) = \begin{vmatrix} |\xi|^2 & 0 & 0 & i\xi_1 \\ 0 & |\xi|^2 & 0 & i\xi_2 \\ 0 & 0 & |\xi|^2 & i\xi_3 \\ i\xi_1 & i\xi_2 & i\xi_3 & 0 \end{vmatrix} = |\xi|^6.$$

Hence the equation (4.2.26) is elliptic and even properly elliptic (see Example 1).

The Stokes system (4.2.26) together with the Dirichlet boundary condition

$$\underline{\mathfrak{u}} = \underline{g} \quad \text{on } \partial\Omega,$$

or with the Neumann boundary condition

$$-p\,\nu_i + \sum_{j=1}^{3} \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\,\nu_j = g_i \quad \text{on } \partial\Omega,\ i = 1,2,3,$$

forms an elliptic boundary value problem.

4.3. Boundary value problems in the variational form

In this section we study the solutions of a boundary value problem given in the variational form. We show that the solution $u \in W_2^l(\Omega)$ of a variational problem $b(u,v) = (f,v)_\Omega$, where b is a certain sesquilinear form on $W_2^l(\Omega) \times W_2^{2m-l}(\Omega)$, can be understood as a generalized solution of a boundary value problem. Using the results of Chapter 3, we obtain a theorem on the solvability of the variational problem and regularity assertions for the variational solutions.

4.3.1. Introductory example. We consider the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{4.3.1}$$

The solution of this problem can be understood as the solution of the following variational problem:

Find a function $u \in W_2^1(\Omega)$ satisfying the equations

$$\int_\Omega \nabla u \cdot \nabla \overline{v}\, dx = (f,v)_\Omega \qquad \textit{for all } v \in \overset{\circ}{W}{}_2^1(\Omega), \tag{4.3.2}$$

$$u = g \quad \textit{on } \partial\Omega, \tag{4.3.3}$$

where $(\cdot,\cdot)_\Omega$ denotes the scalar product in $L_2(\Omega)$.

Problem (4.3.2), (4.3.3) is uniquely solvable in $W_2^1(\Omega)$ for arbitrary $f \in (\overset{\circ}{W}{}_2^1(\Omega))^*$, $g \in W_2^{1/2}(\partial\Omega)$. Indeed, if g is an arbitrary function in $W_2^{1/2}(\partial\Omega)$ then there exists a function $u^{(0)} \in W_2^1(\Omega)$ satisfying the boundary condition (4.3.3). Furthermore, by the Riesz representation theorem, there exists a function $u^{(1)} \in \overset{\circ}{W}{}_2^1(\Omega)$ uniquely determined by $u^{(0)}$ and f such that

$$\int_\Omega \nabla u^{(1)} \cdot \nabla \overline{v}\, dx = (f,v)_\Omega - \int_\Omega \nabla u^{(0)} \cdot \nabla \overline{v}\, dx \qquad \text{for all } v \in \overset{\circ}{W}{}_2^1(\Omega).$$

Then $u = u^{(0)} + u^{(1)}$ is the uniquely determined solution of the variational problem (4.3.2), (4.3.3).

Let $u \in W_2^1(\Omega)$ be a solution of problem (4.3.2), (4.3.3) and let $F \in W_2^1(\Omega)^*$ be an arbitrary functional such that

$$(F,v)_\Omega = (f,v)_\Omega \qquad \text{for all } v \in \overset{\circ}{W}{}_2^1(\Omega).$$

Then

$$(F,v)_\Omega - \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx = 0 \qquad \text{for all } v \in \overset{\circ}{W}{}_2^1(\Omega).$$

Hence (see Lemma 4.3.1 below) there exists a uniquely determined functional $u_1 \in W_2^{1/2}(\partial\Omega)^*$ such that

$$(F,v)_\Omega - \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx = (u_1, v)_{\partial\Omega} \quad \text{for all } v \in W_2^1(\Omega).$$

Consequently, the pair (u, u_1) satisfies the equations

$$\int_\Omega \nabla u \cdot \nabla \overline{v}\, dx + (u_1, v)_{\partial\Omega} = (F,v)_\Omega \qquad \text{for all } v \in W_2^1(\Omega),$$

$$u = g \quad \text{on } \partial\Omega.$$

This variational problem is equivalent to (4.3.2), (4.3.3).

Now we consider the generalized solution $(w, \phi_1, \phi_2) \in \tilde{W}_2^{1,2}(\Omega)$ of the problem

$$-\Delta w = F \quad \text{in } \Omega, \qquad w = g \quad \text{on } \partial\Omega.$$

Due to the formula

$$-\int_\Omega \Delta w \cdot \overline{v}\, dx = \int_\Omega \nabla w \cdot \nabla \overline{v}\, dx - i \int_{\partial\Omega} D_\nu w \cdot \overline{v}\, d\sigma,$$

this solution satisfies the equations

$$\int_\Omega \nabla w \cdot \nabla \overline{v}\, dx - i\,(\phi_2, v)_{\partial\Omega} = (F,v)_\Omega \qquad \text{for all } v \in W_2^1(\Omega),$$

$$w = \phi_1 = g \quad \text{on } \partial\Omega.$$

Thus, we have obtained a direct connection between the variational solution $u \in W_2^1(\Omega)$ and the generalized solution $(w, \phi_1, \phi_2) \in \tilde{W}_2^{1,2}(\Omega)$. In particular, it holds $w = u$, i.e., w does not depend on the choice of the extension F of the functional f.

4.3.2. Formulation of the variational problem. Let L be an elliptic differential operator of order $2m$ with smooth coefficients on $\overline{\Omega}$ given in the form

$$Lu = \sum_{|\alpha|\le l} \sum_{|\beta|\le 2m-l} D_x^\beta \left(a_{\alpha,\beta}(x)\, D_x^\alpha u\right),$$

where l is an arbitrary nonnegative integer not greater than $2m$. We denote the corresponding sesquilinear form by $a(\cdot,\cdot)$, i.e.,

$$a(u,v) = \int_\Omega \sum_{|\alpha|\le l} \sum_{|\beta|\le 2m-l} a_{\alpha,\beta}(x)\, D_x^\alpha u\, \overline{D_x^\beta v}\, dx\,. \tag{4.3.4}$$

Integrating by parts, we get $(Lu, v)_\Omega = a(u,v)$ for each u, $v \in C_0^\infty(\Omega)$.

Furthermore, let P and G be vectors of linear differential operators P_k, G_k ($k = 1, \dots, N$) with smooth coefficients on $\overline{\Omega}$, $\operatorname{ord} P_k \le l-1$, $\operatorname{ord} G_k \le 2m-l-1$. Then we define the sesquilinear form $b(\cdot,\cdot)$ on $W_2^l(\Omega) \times W_2^{2m-l}(\Omega)$ as follows:

$$b(u,v) = a(u,v) + \int_{\partial\Omega} (Pu, Gv)_{\mathbb{C}^N}\, d\sigma\,. \tag{4.3.5}$$

Moreover, let B and M be vectors of linear differential operators B_j ($j = 1, \dots, J$, $J \ge m-l$) and M_s ($s = 1, \dots, J+l-m$), $\operatorname{ord} B_j = \beta_j \le 2m-l-1$, $\operatorname{ord} M_s = \mu_s \le l-1$. We define the subspace $\mathcal{V}_B^{2m-l}(\Omega)$ of $W_2^{2m-l}(\Omega)$ as the set of all functions

$v \in W_2^{2m-l}(\Omega)$ such that $Bv = 0$ on $\partial\Omega$ and consider the following variational problem.

For a given functional f from the dual space $\mathcal{V}_B^{2m-l}(\Omega)^*$ of $\mathcal{V}_B^{2m-l}(\Omega)$ and given functions $g_s \in W_2^{l-\mu_s-1/2}(\partial\Omega)$, $s = 1, \dots, J+l-m$, find a function $u \in W_2^l(\Omega)$ satisfying the equation

$$b(u,v) = (f,v)_\Omega \qquad \textit{for each } v \in \mathcal{V}_B^{2m-l}(\Omega) \tag{4.3.6}$$

and the boundary condition $Mu = \underline{g}$ on $\partial\Omega$, i.e.,

$$M_s u = g_s \qquad \textit{on } \partial\Omega \quad \textit{for } s = 1, \dots, J+l-m. \tag{4.3.7}$$

The variational problem (4.3.6), (4.3.7) generates a linear operator $\mathcal{B} : u \to (f, \underline{g})$ which continuously maps $W_2^l(\Omega)$ into

$$\mathcal{V}_B^{2m-l}(\Omega)^* \times \prod_{s=1}^{J+l-m} W_2^{l-\mu_s-1/2}(\partial\Omega). \tag{4.3.8}$$

4.3.3. An equivalent formulation of the variational problem. It is natural to suppose that the boundary operators $B_1, \dots, B_J$ in the definition of the space $\mathcal{V}_B^{2m-l}(\Omega)$ are independent. More precisely, we assume that the following condition is satisfied:

(B) *For every vector-function $\underline{g} \in W_2^{2m-l-\underline{\beta}-1/2}(\partial\Omega)$ there exists a function $v \in W_2^{2m-l}(\Omega)$ such that $Bv = \underline{g}$ on $\partial\Omega$.*

Here $W_2^{2m-l-\underline{\beta}-1/2}(\partial\Omega)$ denotes the product of the spaces $W_2^{2m-l-\beta_j-1/2}(\partial\Omega)$, $j = 1, \dots, J$.

Condition (B) means that there exists a right inverse Λ to the operator

$$v \to Bv|_{\partial\Omega}$$

continuously mapping the space $W_2^{2m-l-\underline{\beta}-1/2}(\partial\Omega)$ into $W_2^{2m-l}(\Omega)$. This condition is satisfied, e.g, if the operators $B_1, \dots, B_J$ form a normal system on $\partial\Omega$ (see Definition 3.1.4). In particular, it follows from Condition (B) that $J \leq 2m - l$.

Let Λ be the above mentioned right inverse. Then for every $v \in W_2^{2m-l}(\Omega)$ the difference $v - \Lambda(Bv|_{\partial\Omega})$ is a function from $\mathcal{V}_B^{2m-l}(\Omega)$. Hence for every $f \in \mathcal{V}_B^{2m-l}(\Omega)^*$ we can define the functional $F \in W_2^{2m-l}(\Omega)^*$ by the equality

$$(F,v)_\Omega = \big(f\,,\, v - \Lambda(Bv|_{\partial\Omega})\big)_\Omega, \qquad v \in W_2^{2m-l}(\Omega). \tag{4.3.9}$$

Clearly, F coincides with f on the subset $\mathcal{V}_B^{2m-l}(\Omega)$ of $W_2^{2m-l}(\Omega)$, i.e., the functional (4.3.9) is an extension of f to the whole space $W_2^{2m-l}(\Omega)$.

LEMMA 4.3.1. *Let F be a linear and continuous functional on $W_2^{2m-l}(\Omega)$ satisfying the condition*

$$(F,v)_\Omega = 0 \qquad \textit{for each } v \in \mathcal{V}_B^{2m-l}(\Omega).$$

Then there exists a vector-function $\underline{u} \in W_2^{l-2m+\underline{\beta}+1/2}(\partial\Omega)$ such that

$$(F,v)_\Omega = (\underline{u}, Bv|_{\partial\Omega})_{\partial\Omega} \qquad \textit{for each } v \in W_2^{2m-l}(\Omega). \tag{4.3.10}$$

The vector-function $\underline{u}$ is uniquely determined by F.

Proof: From the assumption of the lemma it follows that

$$(F\,,\,v-\Lambda(Bv|_{\partial\Omega})\,)_\Omega=0$$

for each $v\in W_2^{2m-l}(\Omega)$. If we define the vector $\underline{u}$ by the equality

$$(\underline{u},\underline{\psi})_{\partial\Omega}=(F,\Lambda\underline{\psi})_\Omega\,,\qquad \underline{\psi}\in\prod_{j=1}^{J}W_2^{2m-l-\beta_j-1/2}(\partial\Omega),$$

we get (4.3.10). It remains to show the uniqueness of $\underline{u}$. Suppose that $(\underline{u},Bv|_{\partial\Omega})_{\partial\Omega}$ is equal to zero for each $v\in W_2^{2m-l}(\Omega)$. Then from Condition (B) it follows that $(\underline{u},\underline{g})_{\partial\Omega}=0$ for every vector $\underline{g}\in W_2^{2m-l-\underline{\beta}-1/2}(\partial\Omega)$. Consequently, we get $\underline{u}=0$. ■

Lemma 4.3.1 enables us to give an equivalent formulation of problem (4.3.6), (4.3.7) whose advantage is the absence of restrictions on the function v.

THEOREM 4.3.1. *Suppose that Condition (B) is satisfied. Then the function $u\in W_2^l(\Omega)$ is a solution of the variational problem (4.3.6), (4.3.7) if and only if there exists a vector $\underline{u}=(u_1,\dots,u_J)$ with $u_j\in W_2^{l-2m+\beta_j+1/2}(\partial\Omega)$ such that*

$$b(u,v)+(\underline{u},Bv|_{\partial\Omega})_{\partial\Omega}=(F,v)_\Omega\qquad \textit{for each } v\in W_2^{2m-l}(\Omega), \tag{4.3.11}$$

$$Mu=\underline{g}\qquad \textit{on } \partial\Omega, \tag{4.3.12}$$

where $F\in W_2^{2m-l}(\Omega)^*$ is an extension of the functional $f\in\mathcal{V}_B^{2m-l}(\Omega)^*$ to the whole space $W_2^{2m-l}(\Omega)$.

Proof: Obviously, (4.3.11), (4.3.12) implies (4.3.6), (4.3.7). We assume that u is a solution of the variational problem (4.3.6), (4.3.7) and $F\in W_2^{2m-l}(\Omega)^*$ is a functional which coincides with f on $\mathcal{V}_B^{2m-l}(\Omega)$. Then the functional $\Phi\in W_2^{2m-l}(\Omega)^*$ defined by the equality

$$(\Phi,v)_\Omega=(F,v)_\Omega-b(u,v)\,,\qquad v\in W_2^{2m-l}(\Omega)$$

is equal to zero on $\mathcal{V}_B^{2m-l}(\Omega)$. Hence by Lemma 4.3.1, there exists a vector $\underline{u}$ such that

$$(\Phi,v)_\Omega=(\underline{u},\,Bv|_{\partial\Omega}\,)_{\partial\Omega}$$

for each $v\in W_2^{2m-l}(\Omega)$. Therefore, $(u,\underline{u})$ is a solution of problem (4.3.11), (4.3.12). ■

Note that the vector-function $\underline{u}$ in Theorem 4.3.1 depends on the choice of the extension F, while u depends only on f an $\underline{g}$. Furthermore, the vector $\underline{u}=(u_1,\dots,u_J)$ satisfies the estimate

$$\sum_{j=1}^{J}\|u_j\|_{W_2^{l-2m+\beta_j+1/2}(\partial\Omega)}\le c\Big(\|u\|_{W_2^l(\Omega)}+\|F\|_{W_2^{2m-l}(\Omega)^*}\Big). \tag{4.3.13}$$

Indeed, let $\underline{w}$ be an arbitrary vector-function in $W_2^{2m-l-\underline{\beta}-1/2}(\partial\Omega)$. Inserting $v=\Lambda\underline{w}$ into (4.3.11), where Λ is a continuous right inverse to the operator $v\to Bv|_{\partial\Omega}$, we obtain

$$(\underline{u},\underline{w})_{\partial\Omega}=(F,\Lambda\underline{w})_\Omega-b(u,\Lambda\underline{w})\,.$$

This implies (4.3.13).

4.3.4. The boundary value problem corresponding to the variational problem. We consider the solution $(u, \underline{u})$ of the variational problem (4.3.11), (4.3.12) and suppose that $u \in W_2^{2m}(\Omega)$, while $\underline{u}$ belongs to the space $W_2^{\underline{\beta}+1/2}(\partial\Omega)$. Furthermore, we assume that the functional $F \in W_2^{2m-l}(\Omega)^*$ on the right side of (4.3.12) is an element of the space $D_2^{0,2m-l}(\Omega)$ (see Section 3.3), i.e., F has the form

$$(F, v)_\Omega = (f, v)_\Omega + (\underline{h}\,,\, \mathcal{D}^{(2m-l)} v|_{\partial\Omega}\,)_{\partial\Omega}\,, \qquad v \in W_2^{2m-l}(\Omega), \tag{4.3.14}$$

where $f \in L_2(\Omega)$ and $\underline{h} = (h_1, \dots, h_{2m-l})$ is a vector of functions $h_k \in W_2^{k-1/2}(\partial\Omega)$, $k = 1, \dots, 2m-l$. Here, again $\mathcal{D}^{(2m-l)}$ denotes the vector with the components $1, D_\nu, \dots, D_\nu^{2m-l-1}$. Analogously to formula (3.1.7), it can be shown that there exists a vector S of differential operators S_k, $k = 1, \dots, 2m-l$, with smooth coefficients on $\overline{\Omega}$, $\operatorname{ord} S_k \le 2m - k$, such that

$$a(u, v) = \int\limits_\Omega Lu \cdot \overline{v}\, dx + \int\limits_{\partial\Omega} \big(Su|_{\partial\Omega},\, \mathcal{D}^{(2m-l)} v|_{\partial\Omega}\big)_{\mathbb{C}^{2m-l}}\, d\sigma \tag{4.3.15}$$

for each $u \in W_2^{2m}(\Omega)$, $v \in W_2^{2m-l}(\Omega)$.

LEMMA 4.3.2. *There is the representation*

$$Su|_{\partial\Omega} = Q \cdot \mathcal{D}^{(2m)} u|_{\partial\Omega}\,, \tag{4.3.16}$$

where

$$Q = \begin{pmatrix} Q_{1,1} & \cdots & Q_{1,l+1} & Q_{1,l+2} & \cdots & Q_{1,2m-1} & Q_{1,2m} \\ Q_{2,1} & \cdots & Q_{2,l+1} & Q_{2,l+2} & \cdots & Q_{2,2m-1} & 0 \\ & & & & & & \\ Q_{2m-l,1} & \cdots & Q_{2m-l,l+1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a trapezium matrix of tangential differential operators $Q_{k,j}$ on $\partial\Omega$, $\operatorname{ord} Q_{k,j} \le 2m + 1 - k - j$, $Q_{k,j} \equiv 0$ *if* $k + j > 2m + 1$. *If L is elliptic in $\overline{\Omega}$, then the matrix elements $Q_{1,2m}, Q_{2,2m-1}, \dots, Q_{2m-l,l+1}$ are functions which do not vanish on $\partial\Omega$.*

Proof: Formula (4.3.15) holds if we integrate by parts in (4.3.4). Here the assertion of the lemma on the orders of the operators $Q_{k,j}$ follows from the inequality $\operatorname{ord} S_k \le 2m - k$. Analogously to (4.3.15), we obtain the formula

$$a(u, v) = \int\limits_\Omega u \cdot \overline{L^+ v}\, dx + \int\limits_{\partial\Omega} \big(\mathcal{D}^{(l)} u\,,\, R \cdot \mathcal{D}^{(2m)} v\big)_{\mathbb{C}^l}\, d\sigma\,, \tag{4.3.17}$$

where R is a trapezium matrix of tangential differential operators $R_{k,j}$ on $\partial\Omega$, $1 \le k \le l$, $1 \le j \le 2m$, $\operatorname{ord} R_{k,j} \le 2m + 1 - k - j$, $R_{k,j} \equiv 0$ if $k + j > 2m + 1$. Hence by (4.3.15), (4.3.17), we get

$$\int\limits_\Omega \big(Lu \cdot \overline{v} - u \cdot \overline{L^+ v}\big)\, dx$$

$$= \int\limits_{\partial\Omega} \Big(\big(\mathcal{D}^{(l)} u,\, R\mathcal{D}^{(2m)} v\big)_{\mathbb{C}^l} - \big(\mathcal{D}^{(2m)} u,\, Q^+ \mathcal{D}^{(2m-l)} v\big)_{\mathbb{C}^{2m}}\Big)\, d\sigma.$$

However, by (3.1.7) and Remark 3.1.2, we have

$$\int\limits_\Omega \big(Lu \cdot \overline{v} - u \cdot \overline{L^+ v}\big)\, dx = \int\limits_{\partial\Omega} \big(\mathcal{D}^{(2m)} u,\, T\,\mathcal{D}^{(2m)} v\big)_{\mathbb{C}^{2m}}\, d\sigma$$

with a triangular matrix T of tangential differential operators $T_{k,j}$, $1 \le k,\ j \le 2m$, where the elements $T_{1,2m}, T_{2,2m-1}, \dots, T_{2m,1}$ are functions not vanishing on $\partial\Omega$. Since the matrix T is uniquely determined, we have $Q_{k,j} = T^+_{j,k}$ for $j \ge l+1$. This proves the lemma. ∎

Furthermore, there exist matrices

$$C = \big(C_{j,k}\big)_{\substack{1\le j\le J\\ 1\le k\le 2m-l}} \qquad \text{and} \qquad H = \big(H_{j,k}\big)_{\substack{1\le j\le N\\ 1\le k\le 2m-l}}$$

of tangential differential operators $C_{j,k}$ and $H_{j,k}$ on $\partial\Omega$, $\operatorname{ord} C_{j,k} \le \beta_j - k + 1$, $\operatorname{ord} H_{j,k} \le 2m - l - k$, $C_{j,k} \equiv 0$ if $k > \beta_j + 1$, such that

$$Bv|_{\partial\Omega} = C \cdot \mathcal{D}^{(2m-l)} v\,|_{\partial\Omega} \tag{4.3.18}$$

and

$$Gv|_{\partial\Omega} = H \cdot \mathcal{D}^{(2m-l)} v\,|_{\partial\Omega}\,. \tag{4.3.19}$$

Hence by means of (4.3.15), we get

$$\begin{aligned} &b(u,v) + \int\limits_{\partial\Omega} (\underline{u}, Bv)_{\mathbb{C}^J}\, d\sigma \\ &= \int\limits_{\Omega} Lu\cdot \overline{v}\, dx + \int\limits_{\partial\Omega} \Big((S + H^+P)u + C^+\underline{u}\,,\ \mathcal{D}^{(2m-l)} v\Big)_{\mathbb{C}^{2m-l}}\, d\sigma. \end{aligned} \tag{4.3.20}$$

This leads to the following result.

LEMMA 4.3.3. *Suppose that Condition (B) is satisfied and*

$$(u, \underline{u}) \in W_2^{2m}(\Omega) \times \prod_{j=1}^{J} W_2^{\beta_j+1/2}(\partial\Omega)$$

is a solution of the variational problem (4.3.11), (4.3.12), where F is a functional of the form (4.3.14) with $f \in L_2(\Omega)$, $h_k \in W_2^{k-1/2}(\partial\Omega)$, and $\underline{g}$ is a vector-function from $W_2^{2m-\underline{\mu}-1/2}(\partial\Omega)$. Then $(u,\underline{u})$ is a solution of the boundary value problem

$$Lu = f \qquad \text{in } \Omega, \tag{4.3.21}$$

$$Mu = \underline{g}\,, \qquad (S + H^+P)u + C^+\underline{u} = \underline{h} \qquad \text{on } \partial\Omega\,. \tag{4.3.22}$$

In problem (4.3.21), (4.3.22) two groups of boundary conditions occur. The conditions $Mu = \underline{g}$ on $\partial\Omega$ are called *stable* boundary conditions, whereas the conditions $(S + H^+P)u + C^+\underline{u} = \underline{h}$ are called *natural* boundary conditions. In contrast to the natural boundary conditions, the stable conditions contain only derivatives of u up to order $l-1$.

By Theorem 3.2.1, the operator $\mathcal{A}$ of the boundary value problem (4.3.21), (4.3.22) continuously maps the space

$$\tilde{W}_2^{l,2m}(\Omega) \times \prod_{j=1}^{J} W_2^{l-2m+\beta_j+1/2}(\partial\Omega) \tag{4.3.23}$$

into

$$W_2^{2m-l}(\Omega)^* \times \prod_{s=1}^{J+l-m} W_2^{l-\mu_s-1/2}(\partial\Omega) \times \prod_{k=1}^{2m-l} W_2^{l-2m+k-1/2}(\partial\Omega). \tag{4.3.24}$$

We study the relations between the solution $(u, \underline{\phi}, \underline{u})$ of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (f, \underline{g}, \underline{h})$ and the solution of the variational problem (4.3.11), (4.3.12). For this we need a more precise description of the operator $\mathcal{A}$ on the space (4.3.23). Let $(u, \underline{\phi}, \underline{u})$ be an arbitrary element of the space (4.3.23) and $(f, \underline{g}, \underline{h}) = \mathcal{A}\,(u, \underline{\phi}, \underline{u})$. By (4.3.15) and (4.3.16) the functional $f = L(u, \underline{\phi}) \in W_2^{2m-l}(\Omega)^*$ is defined by the equality

$$(4.3.25) \qquad (f, v)_\Omega = a(u, v) - \big(Q\underline{\phi}, \mathcal{D}^{(2m-l)}v|_{\partial\Omega}\big)_{\partial\Omega}, \qquad v \in W_2^{2m-l}(\Omega).$$

Furthermore, according to (4.3.16), the boundary conditions (4.3.22) can be written in the form

$$(4.3.26) \qquad Mu|_{\partial\Omega} = \underline{g}, \qquad Q\underline{\phi} + H^+Pu|_{\partial\Omega} + C^+\underline{u} = \underline{h}\,.$$

This leads to the following result.

LEMMA 4.3.4. *Suppose that Condition (B) is satisfied and $(u, \underline{\phi}, \underline{u})$, $(f, \underline{g}, \underline{h})$ are elements of the spaces (4.3.23) and (4.3.24), respectively. Furthermore, let the functional $F \in W_2^{2m-l}(\Omega)^*$ be defined as follows:*

$$(4.3.27) \qquad (F, v)_\Omega = (f, v)_\Omega + \big(\underline{h}, \mathcal{D}^{(2m-l)}v|_{\partial\Omega}\big)_{\partial\Omega}, \qquad v \in W_2^{2m-l}(\Omega)^*.$$

Then $(u, \underline{\phi}, \underline{u})$ is a solution of the equation

$$\mathcal{A}\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g}, \underline{h})$$

if and only if $(u, \underline{u})$ is a solution of the variational problem (4.3.11), (4.3.12) and $\underline{\phi} = (\phi_1, \dots, \phi_{2m})$ satisfies the equations

$$(4.3.28) \qquad \phi_j = D_\nu^{j-1}u|_{\partial\Omega} \;\; \textit{for } j = 1, \dots, l, \quad Q\underline{\phi} + H^+Pu|_{\partial\Omega} + C^+\underline{u} = \underline{h}.$$

Proof: Let $(u, \underline{\phi}, \underline{u})$ be a solution of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (f, \underline{g}, \underline{h})$. Then (4.3.28) is satisfied and, by (4.3.18), (4.3.19), (4.3.25), (4.3.26), we have

$$\begin{aligned}
(f, v)_\Omega &= a(u, v) - \big(Q\underline{\phi},\, \mathcal{D}^{(2m-l)}v\big)_{\partial\Omega} \\
&= a(u, v) - \big(\underline{h} - H^+Pu|_{\partial\Omega} - C^+\underline{u}\,,\, \mathcal{D}^{(2m-l)}v\big)_{\partial\Omega} \\
&= a(u, v) - \big(\underline{h},\, \mathcal{D}^{(2m-l)}v\big)_{\partial\Omega} + (Pu, Gv)_{\partial\Omega} + (\underline{u}, Bv)_{\partial\Omega} \\
&= b(u, v) + (\underline{u}, Bv)_{\partial\Omega} - \big(\underline{h}, \mathcal{D}^{(2m-l)}v\big)_{\partial\Omega}\,.
\end{aligned}$$

Consequently, $(u, \underline{u})$ is a solution of the variational problem (4.3.11), (4.3.12).

Analogously, (4.3.11), (4.3.12), and (4.3.28) imply (4.3.25) and (4.3.26). This proves the lemma. ■

Note that by Lemma 4.3.2, the system of the equations (4.3.28) has a unique solution $\underline{\phi}$ for arbitrary given u, $\underline{u}$, and $\underline{h}$ if the operator L is elliptic.

4.3.5. Solvability of the variational problem. Now we investigate the solvability of the variational problem (4.3.6), (4.3.7). For this we use the relations between the variational problem (4.3.11), (4.3.12) and the corresponding boundary value problem which were described in the previous lemma.

LEMMA 4.3.5. *Suppose that Condition (B) is satisfied and that the boundary value problem (4.3.21), (4.3.22) is elliptic. Then the following assertions are valid.*

1) *The pair* $(u,\underline{u}) \in W_2^l(\Omega) \times W_2^{l-2m+\underline{\beta}+1/2}(\partial\Omega)$ *is a solution of the homogeneous variational problem*

$$b(u,v) + (\underline{u}, Bv|_{\partial\Omega})_{\partial\Omega} = 0\,, \qquad v \in W_2^{2m-l}(\Omega), \tag{4.3.29}$$

$$Mu = 0 \qquad \text{on } \partial\Omega \tag{4.3.30}$$

if and only if $u \in C^\infty(\overline{\Omega})$, $\underline{u} \in C^\infty(\partial\Omega)^J$ *and* $(u,\underline{u})$ *is a solution of the homogeneous boundary value problem (4.3.21), (4.3.22).*

2) *The variational problem (4.3.11), (4.3.12) is solvable in the space* $W_2^l(\Omega) \times W_2^{l-2m+\underline{\beta}+1/2}(\partial\Omega)$ *for given* $F \in W_2^{2m-l}(\Omega)^*$, $\underline{g} \in W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$ *if and only if* $(F,\underline{g},0)$ *belongs to the range of the operator* $\mathcal{A}$ *mapping (4.3.23) into (4.3.24).*

Proof: 1) Let $(u,\underline{u})$ be a solution of the problem (4.3.29), (4.3.30). Then by Lemma 4.3.4, there exists a vector $\underline{\phi}$ such that $(u,\underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$ and $\mathcal{A}(u,\underline{\phi},\underline{u}) = 0$. Consequently, by Theorem 3.4.1, u and $\underline{u}$ are infinitely differentiable and satisfy the homogeneous equations (4.3.21), (4.3.22). On the other hand, by Lemma 4.3.4, every solution $(u,\underline{u}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^J$ of the homogeneous boundary value problem (4.3.21), (4.3.22) is a solution of problem (4.3.29), (4.3.30).

2) We assume that there exists a solution $(u,\underline{u})$ of the variational problem (4.3.11), (4.3.12). If $\underline{\phi}$ is the vector satisfying the equations (4.3.28) with $\underline{h} = 0$, then by Lemma 4.3.4, $(u,\underline{\phi},\underline{u})$ is a solution of the equation $\mathcal{A}(u,\underline{\phi},\underline{u}) = (F,\underline{g},0)$, i.e., $(F,\underline{g},0)$ belongs to the range of the operator $\mathcal{A}$.

Conversely, if $(F,\underline{g},0)$ belongs to the range of the operator $\mathcal{A}$, i.e., there exists a solution $(u,\underline{\phi},\underline{u})$ of the equation $\mathcal{A}(u,\underline{\phi},\underline{u}) = (f,\underline{g},\underline{h})$ in the space (4.3.23), then by Lemma 4.3.4), $(u,\underline{u})$ is a solution of the variational problem (4.3.11), (4.3.12) with $f = F$. The proof is complete. ∎

By Theorem 3.4.2, the element $(F,\underline{g},0)$ belongs to the range of $\mathcal{A}$ if and only if

$$(F,v)_\Omega + (\underline{g},\underline{v})_{\partial\Omega} = 0$$

for each solution $(v,\underline{v},\underline{w})$ of the equation $\mathcal{A}^+(v,\underline{v},\underline{w}) = 0$, where $\mathcal{A}^+$ denotes the operator of the formally adjoint problem to (4.3.21), (4.3.22) which acts from

$$W_2^{2m}(\Omega) \times \prod_{s=1}^{J+l-m} W_2^{\mu_s+1/2}(\partial\Omega) \times \prod_{k=1}^{2m-l} W_2^{2m-k+1/2}(\partial\Omega) \tag{4.3.31}$$

into

$$L_2(\Omega) \times \prod_{j=1}^{2m} W_2^{j-1/2}(\partial\Omega) \times \prod_{j=1}^{J} W_2^{2m-\beta_j-1/2}(\partial\Omega).$$

In order to give a condition for the existence of a solution of the starting variational problem (4.3.6), (4.3.7), we need the following lemma.

LEMMA 4.3.6. *Suppose that* L *is elliptic in* $\overline{\Omega}$ *and* $(v,\underline{v},\underline{w})$, *where* $v \in W_2^{2m}(\Omega)$, $\underline{v} \in W_2^{\underline{\mu}+1/2}(\partial\Omega)$, $\underline{w} \in \prod W_2^{2m-k+1/2}(\partial\Omega)$, *is a solution of the homogeneous formally adjoint problem*

$$\mathcal{A}^+(v,\underline{v},\underline{w}) = 0.$$

Then $(v, \underline{v})$ *is a solution of the problem*

$$b(u,v) + \big(Mu|_{\partial\Omega}\,, \underline{v}\big)_{\partial\Omega} = 0 \quad \textit{for all } u \in W_2^l(\Omega), \tag{4.3.32}$$

$$Bv = 0 \quad \textit{on } \partial\Omega \tag{4.3.33}$$

and $\underline{w}$ *coincides with the vector* $\mathcal{D}^{(2m-l)}v|_{\partial\Omega}$.

Proof: If $(v, \underline{v}, \underline{w})$ is a solution of the homogeneous formally adjoint problem to (4.3.21), (4.3.22), then

$$(Lu, v)_\Omega + (Mu, \underline{v})_{\partial\Omega} + \big((S + H^+P)u + C^+\underline{u},\, \underline{w}\big)_{\partial\Omega} = 0 \tag{4.3.34}$$

for all $u \in W_2^{2m}(\Omega)$, $\underline{u} \in W_2^{\beta+1/2}(\partial\Omega)$. By (4.3.15)–(4.3.17), we have

$$(Lu, v)_\Omega = (u, L^+v)_\Omega + \big(\mathcal{D}^{(l)}u,\, R\mathcal{D}^{(2m)}v\big)_{\partial\Omega} - \big(\mathcal{D}^{(2m)}u,\, Q^+\mathcal{D}^{(2m-l)}v\big)_{\partial\Omega}$$

and

$$(Su, \underline{w})_{\partial\Omega} = \big(\mathcal{D}^{(2m)}u,\, Q^+\underline{w}\big)_{\partial\Omega}\,.$$

Hence (4.3.34) yields

$$\begin{aligned}(u, L^+v)_\Omega &+ \big(\mathcal{D}^{(2m)}u\,,\, Q^+(\underline{w} - \mathcal{D}^{(2m-l)}v)\big)_{\partial\Omega} + (\underline{u}, C\underline{w})_{\partial\Omega} \\ &+ \big(\mathcal{D}^{(l)}u\,,\, R\mathcal{D}^{(2m)}v\big)_{\partial\Omega} + (Mu,\, \underline{v})_{\partial\Omega} + (Pu,\, H\underline{w})_{\partial\Omega} = 0.\end{aligned} \tag{4.3.35}$$

This implies $L^+v = 0$ and $C\underline{w} = 0$. Furthermore, since the last three expressions on the left side of (4.3.35) contain only derivatives of u up to order $l-1$, the last $2m - l$ components of the vector $Q^+\big(\underline{w} - \mathcal{D}^{(2m-l)}v|_{\partial\Omega}\big)$ vanish. Hence we have

$$\sum_{k=1}^{2m-l} Q^+_{k,j}\,\big(w_k - D_\nu^{k-1}v|_{\partial\Omega}\big) = 0 \quad \text{for } j = l+1, \dots, 2m,$$

and Lemma 4.3.2 implies $\underline{w} = \mathcal{D}^{(2m-l)}v|_{\partial\Omega}$. Using (4.3.18), we get $Bv = 0$ on $\partial\Omega$. Finally, from (4.3.35) and (4.3.17) we conclude that

$$a(u,v) + \big(Mu|_{\partial\Omega}\,,\, \underline{v}\big)_{\partial\Omega} + \big(Pu|_{\partial\Omega}\,,\, H\mathcal{D}^{(2m-l)}v|_{\partial\Omega}\big)_{\partial\Omega} = 0$$

for arbitrary $u \in W_2^{2m}(\Omega)$. Thus, we have shown the validity of (4.3.32). The prove is complete. ■

Problem (4.3.32), (4.3.33) is said to be *formally adjoint* to the variational problem (4.3.11), (4.3.12).

THEOREM 4.3.2. *Suppose condition (B) is satisfied. Then the operator* $\mathcal{B}$ *of the variational problem (4.3.6), (4.3.7) is a Fredholm operator from* $W_2^l(\Omega)$ *into (4.3.8) if and only if the boundary value problem (4.3.21), (4.3.22) is elliptic.*

Proof: 1) First we assume that the boundary value problem (4.3.21), (4.3.22) is elliptic. Then the kernels of $\mathcal{A}$ and $\mathcal{A}^+$ are finite-dimensional. By Theorem 4.3.1 and Lemma 4.3.5, the kernel of $\mathcal{B}$ consists of all functions $u \in C^\infty(\overline{\Omega})$ such that $(u, \underline{u})$ is a solution of the homogeneous problem (4.3.21), (4.3.22) for at least one vector $\underline{u} \in C^\infty(\partial\Omega)^J$. Consequently, the kernel of $\mathcal{B}$ has also finite dimension.

Furthermore, from Theorem 4.3.1, Lemma 4.3.5, and Lemma 4.3.6 it follows that the variational problem (4.3.6), (4.3.7) has a solution $u \in W_2^l(\Omega)$ if and only if

$$(f, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} = 0$$

for all $(v, \underline{v}, \underline{w}) \in \ker \mathcal{A}^+$. Hence the range of $\mathcal{B}$ is closed and the cokernel of $\mathcal{B}$ is finite-dimensional.

2) Now we assume that $\mathcal{B}$ is a Fredholm operator from $W_2^l(\Omega)$ into (4.3.8). To prove the ellipticity of problem (4.3.21), (4.3.22), it suffices to show that the operator $\mathcal{A}$ is Fredholm (see Theorem 3.2.4, Lemma 3.4.1).

First we show that the operator $\mathcal{A}$ has a finite-dimensional kernel. Let $(u, \underline{\phi}, \underline{u})$ be an element of the kernel of $\mathcal{A}$. Then by Lemma 4.3.4, $(u, \underline{u})$ is a solution of the homogeneous variational problem (4.3.29), (4.3.30) and $\underline{\phi}$ is uniquely determined by the equations (4.3.28) with $\underline{h} = 0$. Hence $u \in \ker \mathcal{B}$, whereas the vector-function $\underline{u}$ is uniquely determined by the equality

$$(\underline{u}, Bv|_{\partial\Omega})_{\partial\Omega} = -b(u, v), \qquad v \in W_2^{2m-l}(\Omega)$$

(see Lemma 4.3.1). Therefore, we get dim ker $\mathcal{A} \leq$ dim ker $\mathcal{B} < \infty$.

Now we show that the range of $\mathcal{A}$ is closed in the space (4.3.24) and the cokernel of $\mathcal{A}$ is finite-dimensional. Let $(F, \underline{g}, \underline{h})$ be an arbitrary element of the space (4.3.24). We define the functional $f \in \mathcal{V}_B^{2m-l}(\Omega)^*$ by the equality

$$(f, v)_\Omega = (F, v)_\Omega + \Big(\underline{h}, \mathcal{D}^{(2m-l)} v|_{\partial\Omega}\Big)_{\partial\Omega}, \qquad v \in \mathcal{V}_B^{(2m-l)}(\Omega).$$

Since $\mathcal{B}$ is a Fredholm operator, there exists a finite-dimensional subspace $\mathcal{X}$ of

$$\mathcal{V}_B^{2m-l}(\Omega) \times \prod_{s=1}^{J+l-m} W_2^{-l+\mu_s+1/2}(\partial\Omega)$$

such that the variational problem (4.3.6), (4.3.7) is solvable in $W_2^l(\Omega)$ if and only if $(f, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} = 0$ for each $(v, \underline{v}) \in \mathcal{X}$. From Theorem 4.3.1 it follows that the variational problem

$$b(u, v) + (\underline{u}, Bv|_{\partial\Omega})_{\partial\Omega} = (F, v)_\Omega + (\underline{h}, \mathcal{D}^{(2m-l)} v|_{\partial\Omega})_{\partial\Omega}, \quad v \in W_2^{2m-l}(\Omega),$$
$$Mu = \underline{g} \qquad \text{on } \partial\Omega$$

is solvable if and only if

$$(F, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} + \Big(\underline{h}, \mathcal{D}^{(2m-l)} v|_{\partial\Omega}\Big)_{\partial\Omega} = 0 \tag{4.3.36}$$

for each $(v, \underline{v}) \in \mathcal{X}$. Consequently, by Lemma 4.3.4, every element $(F, \underline{g}, \underline{h})$ of the space (4.3.24) satisfying condition (4.3.36) is an element of the range of $\mathcal{A}$. Hence the operator $\mathcal{A}$ is Fredholm. The proof is complete. ■

If the boundary value problem (4.3.21), (4.3.22) is elliptic, then according to Lemmas 4.3.5 and 4.3.6, we obtain the following solvability conditions for the variational problems (4.3.6), (4.3.7) and (4.3.11), (4.3.12). Problem (4.3.11), (4.3.12) is solvable if and only if

$$(F, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} = 0$$

for all solutions $(v, \underline{v}) \in W_2^{2m-l}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$ of problem (4.3.32), (4.3.33). Analogously, problem (4.3.6), (4.3.7) is solvable if and only if f and $\underline{g}$ satisfy the condition

$$(f, v)_\Omega + (\underline{g}, \underline{v})_{\partial\Omega} = 0$$

for all solutions $(v, \underline{v}) \in W_2^{2m-l}(\Omega) \times W_2^{-l+\underline{\mu}+1/2}(\partial\Omega)$ of problem (4.3.32), (4.3.33).

4.3.6. A regularity assertion for the solution of the variational problem. Using the regularity assertion for solutions of elliptic boundary problems and the connection between the variational problem (4.3.6), (4.3.7) and the corresponding boundary value problem, we can prove the following theorem.

THEOREM 4.3.3. *Let $u \in W_2^l(\Omega)$ be a solution of problem (4.3.6), (4.3.7). We assume that the corresponding boundary value problem (4.3.21), (4.3.22) is elliptic and that there exists a functional $F \in D_2^{l_1-2m,2m-l}(\Omega)$, $l_1 \geq l$, such that $(F,v)_\Omega = (f,v)_\Omega$ for each $v \in \mathcal{V}_B^{2m-l}(\Omega)$. Furthermore, let the components g_s of the vector-function $\underline{g} = (g_1, \dots, g_{J+l-m})$ be functions from the spaces $W_2^{l_1-\mu_s-1/2}(\partial\Omega)$. Then $u \in W_2^{l_1}(\Omega)$ and*

(4.3.37)

$$\|u\|_{W_2^{l_1}(\Omega)} \leq c\Big(\|F\|_{D_2^{l_1-2m,2m-l}(\Omega)} + \sum_{s=1}^{J+l-m} \|g_s\|_{W_2^{l_1-\mu_s-1/2}(\partial\Omega)} + \|u\|_{W_2^l(\Omega)}\Big)$$

with a constant c independent of u.

Proof: By our assumption on f, there exist a vector $\underline{h} = (h_1, \dots, h_{2m-l})$ with $h_k \in W_2^{l_1+k-2m-1/2}(\partial\Omega)$ and a function (or a functional) $\Phi \in \tilde{W}_2^{l_1-2m}(\Omega)$ such that

$$(f,v)_\Omega = (\Phi,v)_\Omega + \Big(\underline{h}, \mathcal{D}^{(2m-l)}v|_{\partial\Omega}\Big)_{\partial\Omega}$$

for each $v \in \mathcal{V}_B^{2m-l}(\Omega)$. According to Theorem 4.3.1, there exists a vector $\underline{u} \in W_2^{l-2m+\underline{\beta}+1/2}(\partial\Omega)$ such that $(u, \underline{u})$ is a solution of the variational problem (4.3.11), (4.3.12), where $F \in W_2^{2m-l}(\Omega)^*$ is the functional

$$(F,v)_\Omega = (\Phi,v)_\Omega + \Big(\underline{h}, \mathcal{D}^{(2m-l)}v|_{\partial\Omega}\Big)_{\partial\Omega}, \quad v \in W_2^{2m-l}(\Omega).$$

Furthermore, by Lemma 4.3.4, there exists a uniquely determined vector-function $\underline{\phi}$ such that $(u, \underline{\phi}) \in \tilde{W}_2^{l,2m}(\Omega)$ and $(u, \underline{\phi}, \underline{u})$ is a solution of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (\Phi, \underline{g}, \underline{h})$. From Theorem 3.2.3 it follows that $(u, \underline{\phi}) \in \tilde{W}_2^{l_1,2m}(\Omega)$ and the vector-function $\underline{u}$ belongs to the space $W_2^{l_1-2m+\underline{\beta}+1/2}(\partial\Omega)$. Moreover, an analogous inequality to (3.2.27) holds. Since $\underline{\phi}$ is uniquely determined by (4.3.28), the norm of $\underline{\phi}$ on the right side of this inequality can be estimated by the corresponding norms of u, $\underline{u}$, and $\underline{h}$. Using the estimate (4.3.13) for the vector $\underline{u}$, we obtain the inequality (4.3.37). ■

4.3.7. Examples.

Example 1: The Dirichlet problem for a properly elliptic operator. Let L be a properly elliptic differential operator given in the form

$$Lu = \sum_{|\alpha|,|\beta|\leq m} D_x^\beta\big(a_{\alpha,\beta}(x)\, D_x^\alpha u\big)$$

with the corresponding sesquilinear form

$$a(u,v) = \int\limits_\Omega \sum_{|\alpha|,|\beta|\le m} a_{\alpha,\beta}(x)\, D_x^\alpha u\, \overline{D_x^\beta v}\, dx.$$

We consider the following variational problem for the function $u \in W_2^m(\Omega)$:

$$a(u,v) = (f,v)_\Omega \quad \text{for all } v \in \overset{\circ}{W}{}_2^m(\Omega), \tag{4.3.38}$$

$$\mathcal{D}^{(m)} u = \underline{g} \quad \text{on } \partial\Omega, \tag{4.3.39}$$

where f is a given linear and continuous functional on $\overset{\circ}{W}{}_2^m(\Omega)$ and $\underline{g}$ is a vector with components $g_k \in W_2^{m-k+1/2}(\partial\Omega)$, $k = 1, \ldots, m$.

Let $F \in W_2^m(\Omega)^*$ be an arbitrary extension of the functional f. Then $u \in W_2^m(\Omega)$ is a solution of problem (4.3.38), (4.3.39) if and only if there exists a vector-function $\underline{u} = (u_1, \ldots, u_m)$, where $u_k \in W_2^{-m+k-1/2}(\partial\Omega)$, such that $(u, \underline{u})$ is a solution of the problem

$$a(u,v) + \big(\underline{u}, \mathcal{D}^{(m)} v\big)_{\partial\Omega} = (F,v)_\Omega \quad \text{for all } v \in W_2^m(\Omega), \tag{4.3.40}$$

$$\mathcal{D}^{(m)} u = \underline{g} \quad \text{on } \partial\Omega. \tag{4.3.41}$$

Now we consider the generalized solution $(u, \underline{\phi}) \in \tilde{W}_2^{m,2m}(\Omega)$ of the problem

$$L\,(u, \underline{\phi}) = F, \qquad \mathcal{D}^{(m)} u|_{\partial\Omega} = \underline{g}. \tag{4.3.42}$$

By (4.3.15), (4.3.16), we have

$$a(u,v) = \int\limits_\Omega Lu \cdot \overline{v}\, dx + \int\limits_{\partial\Omega} \big(Q \cdot \mathcal{D}^{(2m)} u, \mathcal{D}^{(m)} v\big)_{\mathbb{C}^m}\, d\sigma$$

for $u, v \in C^\infty(\overline{\Omega})$. Hence the solution $(u, \underline{\phi}) \in \tilde{W}_2^{m,2m}(\Omega)$ of problem (4.3.42) satisfies the equation

$$a(u,v) - \big(Q \cdot \underline{\phi}, \mathcal{D}^{(m)} v\big)_{\partial\Omega} = (F,v)_\Omega \quad \text{for all } v \in W_2^m(\Omega) \tag{4.3.43}$$

and the boundary condition (4.3.41). This leads to the following results:

1) *Let f be an arbitrary functional on $\overset{\circ}{W}{}_2^m(\Omega)$ and let $F \in (W_2^m(\Omega))^*$ be an extension of f. Then $u \in W_2^m(\Omega)$ is a solution of the variational problem (4.3.38), (4.3.39) if and only if there exists a vector-function $\underline{\phi} = (\phi_1, \ldots, \phi_{2m})$ such that $\phi_k \in W_2^{m-k+1/2}(\partial\Omega)$ for $k = 1, \ldots, 2m$, $\phi_k = D_\nu^{k-1} u|_{\partial\Omega}$ for $k = 1, \ldots, m$, and $(u, \underline{\phi})$ is a solution of the boundary value problem (4.3.42).*

2) *The operator $\mathcal{B} : W_2^m(\Omega) \to (\overset{\circ}{W}{}_2^m(\Omega))^* \times \prod_{k=1}^m W_2^{m-k+1/2}(\partial\Omega)$ is Fredholm.*

3) *If $u \in W_2^m(\Omega)$ is a solution of the variational problem (4.3.38), (4.3.39) and $f \in W_2^k(\Omega)$, $k > -m$, then $u \in W_2^{m+k}(\Omega)$.*

Example 2: The Neumann problem for the Laplace operator. We consider the variational problem

$$\int\limits_\Omega \nabla u \cdot \nabla \overline{v}\, dx = (F,v)_\Omega \quad \text{for all } v \in W_2^1(\Omega), \tag{4.3.44}$$

where F is a given linear and continuous functional on $W_2^1(\Omega)$ and $u \in W_2^1(\Omega)$ has to be found.

Let (u, ϕ_1, ϕ_2) be an element of the space $\tilde{W}_2^{1,2}(\Omega)$, i.e., $u \in W_2^1(\Omega)$, $\phi_1 = u|_{\partial\Omega}$, $\phi_2 \in W_2^{-1/2}(\partial\Omega)$. We denote the extension of the operator $-\Delta$ to the space $\tilde{W}_2^{1,2}(\Omega)$ also by $-\Delta$. It maps the space $\tilde{W}_2^{1,2}(\Omega)$ into $W_2^1(\Omega)^*$. The functional $f = -\Delta(u, \phi_1, \phi_2)$ is determined by the equality

$$(f, v)_\Omega = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx - i\,(\phi_2, v)_{\partial\Omega}\,, \qquad v \in W_2^1(\Omega).$$

Furthermore, we denote the operator of the Neumann problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega \tag{4.3.45}$$

and its extension to the space $\tilde{W}_2^{1,2}(\Omega)$ by $\mathcal{A}$. Then the following relation between the solution $u \in W_2^1(\Omega)$ of the variational problem (4.3.44) and the generalized solution $(u, \phi_1, \phi_2) \in \tilde{W}_2^{1,2}(\Omega)$ of the boundary value problem (4.3.45) holds:

The function $u \in W_2^1(\Omega)$ is a solution of problem (4.3.44) if and only if for arbitrary $\phi_2 \in W_2^{-1/2}(\partial\Omega)$ the tuple $\big(u, u|_{\partial\Omega}, \phi_2\big)$ is a solution of the equation

$$\mathcal{A}\left(u, u|_{\partial\Omega}, \phi_2\right) = (f, i\phi_2)\,,$$

where the functional $f \in W_2^1(\Omega)^*$ is defined by the equality

$$(f, v)_\Omega = (F, v)_\Omega - i\,(\phi_2, v)_{\partial\Omega}\,, \qquad v \in W_2^1(\Omega).$$

In other words, if F is a linear and continuous functional on $W_2^1(\Omega)$ having the representation

$$(F, v)_\Omega = (f, v)_\Omega + (g, v)_{\partial\Omega}\,, \qquad v \in W_2^1(\Omega), \tag{4.3.46}$$

where $f \in W_2^1(\Omega)^*$, $g \in W_2^{-1/2}(\partial\Omega)$, then $\big(u, u|_{\partial\Omega}, g\big)$ is a solution of the equation

$$\mathcal{A}\left(u, u|_{\partial\Omega}, -ig\right) = (f, g).$$

Moreover, the following assertions follow from the ellipticity of the boundary value problem (4.3.45):

1) *The operator $\mathcal{B}: W_2^1(\Omega) \to W_2^1(\Omega)^*$ of problem (4.3.44) is Fredholm.*

2) *If $u \in W_2^1(\Omega)$ is a solution of problem (4.3.44) and F is a linear and continuous functional on $W_2^1(\Omega)^*$ having the form (4.3.46), where $f \in W_2^k(\Omega)$, $g \in W_2^{k+1/2}(\partial\Omega)$, $k \geq 0$, then $u \in W_2^{k+2}(\Omega)$ and u is a classical solution of problem (4.3.45).*

Note that the kernel of the operator $\mathcal{B}$ is the set of the functions $u = const.$, while the range of the operator $\mathcal{B}$ consists of all $F \in W_2^1(\Omega)^*$ such that $(F, 1)_\Omega = 0$.

Example 3: V-elliptic problems. Let $a(\cdot,\cdot)$ be the sesquilinear form (4.3.4) with $l = m$, i.e.,

$$a(u, v) = \int_\Omega \sum_{|\alpha|,|\beta| \leq m} a_{\alpha,\beta}(x)\, D_x^\alpha u\, \overline{D_x^\beta v}\, dx\,. \tag{4.3.47}$$

Furthermore, let B a vector of differential operators B_k, $k = 1, \dots, J$, $\operatorname{ord} B_k = \beta_k \leq m - 1$. We suppose again that Condition (B) is satisfied and consider the following variational problem:

For given $f \in \mathcal{V}_B^{2m-l}(\Omega)^*$ *and* $\underline{g} \in W_2^{m-\underline{\beta}-1/2}(\partial\Omega)$ *find a function* $u \in W_2^m(\Omega)$ *satisfying the equation*

$$a(u,v) = (f,v)_\Omega \qquad \text{for all } v \in \mathcal{V}_B^m(\Omega). \tag{4.3.48}$$

and the boundary condition

$$Bu = \underline{g} \quad \text{on } \partial\Omega. \tag{4.3.49}$$

Here again $\mathcal{V}_B^m(\Omega)$ denotes the set of all $u \in W_2^m(\Omega)$ satisfying the boundary condition $Bu = 0$ on $\partial\Omega$.

DEFINITION 4.3.1. The sesquilinear form (4.3.47) is said to be *V-coercive* if there exist real constants c_0 and c_1, $c_1 > 0$, such that *Gårding's inequality*

$$\operatorname{Re} a(u,u) + c_0 \|u\|^2_{L_2(\Omega)} \ge c_1 \|u\|^2_{W_2^m(\Omega)} \tag{4.3.50}$$

is satisfied for all $u \in \mathcal{V}_B^m(\Omega)$. If the inequality (4.3.50) with $c_0 = 0$ is valid, then $a(\cdot,\cdot)$ is said to be *V-elliptic.*

We denote the operator

$$W_2^m(\Omega) \ni u \to (f,\underline{g}) \in \mathcal{V}_B^m(\Omega)^* \times W_2^{m-\underline{\beta}-1/2}(\partial\Omega)$$

corresponding to the variational problem (4.3.48), (4.3.49) by $\mathcal{B}$.

For the proof of the Fredholm property of the operator $\mathcal{B}$ we need the following generalization of the Riesz theorem which is due to P. D. Lax and A. Milgram [**125**].

LEMMA 4.3.7. *Let $\mathcal{X}$ be a Hilbert space and $B(\cdot,\cdot)$ a sesquilinear form on $\mathcal{X}\times\mathcal{X}$ satisfying the conditions*

$$\begin{aligned} |b(u,v)| &\le c_1 \|u\|_{\mathcal{X}} \cdot \|v\|_{\mathcal{X}} \qquad \text{for all } u,v \in \mathcal{X}, \\ |b(u,u)| &\ge c_0 \|u\|^2_{\mathcal{X}} \qquad \text{for all } u \in \mathcal{X}, \end{aligned}$$

where c_0, c_1 are positive constants independent of u and v. Then for every linear and continuous functional $f \in \mathcal{X}^*$ there exists a uniquely determined element $u \in \mathcal{X}$ such that

$$b(u,v) = f(\overline{v}) \qquad \text{for all } v \in \mathcal{X}.$$

Here the norm of u in $\mathcal{X}$ does not exceed $c_0^{-1} \|f\|_{\mathcal{X}^*}$.

THEOREM 4.3.4. *If the sesquilinear form $a(\cdot,\cdot)$ is V-coercive, then the operator $\mathcal{B}$ is Fredholm and the index of $\mathcal{B}$ is equal to zero.*

Proof: We define the operator $\mathcal{B}_0 : \mathcal{V}_B^m(\Omega) \to \mathcal{V}_B^m(\Omega)^*$ by the equality

$$(\mathcal{B}_0 u, v)_\Omega = a(u,v) \quad \text{for all } u,v \in \mathcal{V}_B^m(\Omega).$$

Furthermore, let I be the identity operator mapping $\mathcal{V}_B^m(\Omega)$ into $\mathcal{V}_B^m(\Omega)^*$. Then

$$\big((\mathcal{B}_0 + c_0 I)u, v\big)_\Omega = a(u,v) + c_0\,(u,v)_\Omega \quad \text{for } u,v \in \mathcal{V}_B^m(\Omega).$$

According to (4.3.50), the sesquilinear form $a(\cdot,\cdot) + c_0\,(\cdot,\cdot)_\Omega$ is V-elliptic. Consequently, by Lax-Milgram's lemma, the operator $\mathcal{B}_0 + c_0\, I$ is invertible. Since the operator I is compact, we conclude that $\mathcal{B}_0$ is Fredholm and $\operatorname{ind}\mathcal{B}_0 = 0$ (see, e.g., [**92**]): Analogously, the adjoint operator $\mathcal{B}_0^* : \mathcal{V}_B^m(\Omega) \to \mathcal{V}_B^m(\Omega)^*$ defined by the equality

$$(u, \mathcal{B}_0^* v)_\Omega = a(u,v)\,, \quad u,v \in \mathcal{V}_B^m(\Omega)$$

is Fredholm. The range of the operator $\mathcal{B}_0$ consists of all $f \in \mathcal{V}_B^m(\Omega)^*$ satisfying the condition

$$(f, v)_\Omega = 0 \quad \text{for all } v \in \ker \mathcal{B}_0^*.$$

Here, $\dim \ker \mathcal{B}_0^* = \dim \operatorname{coker} \mathcal{B}_0 = \dim \ker \mathcal{B}_0$. Since the kernels of $\mathcal{B}$ and $\mathcal{B}_0$ coincide, we get $\dim \ker \mathcal{B} = \dim \ker \mathcal{B}_0 = \dim \ker \mathcal{B}_0^*$. We consider the range of the operator $\mathcal{B}$. Let $f \in \mathcal{V}_B^m(\Omega)^*$, $\underline{g} \in W_2^{m-\underline{\beta}-1/2}(\partial\Omega)$, and $u^{(0)} = \Lambda \underline{g}$, where $\Lambda : W_2^{m-\underline{\beta}-1/2}(\partial\Omega) \to W_2^m(\Omega)$ is a continuous right inverse to the operator B. Then problem (4.3.48), (4.3.49) can be written in the form

$$a(u - u^{(0)}, v) = (f, v)_\Omega - a(u^{(0)}, v) \quad \text{for all } v \in \mathcal{V}_B^m(\Omega), \quad B(u - u^{(0)}) = 0 \quad \text{on } \partial\Omega.$$

This problem is solvable if and only if the functional $v \to (f, v)_\Omega - a(u^{(0)}, v)$ belongs to the range of the operator $\mathcal{B}_0$, i.e., if

$$(f, v)_\Omega - a(u^{(0)}, v) = 0 \quad \text{for all } v \in \ker \mathcal{B}_0^*.$$

Hence the range of $\mathcal{B}$ is closed and $\dim \operatorname{coker} \mathcal{B} = \dim \ker \mathcal{B}_0^*$. This proves the theorem. ■

REMARK 4.3.1. If the sesquilinear form $a(\cdot, \cdot)$ is V-elliptic, then the variational problem (4.3.48), (4.3.49) is uniquely solvable.

REMARK 4.3.2. If the sesquilinear form $a(\cdot, \cdot)$ is symmetric and V-elliptic, then the solution $u \in \mathcal{V}_B^m(\Omega)$ of problem (4.3.48), (4.3.49) coincides with the uniquely determined solution of the variational problem

$$a(u, u) - 2 \operatorname{Re} (f, u)_\Omega \to \min .$$

As a consequence of Theorems 4.3.2–4.3.4, the following regularity assertion for the solutions of problem (4.3.48), (4.3.49) holds.

THEOREM 4.3.5. *Let $u \in W_2^m(\Omega)$ be a solution of problem (4.3.48), (4.3.49). We suppose that the sesquilinear form (4.3.47) is V-coercive, $\underline{g}$ belongs to the space $W_2^{l-\underline{\beta}-1/2}(\partial\Omega)$, $l \geq m$, and there exists a functional $F \in D_2^{l-2m,m}(\Omega)$ such that $(F, v)_\Omega = (f, v)_\Omega$ for all $v \in \mathcal{V}_B^m(\Omega)$. Then $u \in W_2^l(\Omega)$.*

Finally, we give a necessary and sufficient condition for the V-coercivity. Let

$$L^\circ(x, \xi) = \sum_{|\alpha|, |\beta| = m} a_{\alpha,\beta}(x)\, \xi^{\alpha+\beta}$$

be the principal part of the polynomial $L(x, \xi)$. The differential operator $L(x, D_x)$ is said to be *strongly elliptic* in $\overline{\Omega}$ if there exists a positive constant c such that

$$\operatorname{Re} L^\circ(x, \xi) \geq c\, |\xi|^{2m} \quad \text{for all } x \in \overline{\Omega},\ \xi \in \mathbb{R}^n.$$

For arbitrary $x^{(0)} \in \partial\Omega$ and arbitrary vectors ξ' tangential to $\partial\Omega$ in $x^{(0)}$ we denote by $\mathcal{M}_+(x^{(0)}, \xi')$ the set of the stable solutions of the equation

$$L^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})\, D_t\big)\, u(t) = 0 \quad \text{for } t > 0$$

satisfying the condition $B^\circ\big(x^{(0)}, \xi' + \nu(x^{(0)})\, D_t\big) u|_{t=0} = 0$.

For the proof of the following result we refer to [**70**].

The following assertions are equivalent:

1) *The sesquilinear form (4.3.47) is V-coercive.*
2) *The operator $L(x, D_x)$ is strongly elliptic in $\overline{\Omega}$. Moreover, for all $x^{(0)} \in \partial\Omega$, $\xi' \in \mathbb{R}^n$ tangential to $\partial\Omega$ in $x^{(0)}$, and $u \in \mathcal{M}_+(x^{(0)}, \xi')$ the expression*

$$\int_0^\infty \sum_{|\alpha|,|\beta|=m} a_{\alpha,\beta}(x^{(0)}) \left(\xi + \nu(x^{(0)})\, D_t\right)^\alpha u \cdot \overline{\left(\xi + \nu(x^{(0)})\, D_t\right)^\beta u}\, dt$$

is positive.

4.4. Further results

4.4.1. Solutions of elliptic boundary value problems in Sobolev spaces of real order. For arbitrary real l let $W_2^l(\mathbb{R}^n)$ be the closure of the set $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{W_2^l(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^l\, |\hat{u}(\xi)|^2\, d\xi \Big)^{1/2},$$

where $\hat{u}$ denotes the Fourier transform of the function u. The space $W_2^l(\Omega)$ is defined as the restriction of $W_2^l(\mathbb{R}^n)$ to the domain Ω. Using a sufficiently fine open covering $\{\mathcal{U}_j\}$ of $\partial\Omega$ and homeomorphisms κ_j from $\mathcal{U}_j$ onto open subsets $\Omega_j \subset \mathbb{R}^{n-1}$, we define the Sobolev space $W_2^l(\partial\Omega)$ as the set of all functions u on $\partial\Omega$ such that the function $x \to u(\kappa_j^{-1}(x))$ belongs the space $W_2^l(\Omega_j)$ for every index j. Note that $W_2^{l-1/2}(\partial\Omega)$ coincides with the space of traces of functions from $W_2^l(\Omega)$ on the boundary $\partial\Omega$ if $l > 1/2$. Furthermore, let $\tilde{W}_2^{l,k}(\Omega)$ be the closure of the set

$$\left\{(u, \underline{\phi}) \in C_0^\infty(\overline{\Omega}) \times C_0^\infty(\partial\Omega)^k \; : \; \underline{\phi} = \left(u|_{\partial\Omega}\,, D_\nu u|_{\partial\Omega}\,, \ldots\,, D_\nu^{k-1} u|_{\partial\Omega}\right)\right\}$$

with respect to the norm (3.2.3).

Then the operator $\mathcal{A}$ of the boundary value problem (3.1.1), (3.1.2) realizes a continuous mapping

$$\tilde{W}_2^{l,2m}(\mathbb{R}) \times W_2^{l+\underline{\tau}-1/2}(\partial\Omega) \to \tilde{W}_2^{l-2m,0}(\mathbb{R}) \times W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$$

for arbitrary real l, except for the numbers $\frac{1}{2}, \frac{3}{2}, \ldots, 2m - \frac{1}{2}$ (cf. Remark 1.3.2). In the same way as it was done in Chapter 3 for integer l, it can be shown that this operator is Fredholm if and only if problem (3.1.1), (3.1.2) is elliptic. For boundary value problems without unknowns on the boundary we refer to the books of J.-L. Lions, E. Magenes [**126**], H. Triebel [**242**], and Ya. A. Roĭtberg [**206**].

4.4.2. Estimates in L_p and Hölder spaces. The assertions of the first part of this book (a priori estimate, regularity assertions, Fredholm property of the operator of the boundary value problem) are also valid in L_p Sobolev spaces W_p^l and Hölder spaces $C^{l,\alpha}$. We formulate the main results for the Sobolev spaces.

Let l be a nonnegative integer and let p be a real number, $p > 1$. Then $W_p^l(\Omega)$ is defined as the the space of all functions on Ω such that

$$\|u\|_{W_p^l(\Omega)} = \Big(\int_\Omega \sum_{|\alpha|\le l} |D_x^\alpha u(x)|^p\, dx \Big)^{1/p}.$$

By $W_p^{l-1/p}(\partial\Omega)$ we denote the trace space for $W_p^l(\Omega)$, $l \ge 1$.

We consider the boundary value problem (4.2.5), (4.2.6) in a bounded domain Ω with smooth boundary $\partial\Omega$. In accordance with the notation in Section 4.2, we denote the products of the spaces $W_p^{l+t_i}(\Omega)$ and $W_p^{l-s_i}(\Omega)$, $i=1,\dots,N$, by $W_p^{l+\underline{t}}(\Omega)$ and $W_p^{l-\underline{s}}(\Omega)$, respectively. Analogous notation is used for products of the spaces $W_p^{l+\tau_j-1/p}(\partial\Omega)$ and $W_p^{l-\sigma_k-1/p}(\partial\Omega)$.

THEOREM 4.4.1. *Suppose that the boundary value problem (4.2.5), (4.2.6) is elliptic.*

1) *Then the operator*

$$\mathcal{A}\,:\; W_p^{l+\underline{t}}(\Omega)\times W_p^{l+\underline{\tau}-1/2}(\partial\Omega)\to W_p^{l-\underline{s}}(\Omega)\times W_p^{l-\underline{\sigma}-1/p}(\partial\Omega) \tag{4.4.1}$$

is Fredholm for $l\ge 0$, $l>\max\sigma_k$. *Every solution* $(\underline{\mathrm{u}},\underline{u})\in W_p^{l+\underline{t}}(\Omega)\times W_p^{l+\underline{\tau}-1/2}(\partial\Omega)$ *of this problem satisfies the estimate*

$$\|\underline{\mathrm{u}}\|_{W_p^{l+\underline{t}}(\Omega)}+\|\underline{u}\|_{W_p^{l+\underline{\tau}-1/2}(\partial\Omega)} \le c\Big(\|\underline{\mathrm{f}}\|_{W_p^{l-\underline{s}}(\Omega)}+\|\underline{g}\|_{W_p^{l-\underline{\sigma}-1/p}(\partial\Omega)} + \|\underline{\mathrm{u}}\|_{L(\Omega)^N}+\|\underline{u}\|_{L(\partial\Omega)^J}\Big) \tag{4.4.2}$$

with a constant c independent of $\underline{\mathrm{u}}$, $\underline{u}$.

2) *If* $(\underline{\mathrm{u}},\underline{u})\in W_p^{l+\underline{t}}(\Omega)\times W_p^{l+\underline{\tau}-1/2}(\partial\Omega)$ *is a solution of problem (4.2.5), (4.2.6) with vector-functions* $\underline{\mathrm{f}}\in W_q^{l'-\underline{s}}(\Omega)$, $\underline{g}\in W_q^{l'-\underline{\sigma}-1/2}(\partial\Omega)$ *on the right-hand sides,* $l,l'>\max(\sigma_1,\dots,\sigma_{m+J})$, *then* $\underline{\mathrm{u}}\in W_q^{l'+\underline{t}}(\Omega)$ *and* $\underline{u}\in W_p^{l+\underline{\tau}-1/2}\partial\Omega)$.

We refer to the papers of S. Agmon, A. Douglis, L. Nirenberg [**7, 8**] and V. A. Solonnikov [**236**], where the estimate (4.4.2) and the analogous estimate in the class of the Hölder spaces $C^{l,\alpha}$ was proved for the case $J=0$. Furthermore, V. A. Solonnikov [**236**] and L. R. Volevich [**251**] proved the Fredholm property of the operator (4.4.1). A Fredholm theory for pseudodifferential boundary value problems in L_p Sobolev spaces was established in the paper [**82**] of G. Grubb. As in the case $p=2$ let $\tilde{W}_p^{l,k}(\Omega)$ for nonnegative integer l and k be the set of all pairs $(u,\underline{\phi})$ such that

$$u\in W_p^l(\Omega),\qquad \underline{\phi}=(\phi_1,\dots,\phi_k)\in\prod_{j=1}^k W_p^{l-j+1/2}(\partial\Omega)\ \text{ and}$$
$$\phi_j=D_\nu^{j-1}u|_{\partial\Omega}\ \text{for}\ j=0,\dots,\min(l,k).$$

In the case $l\le 0$ we set $\tilde{W}_p^{l,k}(\Omega)=W_{p'}^{-l}(\Omega)^*\times\prod_{j=1}^k W_p^{l-j+1/2}(\partial\Omega)$, where $p'=p/(p-1)$. In the same way as it was done in Sections 3.2 and 4.2, the operator of the boundary value problem (4.2.5), (4.2.6) can be extended to a continuous operator

$$\tilde{W}_p^{l+\underline{t},\sigma_0+\underline{t}}(\Omega)\times W_p^{l+\underline{\tau}-1/2}(\partial\Omega)\to\tilde{W}_p^{l-\underline{s},\sigma_0-\underline{s}}(\Omega)\times W_p^{l-\underline{\sigma}-1/p}(\partial\Omega)$$

where $\sigma_0=\max(0,\sigma_1+1,\dots,\sigma_{m+J}+1)$, $\tilde{W}_p^{l+\underline{t},\sigma_0+\underline{t}}(\Omega)$ denotes the product of the space $\tilde{W}_p^{l+t_i,\sigma_0+t_i}(\Omega)$, and $\tilde{W}_p^{l-\underline{s},\sigma_0-\underline{s}}(\Omega)$ denotes the product of the spaces $\tilde{W}_p^{l-s_i,\sigma_0-s_i}(\Omega)$, $i=1,\dots,N$. Ya. A. Roĭtberg and Z. G. Sheftel' [**208, 204, 206**] have shown (for the case $J=0$) that this operator is Fredholm for arbitrary l if the boundary value problem (4.2.5), (4.2.6) is elliptic.

Finally, we mention the following result of R. S. Strichartz [**241**] concerning solutions of elliptic equations in Hardy spaces.

Assume a neighbourhood of $\partial\Omega$ can be coordinatized by $\partial\Omega \times [0,1]$, where $\partial\Omega$ is identified with $t = 0$. Then the solution u of the Dirichlet problem

$$Lu = 0 \quad \text{in } \Omega, \qquad D_\nu^{k-1} u = g_k \quad \text{on } \partial\Omega,\ k = 1, \dots, m,$$

satisfies the condition

$$\|u(\cdot,t)\|_{L_p(\partial\Omega)} \le c \qquad \text{for } 0 < t < \infty$$

and for fixed p, $1 < p \le \infty$, if and only if $g_1 \in L_p(\partial\Omega)$ and g_k belongs to the Besov space $B_{p,\infty}^{-k+1}(\partial\Omega)$ for $k = 2, \dots, m$.

4.4.3. The Miranda-Agmon maximum principle. It is well-known that every harmonic function satisfies the maximum principle

$$\max_{x\in\overline{\Omega}} |u(x)| \le \max_{x\in\partial\Omega} |u(x)| \,.$$

If u is a solution of the strongly elliptic equation $Lu = 0$ of order $2m$ in the smooth domain Ω, then the following inequality with a constant c independent of u is valid:

$$\|u\|_{C^{m-1}(\overline{\Omega})} \le c\Big(\sum_{k=1}^{m} \|D_\nu^{k-1}u\|_{C^{m-k}(\partial\Omega)} + \|u\|_{L(\Omega)}\Big) \,.$$

The last term on the right can be omitted if the Dirichlet problem for the equation $Lu = 0$ in Ω is uniquely solvable. This generalization of the classical maximum principle was proved by C. Miranda [**167, 168**] for the case $n = 2$ and by S. Agmon [**4**] for the higher-dimensional case. B.-W. Schulze [**220, 221**] obtained analogous C^k estimates for solutions of strongly elliptic systems and for more general boundary conditions $D_\nu^{\mu_k} u = g_k$ on $\partial\Omega$, $k = 1, \dots, m$, where $\mu_k \le 2m - 1$.

4.4.4. Pointwise estimates of Green's functions. In Section 3.5 we proved the existence of Green or generalized Green functions for elliptic boundary value problems. According to Theorem 3.4.1, Lemma 3.2.4, the $\tilde{W}_2^{l,2m}(\Omega)$-norm (with respect to both variables), $l < 2m - n/2$, of the Green function $G(x,y)$ is bounded. However, many applications require pointwise estimates of Green functions.

We consider the boundary value problem

$$Lu = f \quad \text{in } \Omega, \quad B_k u = g_k \quad \text{on } \partial\Omega, \quad k = 1, \dots, m, \tag{4.4.3}$$

where L is a differential operator of order $2m$ with smooth coefficients and B_k are differential operators of order $\mu_k < 2m$ with smooth coefficients which form a normal system on $\partial\Omega$, and suppose that this problem is elliptic. Then by Theorem 3.5.3, every solution u of problem (4.4.3) with smooth functions f, g_k on the right-hand sides is given by the formula

$$u(y) = \int_\Omega G(y,x)\, f(x)\, dx + \sum_{k=1}^{m+J} \int_{\partial\Omega} \overline{B'_{m+k}(x,D_x)} G(y,x) \cdot g_k(x)\, dx + \sum_{s=1}^{d'} c_s\, u^{(s)}(y) \,,$$

where $\{u^{(s)}\}_{s=1,\dots,d'}$ is a basis in the space of the solutions of the homogeneous problem (4.4.3) and B'_{m+k} are the differential operators which occur in the Green formula (3.1.17). Yu. P. Krasovskiĭ [**117**] and V. A. Solonnikov [**237**] proved that the Green function $G(x,y)$ satisfies the following estimates:

$$\begin{aligned} \big|D_x^\alpha\, D_x^\beta\, G(x,y)\big| &\le \big(|x-y|^{2m-n-|\alpha|-|\gamma|} + 1\big) \quad \text{if } 2m - n \ne |\alpha| + |\gamma|, \\ \big|D_x^\alpha\, D_x^\beta\, G(x,y)\big| &\le \big(\log|x-y| + 1\big) \quad \text{if } 2m - n = |\alpha| + |\gamma|. \end{aligned}$$

In [**237**] pointwise estimates of the Green functions were also obtained for boundary value problems for elliptic systems of differential equations.

4.4.5. Assumptions on the coefficients and on the boundary. In Chapters 2–4 it was always assumed that the coefficients of the differential operators are of the class C^∞. Much less restrictive assumptions are necessary if we deal with solutions of finite smoothness (for older statements of such type see [**7, 8, 236**]).

In fact, for the localization argument applied in Section 2.3 only uniform ellipticity of the problem, locally small oscillation of the coefficients in the principal parts of the differential operators together with some Sobolev multiplier type restrictions on the derivatives of these coefficients and the coefficients of the lower order terms are sufficient.

Let the multiplier space $M(W_p^m(\Omega) \to W_p^l(\Omega))$ be the class of functions γ such that $\gamma u \in W_p^l(\Omega)$ for all $u \in W_p^m(\Omega)$. In the case $m = l$ we use the notation $MW_p^l(\Omega)$. The usefulness of the multipliers in the theory of partial differential equations can be seen by the example of the Schrödinger operator $-\Delta+\gamma(x)$, where γ is a function on Ω. A trivial argument shows that this operator continuously maps $W_p^l(\Omega)$ into $W_p^{l-2}(\Omega)$, $l \geq 2$, if and only if $\gamma \in M(W_p^l(\Omega) \to W_p^{l-2}(\Omega))$.

We consider the differential operators

$$L(x, D_x) = \sum_{|\alpha|\leq 2m} a_\alpha(x)\, D_x^\alpha\,, \quad B_k(x, D_x) = \sum_{|\alpha|\leq \mu_k} b_{k;\alpha}(x)\, D_x^\alpha\,, \quad k = 1, \dots, m.$$

We assume that Ω is a bounded domain in $\mathbb{R}^n$ and that for any point of the boundary $\partial\Omega$ there exists a neighbourhood $\mathcal{U}$, a local Cartesian coordinate system $y = (y_1, \dots, y_n)$, and a Lipschitz function φ such that $\mathcal{U} \cap \Omega = \mathcal{U} \cap \{x : y_n > \varphi(y_1, \dots, y_{n-1})\}$ and

$$\|\nabla\varphi\|_{MW_p^{l-1-1/p}(\mathbb{R}^{n-1})} \leq \delta, \tag{4.4.4}$$

where δ is a constant and l is an integer not less than $2m$. The following theorem due to V. G. Mazya and T. O. Shaposhnikova [**157**] guarantees the Fredholm property of the operator of the elliptic boundary value problem

$$Lu = f \quad \text{in } \Omega, \quad B_k u = g_k \quad \text{on } \partial\Omega, \quad k = 1, \dots, m, \tag{4.4.5}$$

under mild conditions on the coefficients and on the boundary of the domain Ω.

THEOREM 4.4.2. *Suppose that the following conditions are satisfied:*

(i) *For any neighbourhood $\mathcal{U} \subset \mathbb{R}^n$ there exist operators*

$$L^{\mathcal{U}}(D_x) = \sum_{\alpha=2m} a_\alpha^{\mathcal{U}}\, D_x^\alpha\,, \qquad B_k^{\mathcal{U}}(D_x) = \sum_{\alpha=\mu_k} b_{k;\alpha}^{\mathcal{U}}\, D_x^\alpha$$

with constant coefficients such that $\mathcal{A}^{\mathcal{U}} = \left(\mathcal{L}^{\mathcal{U}}, \mathcal{B}_k^{\mathcal{U}}\right)$ is the operator of an elliptic boundary value problem in the half-space $\{x : y_n \geq 0\}$.

(ii) *For the coefficients a_α the inequality*

$$\sum_{|\alpha|=2m} \|a_\alpha - a_\alpha^{\mathcal{U}}\|_{L_\infty(\mathcal{U}\cap\Omega)} + \sum_{|\alpha|\leq 2m} \mathrm{ess}\|a_\alpha\|_{M(W_p^{l-|\alpha|}(\Omega)\to W_p^{l-2m}(\Omega))} \leq \delta \tag{4.4.6}$$

is valid, where l is an integer, $l \geq 2m$, and $1 < p < \infty$. A similar inequality with $2m$ replaced by μ_k is valid for the coefficients $b_{k;\alpha}$. Here the constant δ in (4.4.4), (4.4.6) is assumed to be small in comparison with the norms of the inverse operators $\mathcal{A}^{\mathcal{U}}$ for all $\mathcal{U}$.

Then the operator

$$\mathcal{A} : W_p^l(\Omega) \to W_p^{l-2m}(\Omega) \times \prod_{k=1}^{m} W_p^{l-\mu_k-1/p}(\partial\Omega)$$

of problem (4.4.5) is Fredholm. The smallness condition on δ can not be omitted.

The conditions stated in terms of the multipliers can be reformulated in analytical form. In this way, various sufficient analytic conditions for the Fredholm property of the operator $\mathcal{A}$ can be given (see [**157, 158**]. In particular, condition (4.4.4) becomes $\partial\Omega \in W_p^{l-1/p}$ if $p(l-1) > n$.

As it was shown in papers of F. Chiarenza, M. Frasca, P. Longo [**46**]-[**48**] and G. Di Fazio [**60**], the requirement (4.4.6) of small local oscillation is not necessary for the L_p regularity. It suffices to assume that the coefficients of the principal parts belong to the class *VMO* of functions of vanishing mean oscillation defined as follows.

DEFINITION 4.4.1. [**91, 212**] The locally integrable function f is in the space *BMO* if

$$\|f\|_* \stackrel{def}{=} \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < +\infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$ and

$$f|_B = \frac{1}{|B|} \int_B f(x)\, dx$$

is the average of f in B. If, moreover,

$$\sup_{\rho \le r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_{B_\rho}| dx \to 0 \qquad \text{as } r \to 0,$$

where this time the supremum is taken over all balls with radius $\rho \le r$, then we say that f is in *VMO*.

The basis of the just mentioned results of F. Chiarenza, M. Frasca, P. Longo G. Di Fazio is a deep L_p estimate for the commutator of a singular integral operator and a function in *VMO* which was obtained by R. Coifman, R. Rochberg, and G. Weiss [**49**]

In the case of second order elliptic operators with real coefficients the restrictions on the coefficients ensuring different regularity properties can be relaxed even further. We mention the De Giorgi [**57**]-Nash [**176**] and Krylov-Safonov [**120**] Hölder regularity results for divergence and nondivergence elliptic operators with bounded measurable coefficients (see also [**122, 75, 50, 133**]). There exist counterexamples showing that for higher order elliptic equations additional restrictions to the coefficients are necessary. Here we refer to the papers of E. De Giorgi [**58**] and V. G. Maz'ya [**132**].

4.5. Notes

1. In 1953 Ya. B. Lopatinskiĭ [**127**] (for general boundary value problems) and Z. Ya. Shapiro [**231**] (in the case of the Dirichlet problem for systems of differential equations of second order) established conditions under which the boundary value problem can be transformed into Fredholm integral equations on the boundary of the domain. These requirements are often called *complementary condition*, *Lopatinskiĭ condition*, or *Shapiro-Lopatinskiĭ condition* (see Remark 2.2.1).

2. A priori estimates for solutions of elliptic boundary value problems for a $2m$ order differential equation in the class of the Sobolev spaces W_2^l and in other function spaces were obtained 1958-1960 by S. Agmon, A. Douglis, L. Nirenberg [**7**], F. E. Browder [**37, 38**], M. Schechter [**213**], L. N. Slobodetskiĭ [**234**].

Somewhat later M. S. Agranovich, A. S. Dynin [**12**], V. A. Solonnikov [**236**], L. R. Volevich [**250, 251**], L. Hörmander [**86**], S. Agmon, A. Douglis, L. Nirenberg [**8**] extended these estimates to solutions of elliptic boundary value problems for systems of differential equations. Note that a special class of elliptic systems of partial differential equations was introduced in 1939 by I. G. Petrovskiĭ [**194**]. Such systems are called Petrovskiĭ-elliptic. In 1955 A. Douglis and L. Nirenberg [**63**] generalized this notion of ellipticity. In Section 4.2 we have used their definition. Furthermore, in 1951 the definition of strong ellipticity for systems of differential equations was given by M. I. Vishik [**246**].

There are two methods for obtaining estimates for the solutions. One method consists in the use of Poisson kernels and the theory of singular integral operators of A. P. Cálderon, A. Zygmund [**42**] (see S. Agmon [**3**], S. Agmon, A. Douglis, L. Nirenberg [**7**], F. E. Browder [**37, 38**], L. Nirenberg [**189**]). In contrast to these authors, J. Peetre [**193, 192**], L. Arkeryd [**15**], and H. Triebel [**242**] obtained estimates for the solutions by means of theorems on multipliers. For the case $p = 2$ we refer also to the books of L. Hörmander [**87**], Yu. M. Berezanskiĭ [**27**], J.-L. Lions, E. Magenes [**126**], and J. Wloka [**256**].

From the a priori estimates for the solutions of elliptic problems obtained by the above mentioned authors it follows that the range of the operator $\mathcal{A}$ of the boundary value problem is closed and its kernel has finite dimension. To obtain the Fredholm property of this operator it remained to show that the cokernel has finite dimension. This was carried out by M. Schechter [**214, 215**] for boundary value problems with normal boundary conditions. He described the range of the operator $\mathcal{A}$ with the solutions of the homogeneous formally adjoint problem. The same method was used in the book of J.-L. Lions, E. Magenes [**126**]. J. Peetre [**192**] proved estimates for the solution of the adjoint equation without studying the nature of the operator $\mathcal{A}^*$.

Another method for proving the finiteness of the dimension of the cokernel is the construction of left and right regularizers (see Definition 3.4.2). This method was applied to different situations in the works of F. E. Browder [**37**], A. S. Dynin [**64**], M. S. Agranovich, A. S. Dynin [**12**], L. Hörmander [**87**], L. R. Volevich [**250, 251**], and M. S. Agranovich [**10**].

3. The classical Green formula (3.1.17) was proved first by N. Aronszajn and A. N. Milgram [**16**]. This formula made it possible to introduce the formally adjoint problem for boundary value problems for a differential equation of order $2m$ if the boundary operators are *normal* (see Definition 3.1.4) and their orders are less than $2m$. M. Schechter [**214**] proved for such problems that the formally adjoint problem is elliptic if and only if the starting problem is elliptic. A detailed proof of this result can also be found in the books of J.-L. Lions, E. Magenes [**126**] and J. Wloka [**256**].

Ya. A. Roĭtberg [**202**] generalized the Green formula (3.1.17) to nonnormal boundary operators of order less than $2m$. Here, in general, the boundary operators of the adjoint problem are pseudodifferential operators. The Green formula (3.1.9) we have used in Chapter 3 was first obtained by B. Lawruk [**124**]. He considered a special class of elliptic boundary value problems for systems of differential operators. In particular, in [**124**] it was assumed that all differential operators $L_{i,j}$ have the same order s and the boundary conditions contain only normal derivatives of order less than s. Then a Green formula analogous to (3.1.9) is valid.

Furthermore, we refer to the paper of A. S. Dikanskiĭ [**61**] who considered the adjoint boundary value problem to general pseudodifferential boundary problems.

4. In the theory of variational problems the Gårding inequality (4.3.50) plays an important role. This inequality was proved by L. Gårding [**73**] for sesquilinear forms corresponding to strongly elliptic differential operators and functions from $C_0^\infty(\Omega)$. Using this inequality it can be easily shown (see Theorem 4.3.4) that the variational problem corresponding to the Dirichlet problem for a strongly elliptic differential operator of order $2m$ is equivalent to a Fredholm operator in the space $\overset{\circ}{W}{}_2^m(\Omega)$. To prove that the solutions belong to the space $W_2^l(\Omega)$ with $l > m$ if the boundary of the domain and the right-hand side of the differential equation are sufficiently smooth, is more complicated. One method applied by K. O. Friedrichs [**71**], F. E. Browder [**36**], L. Nirenberg [**188**] and S. Agmon [**6**] consists in the estimation of difference quotients of the solution by means of Gårding's inequality. C. G. Simader [**232**] established a L_p theory for solutions of the Dirichlet problem based on a generalization of Gårding's inequality. Here he did not need the strong ellipticity of the differential operator.

More general V-elliptic or V-coercive problems have been considered, e.g., in the books of J.-L. Lions, E. Magenes [**126**] and J. Nečas [**184**].

5. Generalized solutions (which belong to Sobolev spaces of negative or small positive order) of elliptic boundary value problems have been considered by Yu. M. Berezanskiĭ [**25**], M. Schechter [**216**], E. Magenes [**129**], J.-L. Lions, E. Magenes [**126**], Ya. A. Roĭtberg, Z. G. Sheftel' [**207, 208**], Ya. A. Roĭtberg [**202, 204, 206**]. Yu. M. Berezanskiĭ and M. Schechter considered generalized solutions of the $2m$ order elliptic equation $Lu = f$ in Ω with the homogeneous boundary conditions $Bu|_{\partial\Omega} = 0$, where B is a vector of differential operators $B_1, \dots, B_m$ of order less than $2m$ which form a normal system on $\partial\Omega$ and satisfy Lopatinskiĭ's condition. They proved a priori estimates and regularity assertions for the solutions. The disadvantage of their methods consists in the restriction to homogeneous boundary conditions.

In the book of J.-L. Lions and E. Magenes [**126**] generalized solutions of the equation $Lu = f$ with inhomogeneous boundary conditions were considered. However, here the function f on the right-hand side of the differential equation has to belong to some space $K^{-r}(\Omega)$. The role of this space can be played, e.g., by weighted Sobolev spaces.

In Chapters 1-3 and in the first two sections of Chapter 4 we followed the concept of Ya. A. Roĭtberg and Z. G. Sheftel' to extend the operator of the boundary value problem to Sobolev spaces of negative order. The spaces $\tilde{W}_2^{l,k}$ were introduced first by Ya. A. Roĭtberg [**201, 203**]. The solvability of elliptic boundary value problems in the class of these spaces has been studied by Ya. A. Roĭtberg, Z. G. Sheftel' [**207, 208**], Ya. A. Roĭtberg [**202, 204, 206**]. The most general result was obtained in [**204, 206**]. Here Ya. A. Roĭtberg showed the Fredholm property of the operators of elliptic boundary value problems for systems of differential equations (see Theorem 4.2.4). The method in [**204, 206**] differs from our method in Chapters 3, 4. While we have studied the adjoint operator to prove the finiteness of the dimension of the cokernel, in [**204, 206**] the Fredholm property was shown by means of a regularizer.

6. The extension of the operator of the boundary value problem to Sobolev spaces of negative order enabled us to introduce the Green functions in Section 3.5 as generalized solutions of the boundary value problem with δ-distributions on the right-hand side. Similarly, the Green functions were handled by Yu. M. Berezanskiĭ, Ya. A. Roĭtberg [**28**], Yu. M. Berezanskiĭ [**27**], I. A. Kovalenko, Ya. A. Roĭtberg [**103**], and I. A. Kovalenko, Ya. A. Roĭtberg, Z. G. Sheftel' [**104**]. Pointwise estimates of Green functions are given in the papers of Yu. P. Krasovskiĭ [**117, 118**] and V. A. Solonnikov [**237**].

7. In Section 3.6 we have considered elliptic problems with parameter which are uniquely solvable for large values of the parameter. Such problems were first studied in the papers of S. Agmon [**5**], S. Agmon, L. Nirenberg [**9**], M. S. Agranovich, M. I. Vishik [**13**], and M. S. Agranovich [**11**].

Part 2

Elliptic problems in domains with conical points

CHAPTER 5

Elliptic boundary value problems in an infinite cylinder

In this chapter we consider elliptic boundary value problems in an infinite cylinder $\mathcal{C} = \{(x,t) : x \in \Omega,\ t \in \mathbb{R}\}$, where Ω is a bounded domain in $\mathbb{R}^n$. We assume that the coefficients of the differential operators are independent of t or satisfy a stabilization condition at infinity.

In the case of t-independent coefficients we obtain necessary and sufficient conditions for the unique solvability of the boundary value problem in weighted Sobolev spaces of arbitrary integer order. Additionally to the ellipticity, the nonexistence of eigenvalues of an operator pencil on a line parallel to the imaginary axis is necessary for the unique solvability of the boundary value problem in corresponding weighted Sobolev spaces. This operator pencil arises if one applies the Laplace transformation with respect to t to the differential operators of the boundary value problem. We show that analogous conditions ensure the Fredholm property of the operator of the boundary value problem if the coefficients satisfy the mentioned stabilization condition at infinity.

Another goal of this chapter is to describe the asymptotics of the solutions at infinity.

5.1. Operator-valued polynomials and applications to ordinary differential equations with operator coefficients

Boundary value problems on the cylinder $\mathcal{C} = \Omega \times \mathbb{R}$ can be considered as ordinary differential equations with operator coefficients on the interval $(-\infty, +\infty)$. One of the goals of this section is the description of all power-exponential solutions of such equations. We show that these solutions are determined by the eigenvalues, eigenvectors and generalized eigenvectors of the operator pencil which is obtained from the ordinary differential operator via the Laplace transformation.

In the beginning of the section we recall some well-known spectral properties of operator pencils. We refer the reader to the book of I. Gohberg, S. Goldberg, M. A. Kaashoek [**77**] and to the papers of M. V. Keldysh [**93, 94**] and I. Gohberg, E. I. Sigal [**76**].

5.1.1. Eigenvalues and eigenvectors of operator pencils polynomially depending on a complex parameter.

Notation, definitions. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces with the norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively. We consider the operator pencil

$$\mathfrak{A}(\lambda) = \sum_{j=0}^{l} A_j \lambda^j , \tag{5.1.1}$$

where A_j are linear and continuous operators from $\mathcal{X}$ into $\mathcal{Y}$. The subset of the complex plane, where $\mathfrak{A}(\lambda)$ is not invertible, is called the *spectrum* of $\mathfrak{A}(\lambda)$. All points in the complement of the spectrum are said to be *regular*. Clearly, the spectrum of $\mathfrak{A}(\lambda)$ is a closed set.

Every number $\lambda_0 \in \mathbb{C}$ such that $\ker \mathfrak{A}(\lambda_0) \neq \{0\}$ is said to be an *eigenvalue* of $\mathfrak{A}(\lambda)$ and the dimension of $\ker \mathfrak{A}(\lambda_0)$ is called the *geometric multiplicity* of the eigenvalue λ_0. Furthermore, every nonzero element of the kernel of $\mathfrak{A}(\lambda_0)$ is called an *eigenvector* of $\mathfrak{A}$.

Let λ_0 be an eigenvalue and let φ_0 be an eigenvector corresponding to λ_0. If the elements $\varphi_1, \dots, \varphi_s$ satisfy the equations

$$(5.1.2) \qquad \sum_{q=0}^{\sigma} \frac{1}{q!}\, \mathfrak{A}^{(q)}(\lambda_0)\, \varphi_{\sigma-q} = 0 \qquad \text{for } \sigma = 1, \dots, s,$$

where $\mathfrak{A}^{(q)}(\lambda) = \frac{d^q}{d\lambda^q}\, \mathfrak{A}(\lambda)$, then the ordered collection $\varphi_0, \varphi_1, \dots, \varphi_s$ is said to be a *Jordan chain* corresponding to the eigenvalue λ_0 of the length $s+1$. The elements $\varphi_1, \dots, \varphi_s$ are called *generalized eigenvectors*. The maximal length of the Jordan chains corresponding to the eigenvector φ_0 is called the *rank* of the eigenvector φ_0 ($\operatorname{rank} \varphi_0$).

Let $I = \dim \ker \mathfrak{A}(\lambda_0)$. A *canonical system of eigenvectors* of $\mathfrak{A}(\lambda)$ corresponding to the eigenvalue λ_0 is a system of eigenvectors $\varphi_{1,0}, \dots, \varphi_{I,0}$ such that $\operatorname{rank} \varphi_{1,0}$ is maximal among the ranks of all eigenvectors corresponding to λ_0 and $\operatorname{rank} \varphi_{j,0}$ is maximal among the ranks of all eigenvectors in any direct complement in $\ker \mathfrak{A}(\lambda_0)$ to the linear span of the vectors $\varphi_{1,0}, \dots, \varphi_{j-1,0}$ ($j = 2, \dots, I$). The numbers $\kappa_j = \operatorname{rank} \varphi_{j,0}$ ($j = 1, \dots, I$) are called the *partial multiplicities* and the sum $\kappa = \kappa_1 + \dots + \kappa_I$ is called the *algebraic multiplicity* of the eigenvalue λ_0. Note that the partial and algebraic multiplicities are independent of the choice of the canonical system of eigenvectors.

We suppose that $\varphi_{1,0}, \dots, \varphi_{I,0}$ is a canonical system of eigenvectors of $\mathfrak{A}(\lambda)$ corresponding to the eigenvalue λ_0. If the vectors $\varphi_{j,0}, \varphi_{j,1}, \dots, \varphi_{j,\kappa_j-1}$ form a Jordan chain for each $j = 1, \dots, I$, then the system of the vectors

$$\varphi_{j,0}\,,\ \varphi_{j,1}\,,\ \dots\,,\ \varphi_{j,\kappa_j-1} \qquad j = 1, \dots, I,$$

is called a *canonical system of Jordan chains* corresponding to the eigenvalue λ_0.

Fredholm operator pencils.

DEFINITION 5.1.1. The pencil (5.1.1) is said to be a *Fredholm operator pencil* if

(i) the operator $\mathfrak{A}(\lambda)$ is Fredholm for every fixed $\lambda \in \mathbb{C}$,
(ii) the operator $\mathfrak{A}(\lambda)$ is invertible for at least one λ.

LEMMA 5.1.1. *If $\mathfrak{A}(\lambda)$ is a Fredholm operator pencil, then its spectrum consists only of isolated eigenvalues. All eigenvalues have finite algebraic multiplicities.*

We introduce the adjoint operator pencil $\mathfrak{A}^*(\lambda)$ to $\mathfrak{A}(\lambda)$. Let $\mathcal{X}^*$ be the dual space to $\mathcal{X}$, i.e., the set of all linear and continuous functionals on $\mathcal{X}$. We suppose that there is a duality between $\mathcal{X}$ and $\mathcal{X}^*$, i.e., there exists a sesquilinear form $\langle\cdot,\cdot\rangle_1$ which maps $\mathcal{X} \times \mathcal{X}^*$ into $\mathbb{C}$ such that

$$|\langle \varphi, \varphi^* \rangle_1| \leq \|\varphi\|_{\mathcal{X}}\, \|\varphi^*\|_{\mathcal{X}^*}$$

for each $\varphi \in \mathcal{X}$, $\varphi^* \in \mathcal{X}^*$ and every functional $h \in \mathcal{X}^*$ can be uniquely represented in the form

$$h(\varphi) = \langle \varphi, \varphi^* \rangle_1 .$$

The same condition is imposed on the space $\mathcal{Y}$. Here $\langle \cdot, \cdot \rangle_2$ denotes the corresponding sesquilinear form on $\mathcal{Y} \times \mathcal{Y}^*$.

If A is an arbitrary linear and continuous operator from $\mathcal{X}$ into $\mathcal{Y}$, then we define the adjoint operator A^* as the linear and continuous operator from $\mathcal{Y}^*$ into $\mathcal{X}^*$ which satisfies the equality

$$\langle \varphi, A^* \psi \rangle_1 = \langle A\varphi, \psi \rangle_2$$

for all $\varphi \in \mathcal{X}$, $\psi \in \mathcal{Y}^*$. From this definition it follows immediately that the norm of the operator A^* is equal to the norm of A.

Let $\mathfrak{A}(\lambda)$ be the operator pencil defined in (5.1.1). Then $\mathfrak{A}^*(\lambda)$ denotes the operator pencil

$$\mathfrak{A}^*(\lambda) = \sum_{j=0}^{l} A_j^* \lambda^j .$$

This means that $\mathfrak{A}^*(\lambda)$ is defined by the equality

$$\langle \varphi, \mathfrak{A}^*(\lambda)\psi \rangle_1 = \langle \mathfrak{A}(\overline{\lambda})\varphi, \psi \rangle_2 , \qquad \varphi \in \mathcal{X},\ \psi \in \mathcal{Y}^* . \tag{5.1.3}$$

Obviously, the following assertions are valid.

LEMMA 5.1.2. 1) *$\mathfrak{A}(\lambda)$ is a Fredholm operator pencil if and only if $\mathfrak{A}^*(\lambda)$ is a Fredholm operator pencil.*

2) *The number λ_0 is an eigenvalue of $\mathfrak{A}(\lambda)$ if and only if $\overline{\lambda}_0$ is an eigenvalue of $\mathfrak{A}^*(\lambda)$. The geometric, partial, and algebraic multiplicities of λ_0 and $\overline{\lambda}_0$ coincide.*

The following theorem contains a representation of the resolvent near the eigenvalues.

THEOREM 5.1.1. *Let (5.1.1) be a Fredholm operator pencil. Furthermore, let λ_0 be an eigenvalue of $\mathfrak{A}(\lambda)$ with the geometric multiplicity I and the partial multiplicities $\kappa_1, \dots , \kappa_I$. We suppose that $\{\varphi_{j,s}\}_{j=1,\dots,I,\, s=0,\dots,\kappa_j-1}$ is a canonical system of Jordan chains corresponding to λ_0. Then the following assertions are valid.*

1) *There exists a canonical system $\{\psi_{j,s}\}_{j=1,\dots,I,\, s=0,\dots,\kappa_j-1}$ of Jordan chains of $\mathfrak{A}^*(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_0$ such that $\mathfrak{A}(\lambda)^{-1}$ has the following representation*

$$\mathfrak{A}(\lambda)^{-1} = \sum_{j=1}^{I} \sum_{s=0}^{\kappa_j-1} \frac{P_{j,s}}{(\lambda-\lambda_0)^{\kappa_j-s}} + \Gamma(\lambda) \tag{5.1.4}$$

in a neighbourhood of λ_0, where $P_{j,s}$ are linear and continuous operators from $\mathcal{Y}$ into $\mathcal{X}$,

$$P_{j,s}\, v = \sum_{\sigma=0}^{s} \langle v, \psi_{j,\sigma} \rangle_2\, \varphi_{j,s-\sigma} , \qquad v \in \mathcal{Y}, \tag{5.1.5}$$

and Γ is a holomorphic operator function in a neighbourhood of λ_0.

2) *The canonical system* $\{\psi_{j,s}\}$ *of the first assertion is uniquely determined and satisfies the biorthonormality conditions*

$$\sum_{p=0}^{\sigma}\sum_{q=p+1}^{p+s+1}\frac{1}{q!}\langle \mathfrak{A}^{(q)}(\lambda_0)\varphi_{j,p+s+1-q}\,,\,\psi_{k,\sigma-p}\rangle_2 = \delta_{j,k}\,\delta_{s,\kappa_k-1-\sigma} \tag{5.1.6}$$

for $j,k=1,\dots,I$, $s=0,\dots,\kappa_j-1$, $\sigma=0,\dots,\kappa_k-1$.

3) *If* $\{\psi_{j,s}\}_{j=1,\dots,I,\,s=0,\dots,\kappa_j-1}$ *is a collection of Jordan chains of* $\mathfrak{A}^*(\lambda)$ *corresponding to* $\overline{\lambda}_0$ *which is subject to (5.1.6), then this collection is the canonical system given in the first assertion of the theorem.*

5.1.2. Power-exponential zeros of ordinary differential operators. Let $\mathfrak{A}(\lambda)$ be a Fredholm operator pencil of the form (5.1.1) and let λ_μ, $\mu=1,2,\dots$, be eigenvalues of this pencil with the geometric multiplicity I_μ and the partial multiplicities $\kappa_{\mu,j}$ ($j=1,\dots,I_\mu$). Furthermore, let

$$\{\varphi_{j,s}^{(\mu)}\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1} \quad \text{and} \quad \{\psi_{j,s}^{(\mu)}\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

be canonical systems of Jordan chains of $\mathfrak{A}(\lambda)$, $\mathfrak{A}^*(\lambda)$ corresponding to λ_μ and $\overline{\lambda}_\mu$, respectively, such that

$$\sum_{p=0}^{\sigma}\sum_{q=p+1}^{p+s+1}\frac{1}{q!}\,\langle \mathfrak{A}^{(q)}(\lambda_\mu)\,\varphi^{(\mu)}_{j,p+s+1-q}\,,\,\psi^{(\mu)}_{k,\sigma-p}\rangle_2 = \delta_{j,k}\,\delta_{s,\kappa_{\mu,k}-1-\sigma} \tag{5.1.7}$$

for $j,k=1,\dots,I_\mu$, $s=0,\dots,\kappa_{\mu,j}-1$, $\sigma=0,\dots,\kappa_{\mu,k}-1$.

By $\mathcal{N}(\mathfrak{A}(\partial_t),\lambda_\mu)$ we denote the set of all solutions $u=u(t)$ of the differential equation

$$\mathfrak{A}(\partial_t)\,u(t)=\sum_{j=0}^{l}A_j\,\partial_t^j u(t)=0\,,\qquad t\in\mathbb{R}, \tag{5.1.8}$$

which have the "power-exponential" form

$$u(t)=e^{\lambda_\mu t}\sum_{\sigma=0}^{s}\frac{t^\sigma}{\sigma!}\varphi_{s-\sigma}\,, \tag{5.1.9}$$

where s is an arbitrary nonnegative integer and $\varphi_0,\dots,\varphi_s$ are elements of the space $\mathcal{X}$, $\varphi_0\neq 0$.

LEMMA 5.1.3. *The function (5.1.9) is a solution of the equation (5.1.8) if and only if* λ_μ *is an eigenvalue of* $\mathfrak{A}(\lambda)$ *and* $\varphi_0,\varphi_1,\dots,\varphi_s$ *is a Jordan chain corresponding to this eigenvalue.*

Proof: Let $u=e^{\lambda_\mu t}\varphi$ be a function of the form (5.1.9). We have

$$e^{-\lambda_\mu t}\,\mathfrak{A}(\partial_t)\,\left(e^{\lambda_\mu t}\,\varphi\right)=\mathfrak{A}(\partial_t+\lambda_\mu)\,\varphi=\sum_{q=0}^{l}\frac{1}{q!}\,\mathfrak{A}^{(q)}(\lambda_\mu)\,\partial_t^q\varphi\,. \tag{5.1.10}$$

Obviously, the expression (5.1.10) is equal to zero if and only if

$$\partial_t^\sigma\sum_{q=0}^{l}\frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_\mu)\partial_t^q\varphi\Big|_{t=0}=\sum_{q=0}^{s-\sigma}\frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_\mu)\,\partial_t^{q+\sigma}\varphi\Big|_{t=0}=0$$

for $\sigma=0,1,\dots,s$. This is equivalent to the condition (5.1.2). ∎

We set

$$\Phi_{\mu,j}(t) = \sum_{s=0}^{\kappa_{\mu,j}-1} \frac{t^s}{s!}\, \varphi^{(\mu)}_{j,\kappa_{\mu,j}-1-s} \tag{5.1.11}$$

for $j = 1, \dots, I_\mu$. Then the following assertion holds.

LEMMA 5.1.4. *The functions*

$$e^{\lambda_\mu t}\, \partial_t^{\kappa_{\mu,j}-1-s}\, \Phi_{\mu,j} = e^{\lambda_\mu t} \sum_{\sigma=0}^{s} \frac{t^\sigma}{\sigma!} \varphi^{(\mu)}_{j,s-\sigma}\,, \tag{5.1.12}$$

$j = 1, \dots, I_\mu,\ s = 0, \dots, \kappa_{\mu,j} - 1$, *form a basis in* $\mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$. *In particular, the dimension of* $\mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$ *is equal to the algebraic multiplicity* $\kappa_{\mu,1} + \cdots + \kappa_{\mu,I_\mu}$ *of the eigenvalue* λ_μ.

Proof: The linear independence of the functions (5.1.12) follows from the observation that the leading coefficients of the vector-polynomials of the same degree are linear independent. We prove by induction in k that every element

$$u(t) = e^{\lambda_\mu t} \sum_{\sigma=0}^{k} \frac{t^\sigma}{\sigma!} \varphi_{k-\sigma} \tag{5.1.13}$$

($\varphi_0 \neq 0$) of the set $\mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$ can be represented as a linear combination of the functions (5.1.12). For $k = 0$ this assertion is trivial. Let $k > 0$ and let $u \in \mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$ be the function (5.1.13) with $\varphi_0 \neq 0$. Then by Lemma 5.1.3, $\varphi_0, \varphi_1, \dots, \varphi_k$ is a Jordan chain. Hence there exists an integer $q \geq 1$ such that $\operatorname{rank} \varphi^{(\mu)}_{j,0} \geq k + 1$ for $j = 1, \dots, q$ and $\operatorname{rank} \varphi^{(\mu)}_{j,0} \leq k$ for $j = q + 1, \dots, I_\mu$. Since $\{\varphi^{(\mu)}_{j,s}\}$ is a canonical system of Jordan chains, the eigenvector φ_0 lies in the linear span of the eigenvectors $\varphi^{(\mu)}_{1,0}, \dots, \varphi^{(\mu)}_{q,0}$, i.e.,

$$\varphi_0 = c_1\, \varphi^{(\mu)}_{1,0} + \cdots + c_q\, \varphi^{(\mu)}_{q,0}. \tag{5.1.14}$$

Obviously, the elements

$$\eta_\sigma = c_1\, \varphi^{(\mu)}_{1,\sigma} + \cdots + c_q\, \varphi^{(\mu)}_{q,\sigma}\,,$$

$\sigma = 0, 1, \dots, k$, with the same coefficients c_j as in (5.1.14) form a Jordan chain of $\mathfrak{A}(\lambda)$ to the eigenvalue λ_μ. Consequently, the function

$$v(t) = e^{\lambda_\mu t} \sum_{\sigma=0}^{k} \frac{t^\sigma}{\sigma!} \eta_{k-\sigma} = \sum_{j=1}^{q} c_j\, e^{\lambda_\mu t}\, \partial_t^{\kappa_{\mu,j}-1-k}\, \Phi_{\mu,j}$$

is an element of $\mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$. Thus,

$$u(t) - v(t) = e^{\lambda_\mu t} \sum_{\sigma=0}^{k-1} \frac{t^\sigma}{\sigma!} \left(\varphi_{k-\sigma} - \eta_{k-\sigma}\right)$$

is also an element of $\mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$. We assume that our assertion is true for $k - 1$. Then $u - v$ is a linear combination of the functions (5.1.12). This proves the lemma. ■

As a consequence of Lemmas 5.1.3 and 5.1.4, we get the following assertion.

COROLLARY 5.1.1. *Let λ_μ be an eigenvalue of $\mathfrak{A}(\lambda_\mu)$ with the geometric multiplicity I_μ and the partial multiplicities $\kappa_{\mu,j}$ $(j=1,\dots,I_\mu)$. Furthermore, let*

$$e^{\lambda_\mu t}\,\Phi_{\mu,j}(t)\,, \qquad j=1,\dots,I_\mu,$$

be elements of the set $\mathcal{N}(\mathfrak{A}(\partial_t),\lambda_\mu)$, where $\Phi_{\mu,j}$ are polynomials in t of degree $\kappa_{\mu,j}-1$ with coefficients from the space $\mathcal{X}$. Suppose that the leading coefficients of the polynomials $\Phi_{\mu,j}$ are linearly independent. Then the elements

$$(\partial_t^{\kappa_{\mu,j}-1-s}\,\Phi_{\mu,j})(0)\,, \qquad j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1,$$

form a canonical system of Jordan chains corresponding to the eigenvalue λ_μ.

Analogously to $\mathcal{N}(\mathfrak{A}(\partial_t),\lambda_\mu)$ we define $\mathcal{N}(\mathfrak{A}^*(-\partial_t),-\overline{\lambda}_\mu)$ as the set of all solutions v of the equation

$$\mathfrak{A}^*(-\partial_t)\,v(t)=\sum_{j=0}^{l}A_j^*\,(-\partial_t)^j v(t)=0\,, \qquad t\in\mathbb{R}, \tag{5.1.15}$$

which have the form

$$v(t)=e^{-\overline{\lambda}_\mu t}\sum_{\sigma=0}^{s}\frac{t^\sigma}{\sigma!}\,\psi_{s-\sigma}\,,$$

where $\psi_0,\dots,\psi_s$ are elements from $\mathcal{Y}^*$. A basis in $\mathcal{N}(\mathfrak{A}^*(-\partial_t),-\overline{\lambda}_\mu)$ is given by the functions

$$e^{-\overline{\lambda}_\mu t}\,(-\partial_t)^{\kappa_{\mu,j}-1-s}\,\Psi_{\mu,j}(t), \tag{5.1.16}$$

$j=1,\dots,I_\mu$, $s=0,\dots,\kappa_{\mu,j}-1$, where

$$\Psi_{\mu,j}(t)=\sum_{\sigma=0}^{\kappa_{\mu,j}-1}\frac{(-t)^\sigma}{\sigma!}\,\psi^{(\mu)}_{j,\kappa_{\mu,j}-1-\sigma}\,. \tag{5.1.17}$$

In the following lemma we give a useful variant of the biorthonormality condition (5.1.7) in terms of the bases of $\mathcal{N}\big(\mathfrak{A}(\partial_t),\lambda_\mu\big)$ and $\mathcal{N}\big(\mathfrak{A}^*(-\partial_t),-\overline{\lambda}_\mu\big)$.

LEMMA 5.1.5. *Let ζ be a smooth function on $\mathbb{R}$ equal to one in a neighbourhood of $+\infty$ and to zero in a neighbourhood of $-\infty$. Then*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\Big\langle \mathfrak{A}(\partial_t)\,\big(\zeta(t)\,e^{\lambda_\mu t}\,\partial_t^s\,\Phi_{\mu,j}(t)\big)\,,\,e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\,\Psi_{\nu,k}(t)\Big\rangle_2\,dt\\ &=\delta_{\mu,\nu}\,\delta_{j,k}\,\delta_{\kappa_{\mu,j}-1-s,\sigma}\end{aligned} \tag{5.1.18}$$

for $j=1,\dots,I_\mu$, $k=1,\dots,I_\nu$, $s=0,\dots,\kappa_{\mu,j}-1$, $\sigma=0,\dots,\kappa_{\nu,k}-1$.

If ζ is equal to one in a neighbourhood of $-\infty$ and to zero in a neighbourhood of $+\infty$, then the left-hand side of (5.1.18) is equal to $-\delta_{\mu,\nu}\,\delta_{j,k}\,\delta_{\kappa_{\mu,j}-1-s,\sigma}$.

(Since (5.1.12) is a solution of the equation (5.1.8), the integration on the left-hand side of (5.1.18) in both cases is effected only on a bounded interval, where the derivative of ζ does not vanish.)

Proof: We denote the left-hand side of (5.1.18) by $\mathcal{I}$. The proof proceeds in three steps.

1) Let ζ_1, ζ_2 be functions with the same properties as ζ. Since the functions

$$e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\,\Psi_{\nu,k}(t)$$

are elements of $\mathcal{N}(\mathfrak{A}^*(-\partial_t), -\overline{\lambda}_\nu)$, we get

$$\int_{-\infty}^{+\infty} \Big\langle \mathfrak{A}(\partial_t)\left((\zeta_1-\zeta_2)\, e^{\lambda_\mu t}\,\partial_t^s\Phi_{\mu,j}\right),\ e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\Psi_{\nu,k}\Big\rangle_2\, dt$$

$$= \int_{-\infty}^{+\infty} \Big\langle (\zeta_1-\zeta_2)\, e^{\lambda_\mu t}\,\partial_t^s\Phi_{\mu,j}\,,\ \mathfrak{A}^*(-\partial_t)\left(e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\Psi_{\nu,k}\right)\Big\rangle_1\, dt = 0$$

Consequently, $\mathcal{I}$ is independent of the choice of ζ.

2) We show that $\mathcal{I} = 0$ if $\mu \neq \nu$. By the first part of the proof, the integral

$$(5.1.19)\qquad \int_{-\infty}^{+\infty} \Big\langle \mathfrak{A}(\partial_t)\left(\zeta(t+\tau)\, e^{\lambda_\mu t}\,\partial_t^s\Phi_{\mu,j}(t)\right),\ e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\Psi_{\nu,k}(t)\Big\rangle_2\, dt$$

$$= \int_{-\infty}^{+\infty} \Big\langle \mathfrak{A}(\partial_t)\ \left(\zeta(t)\, e^{\lambda_\mu (t-\tau)}\,\partial_t^s\Phi_{\mu,j}(t-\tau)\right),e^{-\overline{\lambda}_\nu (t-\tau)}\,(-\partial_t)^\sigma\Psi_{k,\nu}(t-\tau)\Big\rangle_2\, dt$$

is independent of τ. However, the expression (5.1.19) has the form

$$e^{(\lambda_\mu-\lambda_\nu)\tau}\, P(\tau),$$

where P is a polynomial of τ. Hence $\mathcal{I} = 0$ if $\mu \neq \nu$.

3) Let $\mu = \nu$. By (5.1.10), we have

$$(5.1.20)\qquad \mathcal{I} = \int_{-\infty}^{+\infty} \langle \mathfrak{A}(\partial_t+\lambda_\mu)\ (\zeta(t)\,\partial_t^s\Phi_{\mu,j}(t))\ ,\ (-\partial_t)^\sigma\Psi_{\mu,k}(t)\rangle_2\, dt.$$

Using the equality $\mathfrak{A}(\partial_t+\lambda_\mu)\,\partial_t^s\Phi_{\mu,j}(t) = e^{-\lambda_\mu t}\,\mathfrak{A}(\partial_t)\ \left(e^{\lambda_\mu t}\,\partial_t^s\Phi_{\mu,j}(t)\right) = 0$ and the decomposition

$$\mathfrak{A}(\partial_t+\lambda_\mu) = \sum_{q=0}^{l} \frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_\mu)\,\partial_t^q$$

we get

$$\begin{aligned}
\mathfrak{A}(\partial_t+\lambda_\mu)\ (\zeta(t)\,\partial_t^s\Phi_{\mu,j}(t)) &= \sum_{q=1}^{l}\frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_\mu)\ (\partial_t^q(\zeta\partial_t^s\Phi_{\mu,j}) - \zeta\partial_t^q(\partial_t^s\Phi_{\mu,j}))\\
&= \sum_{q=1}^{l}\frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_\mu)\sum_{h=1}^{q}\binom{q}{h}\,(\partial_t^h\zeta)\,\partial_t^{s+q-h}\Phi_{\mu,j}\,.
\end{aligned}$$

From the equality $\binom{q}{h} = \binom{h-1}{h-1} + \binom{h}{h-1} + \cdots + \binom{q-1}{h-1}$ it follows that

$$\begin{aligned}
\sum_{h=1}^{q}\binom{q}{h}\,(\partial_t^h\zeta)\,\partial_t^{s+q-h}\Phi_{\mu,j} &= \sum_{p=0}^{q-1}\sum_{h=0}^{p}\binom{p}{h}\,(\partial_t^{h+1}\zeta)\,\partial_t^{s+q-1-h}\Phi_{\mu,j}\\
&= \sum_{p=0}^{q-1}\partial_t^p\left((\partial_t\zeta)\,\partial_t^{s+q-1-p}\Phi_{\mu,j})\right).
\end{aligned}$$

Therefore, we obtain

$$\mathfrak{A}(\partial_t + \lambda_\mu)\, (\zeta\, \partial_t^s \Phi_{\mu,j}) = \sum_{q=1}^{l} \sum_{p=0}^{q-1} \frac{1}{q!} \mathfrak{A}^{(q)}(\lambda_\mu)\, \partial_t^p \left((\partial_t \zeta)\, \partial_t^{s+q-1-p} \Phi_{\mu,j} \right).$$

Integrating by parts in (5.1.20), we arrive at

$$\mathcal{I} = \int_{-\infty}^{+\infty} (\partial_t \zeta) \sum_{q=1}^{l} \sum_{p=0}^{q-1} \frac{1}{q!} \left\langle \mathfrak{A}^{(q)}(\lambda_\mu)\, \partial_t^{s+q-1-p} \Phi_{\mu,j} \,,\, (-\partial_t)^{\sigma+p} \Psi_{\mu,k} \right\rangle_2 dt$$

We replace ζ by the Heaviside function χ. Since $\partial_t \chi(t) = \delta(t)$, it holds

$$\mathcal{I} = \sum_{q=1}^{l} \sum_{p} \frac{1}{q!} \left\langle \mathfrak{A}^{(q)}(\lambda_\mu) \varphi^{(\mu)}_{j,\kappa_{\mu,j}-s-q+p} \,,\, \psi^{(\mu)}_{k,\kappa_{\mu,k}-1-\sigma-p} \right\rangle_2 ,$$

where the inner summation is extended over the set of all integer p such that $0 \le p \le q-1$, $s+q-\kappa_{\mu,j} \le p \le \kappa_{\mu,k} - 1 - \sigma$. Changing the order of the summation, we get

$$\mathcal{I} = \sum_{p=0}^{\kappa_{\mu,k}-1-\sigma} \; \sum_{q=p+1}^{p+\kappa_{\mu,j}-s} \frac{1}{q!} \left\langle \mathfrak{A}^{(q)}(\lambda_\mu) \varphi^{(\mu)}_{j,\kappa_{\mu,j}-s-q+p} \,,\, \psi^{(\mu)}_{k,\kappa_{\mu,k}-p-1-\sigma} \right\rangle_2$$

and the biorthonormality condition (5.1.7) implies (5.1.18).

If $\zeta = 1$ in a neighbourhood of $-\infty$ and $\zeta = 0$ in a neighbourhood of $+\infty$, then we obtain the assertion of the lemma replacing ζ by $1 - \zeta$. The proof is complete. ∎

5.1.3. Power-exponential solutions of the inhomogeneous differential equation. Let $\mathfrak{A}(\lambda)$ be a Fredholm operator pencil of the form (5.1.1). We consider the inhomogeneous differential equation

$$\mathfrak{A}(\partial_t)\, u(t) = f(t)\,, \qquad t > 0\,, \tag{5.1.21}$$

where f has the form

$$f(t) = e^{\lambda_0 t} \sum_{\sigma=0}^{s} \frac{t^\sigma}{\sigma!} v_{s-\sigma} \tag{5.1.22}$$

with coefficients $v_\sigma \in \mathcal{Y}$.

LEMMA 5.1.6. *Let f be a function of the form (5.1.22). Then there exists a solution u of the equation (5.1.21) which has the representation*

$$u(t) = e^{\lambda_0 t} \sum_{\sigma=0}^{s+\kappa_1} \frac{t^\sigma}{\sigma!} \varphi_{s+\kappa_1-\sigma}\,, \tag{5.1.23}$$

where $\varphi_\sigma \in \mathcal{X}$, $\sigma = 0, \dots, s+\kappa_1$. Here $\kappa_1 = 0$ if λ_0 is a regular point. Otherwise, κ_1 is the maximal partial multiplicity of λ_0.

Proof: Let u be the function given in (5.1.23). Then by (5.1.10), we have

$$\begin{aligned} e^{-\lambda_0 t}\, \mathfrak{A}(\partial_t)\, u(t) &= \sum_{q=0}^{l} \frac{1}{q!} \mathfrak{A}^{(q)}(\lambda_0)\, \partial_t^q \sum_{\sigma=0}^{s+\kappa_1} \frac{t^\sigma}{\sigma!} \varphi_{s+\kappa_1-\sigma} \\ &= \sum_{\sigma=0}^{s+\kappa_1} \frac{t^\sigma}{\sigma!} \sum_{q=0}^{s+\kappa_1-\sigma} \frac{1}{q!} \mathfrak{A}^{(q)}(\lambda_0)\, \varphi_{s+\kappa_1-\sigma-q}\,. \end{aligned}$$

Hence the equation (5.1.21) is satisfied if and only if

$$\sum_{q=0}^{s+\kappa_1-\sigma} \frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_0)\,\varphi_{s+\kappa-\sigma-q} = \begin{cases} v_{s-\sigma} & \text{for } \sigma = 0,1,\dots,s, \\ 0 & \text{for } \sigma = s+1,\dots,s+\kappa_1\,. \end{cases} \tag{5.1.24}$$

By Theorem 5.1.1, the inverse $\mathfrak{A}(\lambda)^{-1}$ has the representation

$$\mathfrak{A}(\lambda)^{-1} = \sum_{j=-\kappa_1}^{\infty} P_j\,(\lambda-\lambda_0)^j$$

in a neighbourhood of λ_0. From

$$\mathfrak{A}(\lambda)\cdot\mathfrak{A}(\lambda)^{-1} = \sum_{j=-\kappa_1}^{\infty}\Big(\sum_{q=0}^{\kappa_1+j}\frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_0)\,P_{j-q}\Big)(\lambda-\lambda_0)^j = I$$

it follows that

$$\sum_{q=0}^{\kappa_1+j}\frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_0)\,P_{j-q} = \delta_{j,0}\,.$$

Consequently, for

$$\varphi_j \overset{def}{=} \sum_{p=0}^{\min(j,s)} P_{-\kappa_1+j-p}\,v_p\,, \qquad j=1,\dots,s+\kappa_1,$$

the left-hand side of (5.1.24) is equal to

$$\sum_{p=0}^{s+\kappa_1-\sigma}\Big(\sum_{q=0}^{\kappa_1+s-\sigma-p}\frac{1}{q!}\mathfrak{A}^{(q)}(\lambda_0)\,P_{s-\sigma-p-q}\Big)v_p = \sum_{p=0}^{s+\kappa_1-\sigma}\delta_{s-\sigma-s-p,0}\,v_p\,.$$

This proves the lemma. ∎

REMARK 5.1.1. The solution (5.1.23) is unique if λ_0 is a regular point of $\mathfrak{A}(\lambda)$. If λ_0 is an eigenvalue, then this solution is unique up to an arbitrary solution of the homogeneous equation (5.1.8) having the same form (5.1.23).

5.2. Solvability of the model problem in an infinite cylinder

In this section elliptic boundary value problems in an infinite cylinder $\mathcal{C} = \Omega\times\mathbb{R}$ are considered, where the coefficients of the operators do not depend on the last variable. We call such problems *model problems.*

The principal idea for solving the model problem is the following. Applying the Laplace transformation $t\to\lambda$ to the boundary value problem, we obtain a parameter-depending problem in the bounded domain Ω. Under the assumption of ellipticity, this problem is solvable for all complex λ except a countable set of isolated points, the eigenvalues of this parameter-depending problem. If we apply the inverse Laplace transformation to the solution of the parameter-depending problem, we obtain the solution of the model problem provided that the line of integration in the formula of the inverse Laplace transformation does not contain eigenvalues and the functions on the right sides of the boundary value problem belong to certain weighted Sobolev spaces.

5.2.1. Weighted Sobolev spaces in an infinite cylinder.

Definition of weighted Sobolev spaces. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let $\mathcal{C}$ be the cylinder

$$\mathcal{C} \stackrel{def}{=} \Omega \times \mathbb{R} = \{ (x,t) \in \mathbb{R}^{n+1} \ : \ x \in \Omega,\, t \in \mathbb{R}^1 \}.$$

We define the weighted Sobolev space $\mathcal{W}^l_{2,\beta}(\mathcal{C})$ for integer $l \ge 0$ and real β as the set of all functions u on $\mathcal{C}$ such that $e^{\beta t}u$ is an element of the usual Sobolev space $W^l_2(\mathcal{C})$. The norm in $\mathcal{W}^l_{2,\beta}(\mathcal{C})$ is

$$\|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} = \|e^{\beta t}u\|_{W^l_2(\mathcal{C})}\,. \tag{5.2.1}$$

In particular, $L_{2,\beta}(\mathcal{C}) = \mathcal{W}^0_{2,\beta}(\mathcal{C})$ is the Hilbert space with the scalar product

$$\big(u,v\big)_{L_{2,\beta}(\mathcal{C})} = \int_{\mathcal{C}} e^{2\beta t}\, u\cdot \overline{v}\, dx\, dt.$$

Furthermore, for $l \ge 1$ we denote the space of traces of functions from $\mathcal{W}^l_{2,\beta}(\partial\mathcal{C})$ on the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ by $\mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})$. This space is equipped with the norm

$$\|u\|_{\mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})} = \inf\{ \|v\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \, : \, v \in \mathcal{W}^l_{2,\beta}(\mathcal{C}),\ v = u \text{ on } \partial\mathcal{C}\}. \tag{5.2.2}$$

Obviously, the space $\mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})$ is the set of all functions u on $\partial\mathcal{C}$ such that $e^{\beta t}u \in W^{l-1/2}_2(\partial\mathcal{C})$, and the norm (5.2.2) is equivalent to

$$\|u\| = \|e^{\beta t}u\|_{W^{l-1/2}_2(\partial\mathcal{C})}\,. \tag{5.2.3}$$

The last norm is defined for arbitrary integer l. Hence it makes sense to define $\mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})$ for arbitrary integer l as the set of all functions u such that $e^{\beta t}u \in W^{l-1/2}_2(\partial\mathcal{C})$. Equipped with the norm (5.2.3), the space $\mathcal{W}^{l-1/2}_{2,\beta}(\mathcal{C})$ is complete for arbitrary integer l.

It can be easily seen that the norm (5.2.1) is equivalent to

$$\begin{aligned} \|u\| &= \Big(\int_{\mathcal{C}} \sum_{|\alpha|+j\le l} e^{2\beta t}\, |D^\alpha_x D^j_t u|^2\, dx\, dt\Big)^{1/2} \\ &= \Big(\int_{-\infty}^{+\infty} e^{2\beta t} \sum_{j=0}^{l} \|D^j_t u(\cdot,t)\|^2_{W^{l-j}_2(\Omega)}\, dt\Big)^{1/2}. \end{aligned} \tag{5.2.4}$$

LEMMA 5.2.1. *Let $\beta_1 \le \beta \le \beta_2$. Then there are the continuous imbeddings* [1]

$$\mathcal{W}^l_{2,\beta_1}(\mathcal{C}) \cap \mathcal{W}^l_{2,\beta_2}(\mathcal{C}) \subset \mathcal{W}^l_{2,\beta}(\mathcal{C}) \subset \mathcal{W}^l_{2,\beta_1}(\mathcal{C}) + \mathcal{W}^l_{2,\beta_2}(\mathcal{C}) \tag{5.2.5}$$

for $l \ge 0$ and

$$\mathcal{W}^{l-1/2}_{2,\beta_1}(\partial\mathcal{C}) \cap \mathcal{W}^{l-1/2}_{2,\beta_2}(\partial\mathcal{C}) \subset \mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C}) \subset \mathcal{W}^{l-1/2}_{2,\beta_1}(\partial\mathcal{C}) + \mathcal{W}^{l-1/2}_{2,\beta_2}(\partial\mathcal{C}) \tag{5.2.6}$$

for $l \ge 1$.

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[1] If X and Y are Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, then $X+Y$ denotes the set of all elements $z = x+y$, where $x \in X$, $y \in Y$. $X+Y$ is a Banach space with the norm $\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y \, : \, x \in X,\, y \in Y,\, z = x+y\}$, while $X \cap Y$ is a Banach space with the norm $\|x\|_{X\cap Y} = \max(\|x\|_X, \|x\|_Y)$.

Proof: By the equivalence of the norms (5.2.1), (5.2.4), we have

$$\begin{aligned}\|u\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} &\le c_1 \int\limits_{\mathcal{C}} (e^{2\beta_1 t} + e^{2\beta_2 t}) \sum_{|\alpha|+j\le l} |D^\alpha_x D^j_t u(x,t)|^2\, dx\, dt \\ &\le c_2 \left(\|u\|^2_{\mathcal{W}^l_{2,\beta_1}(\mathcal{C})} + \|u\|^2_{\mathcal{W}^l_{2,\beta_2}(\mathcal{C})}\right) \le 2c_2 \max_i \|u\|^2_{\mathcal{W}^l_{2,\beta_i}(\mathcal{C})}\end{aligned}$$

and, consequently,

$$\|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \le c \max_i \|u\|_{\mathcal{W}^l_{2,\beta_i}(\mathcal{C})}$$

with a constant c independent of u.

Let ζ be an arbitrary C^∞ function on $\overline{\mathcal{C}}$ such that $\zeta(x,t) = 0$ for $t < -1$ and $\zeta(x,t) = 1$ for $t > 1$. Then

$$\begin{aligned}\|u\|_{\mathcal{W}^l_{2,\beta_1}(\mathcal{C})+\mathcal{W}^l_{2,\beta_2}(\mathcal{C})} &\le c\left(\|\zeta u\|_{\mathcal{W}^l_{2,\beta_1}(\mathcal{C})} + \|(1-\zeta)u\|_{\mathcal{W}^l_{2,\beta_2}(\mathcal{C})}\right) \\ &\le c\left(\|\zeta u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} + \|(1-\zeta)u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}\right) \le c' \|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}\,.\end{aligned}$$

This proves (5.2.5). The imbedding (5.2.6) is an immediate consequence of (5.2.5). ∎

Equivalent norms. We introduce a partition of unity $\{\zeta_k\}_{k=-\infty}^{+\infty} \subset C_0^\infty(\mathbb{R}^1)$ on the t-axis subordinate to the covering of $\mathbb{R}^1$ by the intervals $k-1 < t < k+1$ satisfying the condition

$$(5.2.7) \qquad |D^j_t \zeta_k(t)| < c_0 \qquad \text{for } t \in \mathbb{R},\ j = 0,1,\dots,l,$$

where c_0 is a constant independent of k.

LEMMA 5.2.2. *Let $\{\zeta_k\}$ be a partition of unity on $\mathbb{R}^1$ with the above properties. Then there exist positive real constants c_1, c_2 depending only on l and c_0 such that*

$$(5.2.8) \qquad c_1 \|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \le \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}\Big)^{1/2} \le c_2 \|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}$$

for each $u \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$. An analogous estimate holds for the norm in the space $\mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})$ if $l \ge 1$.

Proof: It suffices to prove the assertion of the lemma for $\beta = 0$. Otherwise, we substitute $e^{\beta t}u = v$.

1) From (5.2.7) it follows that

$$\|\zeta_k u\|^2_{W^l_2(\mathcal{C})} \le c \int\limits_{k-1}^{k+1} \int\limits_{\Omega} \sum_{|\alpha|+j\le l} |D^\alpha_x D^j_t\, u(x,t)|^2\, dx\, dt$$

for every k, where the constant c depends only on c_0. Hence

$$\sum_{k=-\infty}^{+\infty} \|\zeta_k\, u\|^2_{W^l_2(\mathcal{C})} \le 2c\, \|u\|^2_{W^l_2(\mathcal{C})}\,.$$

Furthermore, since $\zeta_{k-1}+\zeta_k+\zeta_{k+1}=1$ on the interval $k-1<t<k+1$, we get

$$\begin{aligned}\|u\|^2_{W_2^l(\mathcal{C})} &= \frac{1}{2}\sum_{k=-\infty}^{+\infty}\int\limits_{k-1}^{k+1}\int\limits_{\Omega}\sum_{|\alpha|+j\le l}|D_x^\alpha D_t^j u|^2\,dx\,dt\\ &\le \frac{1}{2}\sum_{k=-\infty}^{+\infty}\|(\zeta_{k-1}+\zeta_k+\zeta_{k+1})\,u\|^2_{W_2^l(\mathcal{C})}\le\frac{3}{2}\sum_{k=-\infty}^{+\infty}\|\zeta_k u\|^2_{W_2^l(\mathcal{C})}.\end{aligned}$$

2) Let u be an arbitrary function from $W_2^{l-1/2}(\partial\mathcal{C})$. Then there exists an extension $v\in W_2^l(\mathcal{C})$ of u satisfying the inequality

$$\|v\|^2_{W_2^l(\mathcal{C})}\le 2\,\|u\|^2_{W_2^{l-1/2}(\partial\mathcal{C})}.$$

Consequently, by means of (5.2.8), we obtain

$$\sum_{k=-\infty}^{+\infty}\|\zeta_k u\|^2_{W_2^{l-1/2}(\partial\mathcal{C})}\le\sum_{k=-\infty}^{+\infty}\|\zeta_k v\|^2_{W_2^l(\mathcal{C})}\le c_2^2\,\|v\|^2_{W_2^l(\mathcal{C})}\le 2c_2^2\,\|u\|^2_{W_2^{l-1/2}(\partial\mathcal{C})}.$$

On the other hand, for every $k=0,\pm1,\pm2,\dots$ there exist an extension $v_k\in W_2^l(\mathcal{C})$ of $\zeta_k u$ satisfying the inequality

$$\|v_k\|^2_{W_2^l(\mathcal{C})}\le 2\,\|\zeta_k u\|^2_{W_2^{l-1/2}(\partial\mathcal{C})}.$$

Obviously, the functions $w_k=(\zeta_{k-1}+\zeta_k+\zeta_{k+1})v_k$ are also extensions of $\zeta_k u$ which satisfy the estimate

$$\|w_k\|^2_{W_2^l(\mathcal{C})}\le c\,\|\zeta_k u\|^2_{W_2^{l-1/2}(\partial\mathcal{C})}$$

with a constant c independent of u and k. Since $w=\sum_{k=-\infty}^{+\infty}w_k$ is an extension of u and $w_k(x,t)=0$ for $|t-k|>2$, we obtain

$$\begin{aligned}\|u\|^2_{W_2^{l-1/2}(\partial\mathcal{C})} &\le \Big\|\sum_{k=-\infty}^{+\infty}w_k\Big\|^2_{W_2^l(\mathcal{C})}\\ &= \frac{1}{2}\sum_{\mu=-\infty}^{+\infty}\sum_{|\alpha|+j\le l}\int\limits_{\mu-1}^{\mu+1}\int\limits_{\Omega}\Big|\sum_{k=\mu-2}^{\mu+2}D_x^\alpha D_t^j w_k\Big|^2\,dx\,dt\\ &\le c\sum_{\mu=-\infty}^{+\infty}\sum_{k=\mu-2}^{\mu+2}\|w_k\|^2_{W_2^l(\mathcal{C})}\le c'\sum_{k=-\infty}^{+\infty}\|\zeta_k u\|^2_{W_2^{l-1/2}(\partial\mathcal{C})}.\end{aligned}$$

The proof is complete. ∎

Other analogous norms in $\mathcal{W}_{2,\beta}^l(\mathcal{C})$ and $\mathcal{W}_{2,\beta}^{l-1/2}(\partial\mathcal{C})$ can be given by means of the *Laplace transformation* $\mathcal{L}_{t\to\lambda}$ with respect to the variable t

$$\tilde{u}(\lambda)=(\mathcal{L}_{t\to\lambda}u)(\lambda)=\int\limits_{-\infty}^{+\infty}e^{-\lambda t}\,u(t)\,dt. \tag{5.2.9}$$

Let us recall some well-known properties of this transformation (see, e.g., [**62**]).

LEMMA 5.2.3. 1) *The transformation (5.2.9) defines a linear and continuous mapping from* $C_0^\infty(\mathbb{R})$ *into the space of the analytic functions on the complex plane. Furthermore,* $\mathcal{L}_{t\to\lambda}(\partial_t u) = \lambda\, \mathcal{L}_{t\to\lambda} u$.

2) *For all* $u,\, v \in C_0^\infty(\mathbb{R})$ *the Parseval equality*

$$\int_{-\infty}^{+\infty} e^{2\beta t} u(t)\, \overline{v(t)}\, dt = \frac{1}{2\pi i} \int_{Re\,\lambda=-\beta} \check{u}(\lambda)\, \overline{\check{v}(\lambda)}\, d\lambda \tag{5.2.10}$$

is satisfied. Here the integration takes place over the line $\ell_{-\beta} : \lambda = i\tau - \beta,\ -\infty < \tau < \infty$. *Hence the transformation (5.2.9) can be continuously extended to an isomorphism*

$$L_{2,\beta}(\mathbb{R}) \to L_2(\ell_{-\beta}),$$

where $L_{2,\beta}(\mathbb{R}_+)$ *is the weighted* L_2 *space with scalar product equal to the left-hand side of (5.2.10).*

3) *The inverse Fourier transformation is given by the formula*

$$u(t) = (\mathcal{L}^{-1}_{\lambda\to t}\check{u})(t) = \frac{1}{2\pi i} \int_{\ell_{-\beta}} e^{\lambda t}\, \check{u}(\lambda)\, d\lambda\,.$$

4) *If* $u \in L_{2,\beta_1}(\mathbb{R}) \cap L_{2,\beta_2}(\mathbb{R})$, *where* $\beta_1 < \beta_2$, *then* $\check{u} = \mathcal{L}_{t\to\lambda} u$ *is holomorphic in the strip* $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$.

Using these properties, we can prove the following assertions.

LEMMA 5.2.4. *The norm (5.2.1) with* $l \ge 0$ *is equivalent to the norm*

$$\|u\| = \Big(\frac{1}{2\pi i} \int_{Re\,\lambda=-\beta} \big(\|\check{u}(\cdot,\lambda)\|^2_{W_2^l(\Omega)} + |\lambda|^{2l}\, \|\check{u}(\cdot,\lambda)\|^2_{L_2(\Omega)}\big)\, d\lambda\Big)^{1/2}. \tag{5.2.11}$$

Analogously, an equivalent norm to (5.2.2) for $l \ge 1$ *is*

(5.2.12)

$$\|u\| = \Big(\frac{1}{2\pi i} \int_{Re\,\lambda=-\beta} \big(\|\check{u}(\cdot,\lambda)\|^2_{W_2^{l-1/2}(\partial\Omega)} + |\lambda|^{2l-1}\, \|\check{u}(\cdot,\lambda)\|^2_{L_2(\partial\Omega)}\big)\, d\lambda\Big)^{1/2}.$$

Proof: 1) By the Parseval equality, the norm (5.2.4) is equal to

$$\Big(\frac{1}{2\pi i} \int_{Re\,\lambda=-\beta} \sum_{j=0}^{l} |\lambda|^{2j} \int_\Omega \sum_{|\alpha|\le l-j} |D_x^\alpha \check{u}(x,\lambda)|^2\, dx\, d\lambda\Big)^{1/2}.$$

Using inequality (3.6.14), we get the first assertion.

2) Let u be a function from $\mathcal{W}_{2,\beta}^{l-1/2}(\partial\mathcal{C})$. We suppose first that the support of $u(\cdot,t)$ lies in a sufficiently small neighbourhood $\mathcal{U}$ of the point $x_0 \in \partial\Omega$ for every $t \in \mathbb{R}$. For the sake of simplicity, let $\mathcal{U}$ be a subset of the $(n-1)$-dimensional hyper-plane $x_n = 0$. (Otherwise, there exists a diffeomorphism $x \to \kappa(x) = x'$ taking $\mathcal{U}$ onto a subset of the hyper-plane $x'_n = 0$, and we can consider the norm of the function $(\kappa_* u)(x',t) \stackrel{def}{=} u(\kappa^{-1}(x'),t)$ instead of the norm of u.)

We extend $u = u(y,t)$ to a 2π-periodic function with respect to the variable $y = (x_1, \dots, x_{n-1})$ on $\mathbb{R}^{n-1}$. Then the norm (5.2.2) is equivalent to the norm

$$(5.2.13) \qquad \|u\| = \|e^{\beta t} u\|_{W^{l-1/2}_{2,per}(\mathbb{R}^{n-1}\times\mathbb{R})} = \Big(\sum_{q\in\mathbb{Z}^{n-1}} \langle q\rangle^{2l-2} \int_{\mathbb{R}} (1+\tau^2)^{l-1/2} \, |(\mathcal{F}_{t\to\tau} V)(q,\tau)|^2 \, d\tau \Big)^{1/2},$$

where $V(q,t) = e^{\beta t/\langle q\rangle} \dot u(q, \langle q\rangle^{-1} t)$ and $\dot u(q,t)$ are the Fourier coefficients of $u(\cdot,t)$ (cf. formula (2.2.6)). Substituting $i\langle q\rangle\tau - \beta = \lambda$ and using the equality

$$(\mathcal{F}_{t\to\tau} V)(q,\tau) = \langle q\rangle \, (\mathcal{F}_{t\to\tau} \dot u)(q\,,\, \langle q\rangle\tau + i\beta) = (2\pi)^{-1/2} \, (\mathcal{L}_{t\to\lambda} \dot u)(q, i\langle q\rangle\tau - \beta),$$

we obtain that the norm (5.2.13) is equal to

$$\Big(\frac{1}{2\pi i} \int_{Re\,\lambda=-\beta} \sum_{q\in\mathbb{Z}^{n-1}} (\langle q\rangle^2 + |\lambda+\beta|^2)^{l-1/2} \, |(\mathcal{L}_{t\to\lambda} \dot u)(q,\lambda)|^2 \, d\lambda \Big)^{1/2}.$$

Since

$$c_1 \, (\langle q\rangle^{2l-1} + |\lambda|^{2l-1}) \le (\langle q\rangle^2 + |\lambda+\beta|^2)^{l-1/2} \le c_2 \, (\langle q\rangle^{2l-1} + |\lambda|^{2l-1}),$$

where c_1, c_2 depend only on l and β, we get the equivalence of the norms (5.2.2) and (5.2.12) on the set of all functions $u \in \mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})$ with support in $\mathcal{U}\times\mathbb{R}$. If u is a function with arbitrary support, then this assertion holds by means of a sufficiently fine partition of unity on $\partial\Omega$. ■

Note that the expressions to be integrated in (5.2.11), (5.2.12) can be replaced by the squares of the $W^l_2(\Omega,\lambda)$ and $W^{l-1/2}_2(\partial\Omega,\lambda)$ norms of $\check u$ (see (3.6.18), (3.6.19)).

5.2.2. The operator of the boundary value problem. We consider the boundary value problem

$$(5.2.14) \qquad Lu = f \quad \text{in } \mathcal{C},$$

$$(5.2.15) \qquad Bu + C\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{C}.$$

Here

$$L = L(x,\partial_x,\partial_t) = \sum_{|\alpha|+j\le 2m} a_{\alpha,j}(x)\, \partial_x^\alpha\, \partial_t^j$$

is a differential operator of order $2m$ with coefficients $a_{\alpha,j} \in C^\infty(\overline{\Omega})$, B is a vector of differential operators

$$B_k(x,\partial_x,\partial_t) = \sum_{|\alpha|+j\le\mu_k} b_{k;\alpha,j}(x)\, \partial_x^\alpha\, \partial_t^j, \quad k = 1,\dots,m+J,$$

with coefficients $b_{k;\alpha,j} \in C^\infty(\overline{\Omega})$, and C is a matrix of tangential differential operators

$$C_{k,j}(x,\partial_x,\partial_t) = \sum_{|\alpha|+s\le\mu_k+\tau_j} c_{k,j;\alpha,s}(x)\, \partial_x^\alpha\, \partial_t^s, \quad k = 1,\dots,m+J,\ j = 1,\dots,J,$$

on $\partial\mathcal{C}$ with infinitely differentiable coefficients in a neighbourhood of $\partial\Omega$. (∂_x^α denotes the partial derivative $\partial^{|\alpha|}/\partial^{\alpha_1}_{x_1}\cdots\partial^{\alpha_n}_{x_n}$.)

Throughout this chapter, the orders of the differential operators B_k are assumed to be less than $2m$.

REMARK 5.2.1. In the case $n = 1$, when the boundary $\partial\mathcal{C}$ consists of two parallel lines, we can admit different boundary conditions on both parts of $\partial\mathcal{C}$.

We denote the operator of the boundary value problem (5.2.14), (5.2.15) by $\mathfrak{A}(\partial_t)$. Obviously, $\mathfrak{A}(\partial_t)$ continuously maps the space

$$\mathcal{W}^{l}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C}) \tag{5.2.16}$$

into

$$\mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C}) \tag{5.2.17}$$

for $l \geq 2m$. Here $\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$, $\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})$ denote the products of the spaces $\mathcal{W}^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{C})$, $j = 1, \dots , J$, and $\mathcal{W}^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{C})$, $k = 1, \dots , m + J$, respectively.

5.2.3. The operator pencil generated by the model problem. Applying the Laplace transformation $t \to \lambda$ to the boundary value problem (5.2.14), (5.2.15), we get the parameter-depending problem

$$L(x, \partial_x, \lambda)\, \check{u} = \check{f} \qquad \text{in } \Omega, \tag{5.2.18}$$

$$B(x, \partial_x, \lambda)\, \check{u} + C(x, \partial_x, \lambda)\, \underline{\check{u}} = \underline{\check{g}} \quad \text{on } \partial\Omega. \tag{5.2.19}$$

We denote the operator of this boundary value problem by $\mathfrak{A}(\lambda)$. For every fixed $\lambda \in \mathbb{C}$ this operator continuously maps

$$\mathcal{X} \stackrel{def}{=} W^l_2(\Omega) \times \prod_{j=1}^{J} W^{l+\tau_j-1/2}_2(\partial\Omega) \tag{5.2.20}$$

into

$$\mathcal{Y} \stackrel{def}{=} W^{l-2m}_2(\Omega) \times \prod_{k=1}^{m+J} W^{l-\mu_k-1/2}_2(\partial\Omega), \quad l \geq 2m. \tag{5.2.21}$$

If the boundary value problem (5.2.14), (5.2.15) is elliptic, then the problem (5.2.18), (5.2.19) is elliptic with parameter (cf. Definition 3.6.1). As a consequence of Theorems 3.6.1 and 5.1.1, we get the following assertions.

THEOREM 5.2.1. *Suppose that the boundary value problem (5.2.14), (5.2.15) is elliptic. Then*

1) *the operator* $\mathfrak{A}(\lambda) : \mathcal{X} \to \mathcal{Y}$ *is Fredholm for all* $\lambda \in \mathbb{C}$ *and isomorphic for all* $\lambda \in \mathbb{C}$ *except for a countable number of isolated points, the eigenvalues of* $\mathfrak{A}(\lambda)$,

2) *the eigenvalues, with the possible exception of finitely many points, are situated outside a double sector*

$$\big\{\lambda \in \mathbb{C} : \ |\mathrm{Re}\, \lambda| < \delta\, |\mathrm{Im}\, \lambda|\big\}, \quad \delta > 0,$$

of the complex plane containing the imaginary axis.

3) *If* λ_0 *is an eigenvalue of* $\mathfrak{A}(\lambda)$ *with the geometric multiplicity* I *and the partial multiplicities* $\kappa_1, \dots , \kappa_I$, *then in a neighbourhood of* λ_0 *there is an expansion*

$$\mathfrak{A}(\lambda)^{-1} = \sum_{j=1}^{I} \sum_{s=0}^{\kappa_j-1} \frac{P_{j,s}}{(\lambda - \lambda_0)^{\kappa_j-s}} + \Gamma(\lambda), \tag{5.2.22}$$

where $P_{j,s} : \mathcal{Y} \to \mathcal{X}$ *are linear operators not depending on* λ *and* Γ *is a holomorhic operator function in this neighbourhood.*

REMARK 5.2.2. The operators $P_{j,s}$ map the space $\mathcal{Y}$ into the linear space of the eigenvectors and generalized eigenvectors of $\mathfrak{A}(\lambda)$ corresponding to the eigenvalue λ_0. Since every Jordan chain $\{(\varphi_{j,s}, \underline{\varphi}_{j,s})\}_{s=0,\dots,\kappa_j-1}$ satisfies the equations

$$\mathfrak{A}(\lambda_0)\,(\varphi_{j,s}, \underline{\varphi}_{j,s}) = -\sum_{\sigma=1}^{s} \frac{1}{\sigma!}\mathfrak{A}^{(\sigma)}(\lambda_0)\,(\varphi_{j,s-\sigma}, \underline{\varphi}_{j,s-\sigma}), \quad s = 0, \dots, \kappa_j - 1,$$

and $\mathfrak{A}(\lambda_0)$ is the operator of an elliptic boundary value problem in Ω, it follows by induction that every eigenvector and generalized eigenvector is infinitely differentiable. Thus, the operators $P_{j,s}$ map the space $\mathcal{X}$ into a finite-dimensional subspace of $C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^J$. Moreover, we can conclude that the eigenvalues, eigenvectors, and generalized eigenvectors of $\mathfrak{A}(\lambda)$ do not depend on the number l in the definition of the spaces $\mathcal{X}$ and $\mathcal{Y}$.

Let $\mathfrak{A}^*(\lambda)$ be the adjoint operator pencil defined by (5.1.3), where $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$, are the scalar products in $L_2(\Omega) \times L_2(\partial\Omega)^J$ and $L_2(\Omega) \times L_2(\partial\Omega)^{m+J}$, respectively. Then formula (5.1.5) is valid for the operators $P_{j,s}$ in (5.2.22), where $\psi_{j,\sigma}$ are the eigenvectors and generalized eigenvectors of the pencil $\mathfrak{A}^*(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_0$ satisfying the biorthonormality condition (5.1.6).

5.2.4. Existence and uniqueness of the solution. From the second assertion of Theorem 5.2.1 it follows that every line $\operatorname{Re}\lambda = -\beta$ parallel to the imaginary axis contains only finitely many eigenvalues of $\mathfrak{A}(\lambda)$.

THEOREM 5.2.2. *Suppose that the model problem (5.2.14), (5.2.15) is elliptic and no eigenvalues of* $\mathfrak{A}(\lambda)$ *lie on the line* $\operatorname{Re}\lambda = -\beta$. *Then this problem is uniquely solvable in (5.2.16) for every given pair* $(f, \underline{g})$ *from the space (5.2.17) with* $l \geq 2m$, $l > \max \mu_k$. *Furthermore, the solution* $(u, \underline{u})$ *satisfies the estimate*

$$\|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} + \|\underline{u}\|_{\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})} \leq c\left(\|f\|_{\mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C})} + \|\underline{g}\|_{\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})}\right) \tag{5.2.23}$$

with a constant c *independent of* u *and* $\underline{u}$.

Proof: 1) First we prove the uniqueness of the solution. Let $(u, \underline{u}) \in \mathcal{W}^l_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$ be a solution of the model problem. We denote the Laplace transforms of u, $\underline{u}$, f, and $\underline{g}$ with respect to the variable t by $\check{u}$, $\check{\underline{u}}$, $\check{f}$, and $\check{\underline{g}}$, respectively. Then $(\check{u}, \check{\underline{u}})$ is a solution of the parameter-depending problem (5.2.18), (5.2.19) in the domain Ω.

By Theorem 3.6.1, this solution satisfies the estimate

$$\begin{aligned} &\|\check{u}(\cdot,\lambda)\|^2_{W^l_2(\Omega,\lambda)} + \|\check{\underline{u}}(\cdot,\lambda)\|^2_{W_2^{l+\underline{\tau}-1/2}(\partial\Omega,\lambda)} \\ &\leq c\left(\|\check{f}(\cdot,\lambda)\|^2_{W_2^{l-2m}(\Omega,\lambda)} + \|\check{\underline{g}}(\cdot,\lambda)\|^2_{W_2^{l-\underline{\mu}-1/2}(\partial\Omega,\lambda)}\right) \end{aligned} \tag{5.2.24}$$

for every λ on the line $\operatorname{Re}\lambda = -\beta$, where the constant c is independent of u, $\underline{u}$ and λ. Integrating (5.2.24) with respect to λ over the line $\operatorname{Re}\lambda = -\beta$, we get the estimate (5.2.23). This proves the uniqueness of the solution.

2) The existence of the solution follows from the solvability of the parameter-depending problem (5.2.18), (5.2.19) for $\operatorname{Re}\lambda = -\beta$ and the a priori estimate (5.2.24). Using the formula for the inverse Laplace transformation, we get the

representation

$$(u(\cdot,t),\underline{u}(\cdot,t)) = \frac{1}{2\pi i} \int\limits_{Re\,\lambda=-\beta} e^{\lambda t}\, \mathfrak{A}(\lambda)^{-1} \big(\check{f}(\cdot,\lambda),\check{\underline{g}}(\cdot,\lambda)\big)\, d\lambda \tag{5.2.25}$$

for the solution $(u,\underline{u})$. ■

5.2.5. Necessity of the condition on the eigenvalues. Suppose that the line $\operatorname{Re}\lambda=-\beta$ contains an eigenvalue λ_0 of $\mathfrak{A}(\lambda)$. We consider the vector-function

$$\big(u(x,t)\,,\,\underline{u}(x,t)\big) = \chi_T(t)\, e^{\lambda_0 t}\, \big(\varphi(x),\underline{\varphi}(x)\big),$$

where $(\varphi,\underline{\varphi})$ is an arbitrary eigenvector to λ_0 and $\{\chi_T\}_{T>0}$ is a set of smooth functions on the t-axis satisfying the conditions

$$\chi_T(t) = \begin{cases} 1 & \text{for} \quad |t|<T \\ 0 & \text{for} \quad |t|>T+1 \end{cases}, \qquad |D_t^j \chi_T(t)| < c_j \quad \text{for } j=0,1,\ldots$$

with constants c_j independent of t and T. Obviously, the norm of $(u,\underline{u})$ in the space (5.2.16) tends to infinity for $T\to\infty$. However, from the equation

$$\begin{aligned} (f,\underline{g}) &= \mathfrak{A}(\partial_t)\,(u,\underline{u}) = e^{\lambda_0 t}\, \mathfrak{A}(\partial_t+\lambda_0)\, \chi_T(t)\, (\varphi,\underline{\varphi}) \\ &= e^{\lambda_0 t} \sum_{j\ge 1} \partial_t^j \chi_T(t)\cdot \mathfrak{A}^{(j)}(\lambda_0)\, (\varphi,\underline{\varphi}) \end{aligned}$$

it follows that the norm of $(f,\underline{g})$ in the space (5.2.17) is bounded by a constant c independent of T. Thus, we have shown the following assertion.

LEMMA 5.2.5. *If the line* $\operatorname{Re}\lambda=-\beta$ *contains eigenvalues of* $\mathfrak{A}(\lambda)$, *then there does not exist a finite constant c such that the inequality (5.2.23) is satisfied for all elements* $(u,\underline{u})$ *of the space (5.2.16).*

5.3. Solvability of the model problem in the cylinder in Sobolev spaces of negative order

In the previous section we obtained necessary and sufficient conditions for the unique solvability of the model problem in weighted Sobolev spaces of positive order. Now we show that these conditions ensure the unique solvability of the model problem in weighted Sobolev spaces of arbitrary integer order. The extension of the operator of the boundary value problem to weighted Sobolev spaces of negative order will be constructed in a similar manner as for the case of a bounded domain with smooth boundary.

5.3.1. Weighted Sobolev spaces of negative order. If l is an arbitrary nonnegative integer, then we define the space $\mathcal{W}^l_{2,\beta}(\mathcal{C})^*$ as the dual space of $\mathcal{W}^l_{2,\beta}(\mathcal{C})$ equipped with the norm

$$\|v\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} = \sup\Big\{ |(u,v)_\mathcal{C}| \ :\ v\in \mathcal{W}^l_{2,\beta}(\mathcal{C}),\ \|v\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \le 1\Big\}. \tag{5.3.1}$$

Here $(\cdot,\cdot)_\mathcal{C}$ denotes the extension of the scalar product in $L_2(\mathcal{C})$ to the product of the spaces $\mathcal{W}^l_{2,\beta}(\mathcal{C})^*$ and $\mathcal{W}^l_{2,\beta}(\mathcal{C})$. Obviously, $\mathcal{W}^l_{2,\beta}(\mathcal{C})^*$ is the set of all functionals v such that $e^{-\beta t}u \in W^l_2(\mathcal{C})^*$ and

$$\|v\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} = \|e^{-\beta t}v\|_{W^l_2(\mathcal{C})^*}.$$

In particular, the space $\mathcal{W}^0_{2,\beta}(\mathcal{C})^*$ coincides with $\mathcal{W}^0_{2,-\beta}(\mathcal{C})$. Furthermore, it can be easily seen that the space $\mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})$ defined in Section 5.2 is the dual space of $\mathcal{W}^{-l+1/2}_{2,-\beta}(\partial\mathcal{C})$ if $l \le 0$.

Analogously to the spaces $\tilde{W}^{l,k}_2$ which were defined in Section 3.2, we introduce the class of the spaces $\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C})$ with arbitrary integer l and k, $k \ge 0$. For negative l this space is defined as the set of all pairs $(u, \underline{\phi})$, where

$$u \in \mathcal{W}^{-l}_{2,-\beta}(\mathcal{C})^* \quad \text{and} \quad \underline{\phi} = (\phi_1, \dots, \phi_k) \in \prod_{j=1}^{k} \mathcal{W}^{l-j+1/2}_{2,\beta}(\partial\mathcal{C})$$

If l is a nonnegative integer, then $\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C})$ is the set of all pairs $(u, \underline{\phi})$ such that $u \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$ and the components ϕ_j of the vector-function $\underline{\phi}$ are functions from $\mathcal{W}^{l-j+1/2}_{2,\beta}(\partial\mathcal{C})$, $j = 1, \dots, k$, satisfying the condition

$$D^{j-1}_\nu u\big|_{\partial\mathcal{C}} = \phi_j \quad \text{for } j \le \min(k, l).$$

In particular, with this notation, we have

$$\tilde{\mathcal{W}}^{l,0}_{2,\beta}(\mathcal{C}) = \begin{cases} \mathcal{W}^l_{2,\beta}(\mathcal{C}) & \text{if } l \ge 0, \\ \mathcal{W}^{-l}_{2,-\beta}(\mathcal{C})^* & \text{if } l < 0. \end{cases}$$

The norm in $\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C})$ is defined in a natural way as the sum of the norms of u and ϕ_j in the corresponding weighted spaces. Obviously,

$$\|(u, \underline{\phi})\|_{\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C})} = \|e^{\beta t}\,(u, \underline{\phi})\|_{\tilde{W}^{l,k}_2(\mathcal{C})}\,.$$

Hence $\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C})$ is dense in $\tilde{\mathcal{W}}^{l-1,k}_{2,\beta}(\mathcal{C})$ for arbitrary integer l, and the operator of the imbedding $\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C}) \subset \tilde{\mathcal{W}}^{l-1,k}_{2,\beta}(\mathcal{C})$ is continuous.

LEMMA 5.3.1. *Let $\{\zeta_k\}_{k=-\infty}^{+\infty}$ be a partition of unity on $\mathbb{R}^1$ with the same properties as in Lemma 5.2.2. Then there exist positive constants c_1, c_2 such that*

$$c_1\, \|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} \le \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} \Big)^{1/2} \le c_2\, \|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} \tag{5.3.2}$$

for each $u \in \mathcal{W}^l_{2,\beta}(\mathcal{C})^*$, $l \ge 0$. Analogous estimates are valid for the norms in $\mathcal{W}^{l-1/2}_{2,\beta}(\mathcal{C})$ and $\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C})$ if l, k are arbitrary integers, $k \ge 0$.

Proof: Let u, v be arbitrary elements of the spaces $\mathcal{W}^l_{2,\beta}(\mathcal{C})^*$ and $\mathcal{W}^l_{2,\beta}(\mathcal{C})$, respectively. We set $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$. Using Lemma 5.2.2 and the fact that $\eta_k = 1$ on the support of ζ_k, we get

$$\begin{aligned} |(u, v)_{\mathcal{C}}| &= \Big| \sum_{k=-\infty}^{+\infty} (\zeta_k u, v)_{\mathcal{C}} \Big| = \Big| \sum_{k=-\infty}^{+\infty} \big(\zeta_k u\,,\, \eta_k v\big)_{\mathcal{C}} \Big| \\ &\le 3 \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} \Big)^{1/2} \cdot \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k v\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \Big)^{1/2} \\ &\le c \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} \Big)^{1/2} \cdot \|v\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}\,. \end{aligned}$$

This proves the left inequality in (5.3.2). Furthermore, by the Riesz theorem, for every $u \in \mathcal{W}^l_{2,\beta}(\mathcal{C})^*$ there exists a function $w \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$ such that

$$\|w\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} = \|u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} \quad \text{and} \quad (u,v)_{\mathcal{C}} = (w,v)_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \tag{5.3.3}$$

for all $v \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$. Here $(\cdot,\cdot)_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}$ denotes the scalar product in $\mathcal{W}^l_{2,\beta}(\mathcal{C})$. Moreover, there exist functions $w_k \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$, $k = 0, \pm 1, \dots$, such that

$$\|w_k\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} = \|\zeta_k u\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} \qquad \text{and} \quad (\zeta_k u, v)_{\mathcal{C}} = (w_k\, , v)_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}$$

for each $v \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$. Since $(\zeta_k u, v)_{\mathcal{C}} = (u, \overline{\zeta}_k v)_{\mathcal{C}}$, it follows from (5.3.3) that

$$(w_k, v)_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} = (u, \overline{\zeta}_k v)_{\mathcal{C}} = (w, \overline{\zeta}_k v)_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} = \big(\eta_k w, \overline{\zeta}_k v\big)_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}\,.$$

Hence

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*} &= \sum_{k=-\infty}^{+\infty} \|w_k\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} = \sum_{k=-\infty}^{+\infty} \big(\eta_k w, \overline{\zeta}_k w_k\big)_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \\
&\le 3\Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k w\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}\Big)^{1/2} \cdot \Big(\sum_{k=-\infty}^{+\infty} \|\overline{\zeta}_k w_k\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})}\Big)^{1/2} \\
&\le c\,\|w\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \cdot \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{\mathcal{W}^l_{2,\beta}(\mathcal{C})^*}\Big)^{1/2}.
\end{aligned}$$

This implies the right estimate in (5.3.2).

In the same way, the analogous inequalities for the norm in $\mathcal{W}^{l-1/2}_{2,\beta}(\partial\mathcal{C})$ can be proved if $l \le 0$. As a consequence we get the estimates in $\tilde{\mathcal{W}}^{l,k}_{2,\beta}(\mathcal{C})$. ■

5.3.2. Solvability of the formally adjoint problem. Since $\operatorname{ord} B_k < 2m$ for $k = 1, \dots , m + J$, the vector $B(x, \partial_x, \partial_t)u$ can be written in the form

$$B(x, \partial_x, \partial_t)u\big|_{\partial\mathcal{C}} = Q(x, \partial_x, \partial_t) \cdot \mathcal{D}u\big|_{\partial\mathcal{C}}\,, \tag{5.3.4}$$

where Q is a $(m+J)\times 2m$-matrix of tangential differential operators $Q_{k,j}$, $\operatorname{ord} Q_{k,j} \le \mu_k + 1 - j$, $Q_{k,j} \equiv 0$ if $\mu_k + 1 - j < 0$. Here, as in Chapters 3 and 4, $\mathcal{D}$ denotes the column vector with the components $1, D_\nu, \dots , D_\nu^{2m-1}$.

As in Section 3.6, we denote by $L^+(x, \partial_x, \lambda)$ the formally adjoint differential operator to $L(x, \partial_x, \overline{\lambda})$. Then the operator $L^+(x, \partial_x, -\partial_t)$ is formally adjoint to $L(x, \partial_x, \partial_t)$. Furthermore, let $C^+(x, \partial_x, -\partial_t)$ and $Q^+(x, \partial_x, -\partial_t)$ be the formally adjoint operators to $C(x, \partial_x, \partial_t)$ and $Q(x, \partial_x, \partial_t)$, respectively. Then the following Green formula (cf. (3.6.11)) is valid for smooth functions u, v, $\underline{u}$, $\underline{v}$ with compact supports:

$$\begin{aligned}
&\int_{\mathcal{C}} L(\partial_t)u \cdot \overline{v}\, dx\, dt + \int_{\partial\mathcal{C}} \big(B(\partial_t)u + C(\partial_t)\underline{u}\, ,\, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma\, dt \\
&= \int_{\mathcal{C}} u \cdot \overline{L^+(-\partial_t)v}\, dx\, dt + \int_{\partial\mathcal{C}} \big(\mathcal{D}u,\ P(-\partial_t)v + Q^+(-\partial_t)\underline{v}\big)_{\mathbb{C}^{2m}}\, d\sigma\, dt \\
&\quad + \int_{\partial\mathcal{C}} \big(\underline{u}, C^+(-\partial_t)\underline{v}\big)_{\mathbb{C}^J}\, d\sigma\, dt
\end{aligned} \tag{5.3.5}$$

Here, for the sake of brevity, we have omitted the arguments x and ∂_x in the differential operators. The operator $P(-\partial_t)$ is a vector of differential operators

$P_j(x, \partial_x, -\partial_t)$ of order $2m - j$ ($j = 1, \dots, 2m$). By (5.3.5), the boundary value problem

$$L^+(-\partial_t)v = f \quad \text{in } \mathcal{C}, \tag{5.3.6}$$

$$P(-\partial_t)v + Q^+(-\partial_t)\underline{v} = \underline{g}, \quad C^+(-\partial_t)\underline{v} = \underline{h} \quad \text{on } \partial\mathcal{C} \tag{5.3.7}$$

is *formally adjoint* to problem (5.2.14), (5.2.15). We denote the operator of this problem by $\mathfrak{A}^+(-\partial_t)$ and the corresponding operator pencil (i.e., the operator of the parameter-depending boundary value problem (3.6.12), (3.6.13)) by $\mathfrak{A}^+(-\lambda)$. By the Green formula (3.6.9), the problem $\mathfrak{A}^+(\overline{\lambda})\,(v, \underline{v}) = (f, \underline{g}, \underline{h})$, i.e.,

$$L^+(\overline{\lambda})\, v = f \quad \text{in } \Omega,$$
$$P(\overline{\lambda})\, v + Q^+(\overline{\lambda})\, \underline{v} = \underline{g}, \quad C^+(\overline{\lambda})\, \underline{v} = \underline{h} \quad \text{on } \partial\Omega,$$

is formally adjoint to problem (5.2.18), (5.2.19) for each $\lambda \in \mathbb{C}$.

LEMMA 5.3.2. *If the boundary value problem (5.2.14), (5.2.15) is elliptic, then* $\mathfrak{A}^+(\lambda)$ *is an isomorphism from*

$$W_2^l(\Omega) \times \prod_{k=1}^{m+J} W_2^{l-2m+\mu_k+1/2}(\partial\Omega),$$

$l \geq 2m$, *onto the space*

$$W_2^{l-2m}(\Omega) \times \prod_{j=1}^{2m} W_2^{l-2m+j-1/2}(\partial\Omega) \times \prod_{j=1}^{J} W_2^{l-2m-\tau_j+1/2}(\partial\Omega)$$

for all $\lambda \in \mathbb{C}$ *except for a countable number of isolated points, the eigenvalues of* $\mathfrak{A}^+(\lambda)$. *The complex number* λ_0 *is an eigenvalue of* $\mathfrak{A}^+(\lambda)$ *if and only if* $\overline{\lambda}_0$ *is an eigenvalue of* $\mathfrak{A}(\lambda)$.

Proof: Since the boundary value problem (5.2.14), (5.2.15) and its formally adjoint are simultaneously elliptic, the first assertion follows immediately from Theorem 5.2.1. Furthermore, since the problem $\mathfrak{A}^+(\overline{\lambda})\,(v, \underline{v}) = (f, \underline{g}, \underline{h})$ is formally adjoint to (5.2.18), (5.2.19) for every fixed $\lambda \in \mathbb{C}$, the second assertion is a consequence of Theorems 3.4.1 and 5.2.1. ∎

Using the last lemma and Theorem 5.2.2, we get the following assertion.

COROLLARY 5.3.1. *Suppose that the model problem (5.2.14), (5.2.15) is elliptic and no eigenvalues of* $\mathfrak{A}(\lambda)$ *lie on the line* $\text{Re}\,\lambda = -\beta$. *Then the operator* $\mathfrak{A}^+(-\partial_t)$ *of the formally adjoint boundary value problem (5.3.6), (5.3.7) is an isomorphism from*

$$\mathcal{W}_{2,-\beta}^l(\mathcal{C}) \times \prod_{k=1}^{m+J} \mathcal{W}_{2,-\beta}^{l-2m+\mu_k+1/2}(\partial\mathcal{C}) \tag{5.3.8}$$

onto the space

$$\mathcal{W}_{2,-\beta}^{l-2m}(\mathcal{C}) \times \prod_{j=1}^{2m} \mathcal{W}_{2,-\beta}^{l-2m+j-1/2}(\partial\mathcal{C}) \times \prod_{j=1}^{J} \mathcal{W}_{2,-\beta}^{l-2m-\tau_j+1/2}(\partial\mathcal{C}). \tag{5.3.9}$$

for $l \geq 2m$, $l \geq 2m + \max \tau_j$.

5.3.3. Extension of the operator of the boundary value problem. For $l \ge 2m$ the operator $\mathfrak{A}(\partial_t)$ can be identified with the operator

$$(5.3.10)\quad \tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C}) \ni \big(u, \mathcal{D}u|_{\partial\mathcal{C}}, \underline{u}\big) \\ \to \big(L(\partial_t)u,\ B(\partial_t)u|_{\partial\mathcal{C}} + C(\partial_t)\,\underline{u}\big) \in \mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C}).$$

Our goal is the construction of the extension of the operator (5.3.10) to the space

$$(5.3.11)\qquad \tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$$

with $l < 2m$. We start with the extension of the operator L.

Let l be an integer, $0 < l < 2m$. We write the differential operator L in the form

$$L(x, \partial_x, \partial_t) = \sum_{|\alpha|+j\le 2m-l} \partial_x^\alpha\, \partial_t^j\, L_{\alpha,j}(x, \partial_x, \partial_t),$$

where $L_{\alpha,j}$ are differential operators of order $\le l$. Then the formula

$$\int_{\mathcal{C}} Lu \cdot \overline{v}\, dx\, dt = \sum_{|\alpha|+j\le 2m-l} \int_{\mathcal{C}} L_{\alpha,j}u \cdot \overline{(-\partial_x)^\alpha(-\partial_t)^j v}\, dx\, dt \\ + \sum_{j=l+1}^{2m} \int_{\partial\mathcal{C}} D_\nu^{j-1}u \cdot \overline{P_j(x, \partial_x, -\partial_t)v}\, d\sigma\, dt + \sum_{j=1}^{l} \int_{\partial\mathcal{C}} D_\nu^{j-1}u\ \cdot \overline{P_{l,j}v}\, d\sigma\, dt$$

is valid for all $u, v \in C_0^\infty(\overline{\mathcal{C}})$ (see Lemma 3.2.1). Here P_j are the same operators as in (5.3.5) and $P_{l,j}$ are differential operators of order $\le 2m - j$ with coefficients depending only on x such that the functional

$$v \to \sum_{j=1}^{l} \int_{\partial\Omega} D_\nu^{j-1}u\ \cdot \overline{P_{l,j}v}\, d\sigma$$

is continuous on $\mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C})$ for arbitrary $u \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$.

Analogously to Lemma 3.2.2, the following statement holds.

LEMMA 5.3.3. *The operator*

$$(5.3.12)\qquad \tilde{\mathcal{W}}^{2m,2m}_{2,\beta}(\mathcal{C}) \ni \big(u, \mathcal{D}u|_{\partial\mathcal{C}}\big) \to L(x, \partial_x, \partial_t)\, u = f \in \mathcal{W}^0_{2,\beta}(\mathcal{C})$$

can be uniquely extended to a continuous operator

$$(5.3.13)\qquad \tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \ni (u, \underline{\phi}) \to f \in \mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C})^*, \quad l < 2m.$$

The functional $f = L(u, \underline{\phi})$ in (5.3.13) is given by the equality

$$(5.3.14)\qquad (f, v)_{\mathcal{C}} = \big(u, L^+(x, \partial_x, -\partial_t)v\big)_{\mathcal{C}} + \sum_{j=1}^{2m} \big(\phi_j, P_j(x, \partial_x, -\partial_t)\, v\big)_{\partial\mathcal{C}}$$

if $l \le 0$ and by

$$(5.3.15)\quad (f, v)_{\mathcal{C}} = \sum_{|\alpha|+j\le 2m-l} \int_{\mathcal{C}} L_{\alpha,j}u \cdot \overline{(-\partial_x)^\alpha(-\partial_t)^j v}\, dx\, dt \\ + \sum_{j=l+1}^{2m} \big(\phi_j\,, P_j(x, \partial_x, -\partial_t)v\big)_{\partial\mathcal{C}} + \sum_{j=1}^{l} \big(D_\nu^{j-1}u\,, P_{l,j}v\big)_{\partial\mathcal{C}}$$

if $0 < l < 2m$, *where* v *is an arbitrary function from* $\mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C})$. *Here* $(\cdot,\cdot)_{\mathcal{C}}$ *denotes the extension of the scalar product in* $L_2(\Omega)$ *both to* $\mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C})^* \times \mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C})$ *and* $\mathcal{W}^{-l}_{2,-\beta}(\mathcal{C})^* \times \mathcal{W}^{-l}_{2,-\beta}(\mathcal{C})$, *while* $(\cdot,\cdot)_{\partial\mathcal{C}}$ *is the extension of the scalar product in* $L_2(\partial\mathcal{C})$ *to* $\mathcal{W}^{l-j+1/2}_{2,\beta}(\partial\mathcal{C}) \times \mathcal{W}^{-l+j-1/2}_{2,-\beta}(\partial\mathcal{C})$.

By (5.3.4), the operator

$$\tilde{W}^{l,2m}_2(\Omega) \ni (u,\underline{\phi}) \to Q\underline{\phi} \in W_2^{l-\underline{\mu}-1/2}(\partial\Omega), \quad l < 2m,$$

is a continuous extension of the operator

$$\tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \ni \big(u, \mathcal{D}u|_{\partial\mathcal{C}}\big) \to Bu|_{\partial\mathcal{C}} \in \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C}), \quad l \ge 2m.$$

Consequently, we obtain the following theorem.

THEOREM 5.3.1. *The operator (5.3.10) can be uniquely extended to a linear and continuous operator*

$$(5.3.16) \qquad \mathfrak{A}(\partial_t) \,:\, \tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}_2^{l+\underline{\tau}-1/2}(\partial\mathcal{C}) \to \tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})$$

with $l < 2m$. *This extension has the form*

$$(u,\underline{\phi},\underline{u}) \to \big(L(u,\underline{\phi}),\, Q\underline{\phi} + C\underline{u}\big)\,,$$

where L *is the operator (5.3.13) given in Lemma 5.3.3 and* Q *is the matrix in (5.3.4).*

REMARK 5.3.1. If $(u,\underline{\phi},\underline{u}) \in \tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$, $l \le 0$, then the functional $f = L(u,\underline{\phi}) \in \mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C})^*$ and the vector-function $\underline{g} = Q\underline{\phi} + C\underline{u}$ satisfy the equality

$$(5.3.17) \quad \begin{aligned}(f,v)_{\mathcal{C}} + (\underline{g},\underline{v})_{\partial\mathcal{C}} = \big(u, L^+(-\partial_t)v\big)_{\mathcal{C}} &+ \big(\underline{\phi}\,,\, P(-\partial_t)\,v|_{\partial\mathcal{C}} + Q^+(-\partial_t)\underline{v}\big)_{\partial\mathcal{C}} \\ &+ \big(\underline{u}\,,\, C^+(-\partial_t)\,\underline{v}\big)_{\partial\mathcal{C}}\end{aligned}$$

for arbitrary $v \in \mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C})$ and $\underline{v} \in \mathcal{W}^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{C})$. Consequently, in the case $l \le 0$ the operator (5.3.16) is adjoint to the operator

$$(v,\underline{v}) \to \big(L^+(-\partial_t)v,\, P(-\partial_t)v|_{\partial\mathcal{C}} + Q^+(-\partial_t)\underline{v},\, C^+(-\partial_t)\underline{v}\big)$$

of the formally adjoint problem which maps $\mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C}) \times \mathcal{W}^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{C})$ into

$$\mathcal{W}^{-l}_{2,-\beta}(\mathcal{C}) \times \prod_{j=1}^{2m} \mathcal{W}^{-l+j-1/2}_{2,-\beta}(\partial\mathcal{C}) \times \mathcal{W}^{-l-\underline{\tau}+1/2}_{2,-\beta}(\partial\mathcal{C}).$$

5.3.4. Bijectivity of the operator of the boundary value problem. We prove now that the operator (5.3.16) is an isomorphism for arbitrary integer l if the conditions of Theorem 5.2.2 are satisfied. For the extension of Theorem 5.2.2 to all integer values of l we need the following lemmas.

LEMMA 5.3.4. *If* L *is elliptic, then there exists a constant* c *such that*

$$\sum_{j=1}^{2m} \|\phi_j\|_{\mathcal{W}^{l-j+1/2}_{2,\beta}(\partial\mathcal{C})} \le c\Big(\|u\|_{\tilde{\mathcal{W}}^{l,0}_{2,\beta}(\mathcal{C})} + \|L(\partial_t)\,(u,\underline{\phi})\|_{\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C})}\Big)$$

for all $(u,\underline{\phi}) \in \tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C})$.

Proof: Let $\{\zeta_k\}_{k=-\infty}^{+\infty} \subset C_0^\infty(\mathbb{R}^1)$ be a partition of unity on $\mathbb{R}^1$ subordinate to the covering of $\mathbb{R}^1$ by the intervals $(k-1, k+1)$ satisfying the condition

$$|D_t^j \zeta_k(t)| < c_j \quad \text{for } t \in \mathbb{R},\ j = 0, 1, \dots ,$$

where c_j are constants independent of k and t. Furthermore, let $\eta_k = \zeta_{k-1}+\zeta_k+\zeta_{k+1}$ for $k = 0, \pm 1, \dots$. Then Lemma 3.2.3 yields

$$\sum_{j=1}^{2m} \|\zeta_k \phi_j\|_{W_2^{l-j+1/2}(\partial\mathcal{C})} \le c \Big(\|\eta_k u\|_{\tilde{W}_2^{l,0}(\mathcal{C})} + \|\eta_k L(\partial_t)\,(u, \underline{\phi})\|_{\tilde{W}_2^{l-2m,0}(\mathcal{C})} \Big).$$

Since the coefficients of L, B, and C are independent of t, the constant c in this inequality can be chosen independent of k. Furthermore, by our assumptions on ζ_k, there exist constants c_1, c_2 independent of k such that

$$c_1 \|\zeta_k \phi_j\|_{W_2^{l-j+1/2}(\partial\mathcal{C})} \le e^{-k\beta} \|\zeta_k \phi_j\|_{\mathcal{W}_{2,\beta}^{l-j+1/2}(\partial\mathcal{C})} \le c_2 \|\zeta_k \phi_j\|_{W_2^{l-j+1/2}(\partial\mathcal{C})}$$

for $j = 1, \dots , 2m$. Analogous inequalities are valid for the norm of $\eta_k u$ in $\tilde{\mathcal{W}}_{2,\beta}^{l,0}(\mathcal{C})$ and the norm of $\eta_k L(\partial_t)\,(u, \underline{\phi})$ in $\tilde{\mathcal{W}}_{2,\beta}^{l-2m,0}(\mathcal{C})$. Hence there exists a constant c independent of k such that

$$\sum_{j=1}^{2m} \|\zeta_k \phi_j\|^2_{\mathcal{W}_{2,\beta}^{l-j+1/2}(\partial\mathcal{C})} \le c \Big(\|\eta_k u\|^2_{\tilde{\mathcal{W}}_{2,\beta}^{l,0}(\mathcal{C})} + \|\eta_k L(\partial_t)\,(u, \underline{\phi})\|^2_{\tilde{\mathcal{W}}_{2,\beta}^{l-2m,0}(\mathcal{C})} \Big).$$

Summing up over all integer k and using Lemma 5.3.1, we get the desired inequality. ■

LEMMA 5.3.5. *Suppose that the boundary value problem (5.2.14), (5.2.15) is elliptic. If* $(u, \underline{\phi}, \underline{u}) \in \tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{C})$ *is a solution of the equation*

$$\mathfrak{A}(\partial_t)\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$$

and $(f, \underline{g}) \in \tilde{\mathcal{W}}_{2,\beta}^{l-2m+1,0}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l-\underline{\mu}+1/2}(\partial\mathcal{C})$, *then* $(u, \underline{\phi}, \underline{u})$ *belongs to the space* $\tilde{\mathcal{W}}_{2,\beta}^{l+1,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}+1/2}(\partial\mathcal{C})$. *Furthermore,* $(u, \underline{\phi}, \underline{u})$ *satisfies the estimate*

$$(5.3.18)\quad \|(u, \underline{\phi}, \underline{u})\|_{l+1,\beta} \le c \Big(\|f\|_{\tilde{\mathcal{W}}_{2,\beta}^{l-2m+1,0}(\mathcal{C})} + \|\underline{g}\|_{\mathcal{W}_{2,\beta}^{l-\underline{\mu}+1/2}(\partial\mathcal{C})} + \|(u, \underline{\phi}, \underline{u})\|_{l,\beta} \Big),$$

where $\|\cdot\|_{l,\beta}$ *denotes the norm in (5.3.11). The constant* c *in (5.3.18) is independent of* $(u, \underline{\phi}, \underline{u})$.

Proof: Let ζ_k, η_k be the same functions on the t-axis as in the proof of Lemma 5.3.4. From the assumptions of the lemma it follows that $\zeta_k(u, \underline{\phi}, \underline{u}) \in \tilde{W}_2^{l,2m}(\mathcal{C}) \times W_2^{l+\underline{\tau}-1/2}(\partial\mathcal{C})$ for every index $k = 0, \pm 1, \dots$. Consequently, by Lemma 3.2.4, we get $\zeta_k(u, \underline{\phi}, \underline{u}) \in \tilde{W}_2^{l+1,2m}(\mathcal{C}) \times W_2^{l+\underline{\tau}+1/2}(\partial\mathcal{C})$ and

$$(5.3.19)\quad \|\zeta_k(u, \underline{\phi}, \underline{u})\|^2_{l+1,0} \le c \Big(\|\zeta_k f\|^2_{\tilde{W}_2^{l-2m+1,0}(\mathcal{C})} + \|\zeta_k \underline{g}\|^2_{W_2^{l-\underline{\mu}+1/2}(\partial\mathcal{C})} + \|\eta_k(u, \underline{\phi}, \underline{u})\|^2_{l,0} \Big),$$

where the constant c is independent of $(u, \underline{\phi}, \underline{u})$. Since the coefficients of the operators L, B, and C do not depend on the variable t, the constant c in (5.3.19) is even independent of k. Due to our assumptions on the functions ζ_k, there exist constants c_1, c_2 independent of k such that

$$c_1 \|\zeta_k (u, \underline{\phi}, \underline{u})\|_{l,0} \le e^{-k\beta} \|\zeta_k (u, \underline{\phi}, \underline{u})\|_{l,\beta} \le c_2 \|\zeta_k (u, \underline{\phi}, \underline{u})\|_{l,0} \tag{5.3.20}$$

for all $(u, \underline{\phi}) \in \mathcal{W}^{l,2m}_{2,\beta}(\mathcal{C})$, $\underline{u} \in \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$. Analogous inequalities are valid for f and $\underline{g}$. Consequently, (5.3.19) implies

$$\begin{aligned} \|\zeta_k(u, \underline{\phi}, \underline{u})\|^2_{l+1,\beta} \quad \le \quad & c\Big(\|\zeta_k f\|^2_{\tilde{\mathcal{W}}^{l-2m+1,0}_{2,\beta}(\mathcal{C})} + \|\zeta_k \underline{g}\|^2_{\mathcal{W}^{l-\underline{\mu}+1/2}_{2,\beta}(\partial\mathcal{C})} \\ & + \|\eta_k(u, \underline{\phi}, \underline{u})\|^2_{l,\beta}\Big). \end{aligned} \tag{5.3.21}$$

Summing up over all integer k and using Lemma 5.3.1, we get (5.3.18). ■

REMARK 5.3.2. By Lemma 5.3.4, the term $\|(u, \underline{\phi}, \underline{u})\|_{l,\beta}$ on the right-hand side of the estimate (5.3.18) can be replaced by

$$\|u\|_{\tilde{\mathcal{W}}^{l,0}_{2,\beta}(\mathcal{C})} + \|\underline{u}\|_{\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})},$$

since the norm of $\underline{\phi}$ can be estimated by the norms of u and f.

Now it is easy to obtain the following result.

THEOREM 5.3.2. *Suppose that the boundary value problem (5.2.14), (5.2.15) is elliptic and no eigenvalues of* $\mathfrak{A}(\lambda)$ *lie on the line* $\operatorname{Re}\lambda = -\beta$. *Then the operator (5.3.16) is an isomorphism for arbitrary integer* l.

Proof: For $l \le 0$ the operator $\mathfrak{A}(\partial_t)$ is adjoint to the operator

$$\begin{aligned} \mathfrak{A}^+(-\partial_t) \, : \, & \mathcal{W}^{2m-l}_{2,-\beta}(\mathcal{C}) \times \mathcal{W}^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{C}) \\ & \to \mathcal{W}^{-l}_{2,-\beta}(\mathcal{C}) \times \prod_{j=1}^{2m} \mathcal{W}^{-l+j-1/2}_{2,-\beta}(\partial\mathcal{C}) \times \mathcal{W}^{-l-\underline{\tau}+1/2}_{2,-\beta}(\partial\mathcal{C}). \end{aligned}$$

of the formally adjoint problem (see Remark 5.3.1). According to Corollary 5.3.1, the last operator realizes an isomorphism for arbitrary integer $l \le 0$, $l \le -\max \tau_j$. This proves the validity of our assertion for $l \le 0$, $l \le -\max \tau_j$. Using the regularity assertion of Lemma 5.3.5 and the uniqueness of the solution of the equation

$$\mathfrak{A}(\partial_t)\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$$

in the space (5.3.11) for $f \in \mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C})$, $\underline{g} \in \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})$, $l \ge 2m$, $l > \max \mu_k$ (see Theorem 5.2.2), we obtain the validity of the theorem for arbitrary integer l. ■

5.4. Asymptotics of the solution of the model problem at infinity

In Sections 5.2, 5.3 it was shown that the model problem is uniquely solvable in $\tilde{\mathcal{W}}^{l,2m}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$ if no eigenvalues of the operator pencil $\mathfrak{A}$ are situated on the line $\operatorname{Re}\lambda = -\beta_1$. Now we study the behaviour of the solution at infinity. We assume that the functions f and g_k on the right sides of the model problem belong also to weighted Sobolev spaces with another exponent β_2. Applying Cauchy's residual theorem, we show that then the solution is the sum of a linear combination of finitely many power-exponential solutions of the homogeneous model problem and a remainder belonging to a weighted Sobolev space with the exponent β_2.

Furthermore, we derive a formula for the coefficients of the mentioned power-exponential solutions.

5.4.1. Decomposition of the solution. Formula (5.2.25) in the proof of Theorem 5.2.2 gives a representation for the solution of the boundary value problem (5.2.14), (5.2.15) in terms of the Laplace transformation $\mathcal{L}_{t\to\lambda}$ and its inverse. Naturally, this solution depends on the number β. We assume now that the right-hand side $(f, \underline{g})$ belongs to the space

$$\bigcap_{i=1}^{2} \Big(\mathcal{W}_{2,\beta_i}^{l_i-2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta_i}^{l_i-\underline{\mu}-1/2}(\partial\mathcal{C}) \Big), \tag{5.4.1}$$

where $l_1, l_2 \geq \max(2m, \mu_1+1, \dots, \mu_{m+J}+1)$. Then by Theorem 5.2.2, there exist two solutions $(u, \underline{u})$ and $(w, \underline{w})$ given by formula (5.2.25) with the integral over the lines $\operatorname{Re}\lambda = -\beta_1$ and $\operatorname{Re}\lambda = -\beta_2$, respectively. Our goal is to obtain a relation between these two solutions. To this end, we prove the following lemma.

LEMMA 5.4.1. *Let $(f, \underline{g})$ be an element of the space (5.4.1), where $\beta_1 < \beta_2$ and $l_i \geq \max(2m, \mu_1+1, \dots, \mu_{m+J}+1)$. Then the norm of*

$$\big(v_\rho(x,t), \underline{v}_\rho(x,t)\big) \overset{def}{=} \int_{-\beta_2+i\rho}^{-\beta_1+i\rho} e^{\lambda t}\, \mathfrak{A}(\lambda)^{-1} \big(\check{f}(x,\lambda), \check{g}(x,\lambda)\big)\, d\lambda \tag{5.4.2}$$

in the space $L_2\big(\Omega\times(-N,+N)\big) \times L_2\big(\partial\Omega\times(-N,+N)\big)^J$, where N is an arbitrary positive number, tends to zero as $\rho\to\infty$.

Proof: We write the operator $\mathfrak{A}(\lambda)^{-1} : (\check{f}, \check{g}) \to (\check{u}, \underline{\check{u}})$ as a pair of two operators $\mathfrak{A}_1(\lambda), \mathfrak{A}_2(\lambda)$, where $\mathfrak{A}_1(\lambda)(\check{f}, \check{g}) = \check{u}$ and $\mathfrak{A}_2(\lambda)(\check{f}, \check{g}) = \underline{\check{u}}$. Then

$$v_\rho(x,t) = \int_{-\beta_2+i\rho}^{-\beta_1+i\rho} e^{\lambda t}\, \mathfrak{A}_1(\lambda) \big(\check{f}(x,\lambda), \check{g}(x,\lambda)\big)\, d\lambda$$

and

$$\|v_\rho\|^2_{L_2(\Omega\times(-N,+N))} \leq (\beta_2-\beta_1) \int_{-N}^{N} \int_\Omega \int_{-\beta_2+i\rho}^{-\beta_1+i\rho} |e^{\lambda t}\, \mathfrak{A}_1(\lambda)\, (\check{f}, \underline{\check{g}})|^2\, d\lambda\, dx\, dt$$

$$\leq 2N\, (\beta_2-\beta_1)\, e^{2N|\beta_2|} \int_{-\beta_2+i\rho}^{-\beta_1+i\rho} \int_\Omega |\mathfrak{A}_1(\lambda)\, (\check{f}(x,\lambda), \underline{\check{g}}(x,\lambda)|^2\, dx\, d\lambda.$$

By Theorem 3.6.1, the inequality

$$\|\mathfrak{A}_1(\lambda)\, (\check{f}, \underline{\check{g}})\|^2_{L_2(\Omega)} \leq c\, |\lambda|^{-2l} \Big(\|\check{f}\|^2_{W_2^l(\Omega,|\lambda|)} + \|\underline{\check{g}}\|^2_{W_2^{l-\underline{\mu}-1/2}(\partial\Omega,|\lambda|)} \Big)$$

is valid for all λ in the strip $-\beta_2 \le \operatorname{Re}\lambda \le -\beta_1$ with sufficiently large imaginary part. Hence we obtain

$$\int_{c_1}^{c_2} \|v_\rho\|^2_{L_2(\Omega\times(-N,+N))}\, d\rho$$
$$\le \frac{2cN}{c_1^{2l}}\,(\beta_2-\beta_1)\, e^{2N|\beta_2|} \int_{\beta_1}^{\beta_2} \int_{-\beta+ic_1}^{-\beta+ic_2} \Big(\|\check f\|^2_{W_2^l(\Omega,|\lambda|)} + \|\check{\underline{g}}\|^2_{W_2^{l-\underline{\mu}-1/2}(\partial\Omega,|\lambda|)} \Big)\, |d\lambda|\, d\beta$$
$$= 2cN\,(\beta_2-\beta_1)\, e^{2N|\beta_2|}\, c_1^{-2l} \int_{\beta_1}^{\beta_2} \Big(\|f\|^2_{\mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C})} + \|\underline{g}\|^2_{\mathcal{W}_2^{l-\underline{\mu}-1/2}(\partial\Omega)} \Big)\, d\beta$$
$$\le c'\, c_1^{-2l} \Big(\|f\|^2_{\mathcal{W}^{l-2m}_{2,\beta_1}(\mathcal{C})\cap \mathcal{W}^{l-2m}_{2,\beta_2}(\mathcal{C})} + \|\underline{g}\|^2_{\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_1}(\partial\Omega)\cap \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{C})} \Big)$$

for sufficiently large c_1 and $c_2 > c_1$. Here the constant c' depends only on β_1, β_2, and N. Consequently, the norm of v_ρ in $L_2(\Omega\times(-N,N))$ is square integrable with respect to ρ over the interval $(c_1,+\infty)$. Thus, this norm tends to zero as $\rho\to\infty$.

Analogously, by means of Theorem 3.6.1, we get

$$\|\underline{v}_\rho\|^2_{L_2(\partial\Omega\times(-N,+N))^J}$$
$$\le 2N\,(\beta_2-\beta_1)\, e^{2N|\beta_2|} \int_{-\beta_2+i\rho}^{-\beta_1+i\rho} \|\mathfrak{A}_2(\lambda)\,(\check f(\cdot,\lambda),\check{\underline{g}}(\cdot,\lambda))\|^2_{L_2(\partial\Omega)^J}\, d\lambda$$
$$\le c \int_{-\beta_2+i\rho}^{-\beta_1+i\rho} |\lambda|^{-1} \Big(\|\check f\|^2_{W_2^l(\Omega,|\lambda|)} + \|\check{\underline{g}}\|^2_{W_2^{l-\underline{\mu}-1/2}(\partial\Omega,|\lambda|)} \Big)\, d\lambda.$$

with a constant c independent of ρ. This yields

$$\int_{c_1}^{c_2} \|\underline{v}_\rho\|^2_{L_2(\partial\Omega\times(-N,+N))^J}\, d\rho$$
$$\le c\, c_1^{-1} \int_{\beta_1}^{\beta_2} \int_{-\beta-i\infty}^{-\beta+i\infty} \Big(\|\check f\|^2_{W_2^l(\Omega,|\lambda|)} + \|\check{\underline{g}}\|^2_{W_2^{l-\underline{\mu}-1/2}(\partial\Omega,|\lambda|)} \Big)\, |d\lambda|\, d\beta$$
$$= c\, c_1^{-1} \int_{\beta_1}^{\beta_2} \Big(\|f\|^2_{\mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C})} + \|\underline{g}\|^2_{\mathcal{W}_2^{l-\underline{\mu}-1/2}(\partial\Omega)} \Big)\, d\beta.$$

Hence the norm of $\underline{v}_\rho$ in $L_2(\partial\Omega\times(-N,N))^J$ is also square integrable over the interval $(c_1,+\infty)$. From this it follows that the norm of $\underline{v}_\rho$ tends to zero as $\rho\to\infty$. The proof is complete. ∎

Le λ_μ be the eigenvalues of the operator pencil $\mathfrak{A}(\lambda)$. Furthermore, let

$$\{(\varphi^{(\mu)}_{j,s},\, \underline{\varphi}^{(\mu)}_{j,s})\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

be canonical systems of Jordan chains of $\mathfrak{A}(\lambda)$ corresponding to the eigenvalues λ_μ. Then the pairs

$$(5.4.3)\qquad (u_{\mu,j,s}(x,t)\,,\,\underline{u}_{\mu,j,s}(x,t)) = e^{\lambda_\mu t}\sum_{\sigma=0}^{s}\frac{1}{\sigma!}\,t^\sigma\,(\varphi^{(\mu)}_{j,s-\sigma}(x),\underline{\varphi}^{(\mu)}_{j,s-\sigma}(x))$$

are power-exponential zeros of the operator $\mathfrak{A}(\partial_t)$ (see Lemma 5.1.3).

THEOREM 5.4.1. *Suppose that the model problem (5.2.14), (5.2.15) is elliptic and no eigenvalues of* $\mathfrak{A}(\lambda)$ *lie on the lines* $\operatorname{Re}\lambda = -\beta_1$ *and* $\operatorname{Re}\lambda = -\beta_2$, $\beta_1 < \beta_2$. *Let* $\lambda_1, \dots, \lambda_N$ *be the eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *in the strip* $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$. *If*

$$(5.4.4)\qquad (u,\underline{u}) \in \mathcal{W}^{l_1}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$$

is a solution of the boundary value problem (5.2.14), (5.2.15) with the right-hand side $(f,\underline{g})$ *from the space (5.4.1), then there is the representation*

$$(5.4.5)\qquad (u,\underline{u}) = \sum_{\mu=1}^{N}\sum_{j=1}^{I_\mu}\sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\,(u_{\mu,j,s}\,,\,\underline{u}_{\mu,j,s}) + (w,\underline{w}),$$

where $c_{\mu,j,s}$ *are constants and* $(w,\underline{w})$ *belongs to the space*

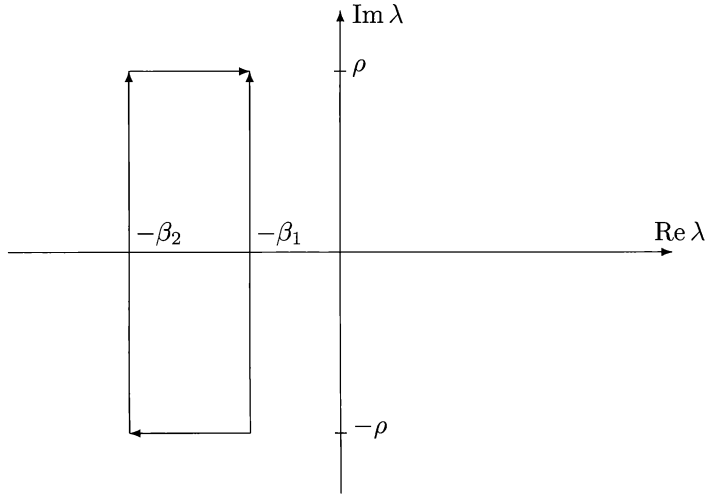
$$(5.4.6)\qquad \mathcal{W}^{l_2}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C}).$$

The constants $c_{\mu,j,s}$ *depend on* f, $\underline{g}$ *and on the choice of the above mentioned system of Jordan chains.*

Proof: As it was shown in the proof of Theorem 5.2.2, the solution $(u,\underline{u})$ has the form (5.2.25), where β has to be replaced by β_1. From the condition on f and $\underline{g}$ it follows that $\check{f}(x,\cdot)$ and $\check{\underline{g}}(x,\cdot)$ are holomorphic in the strip $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$. Hence the only singularities of the function $e^{\lambda t}\,\mathfrak{A}(\lambda)^{-1}(\check{f},\check{\underline{g}})$ in the strip $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$ are the poles of $\mathfrak{A}(\lambda)^{-1}$, i.e., the eigenvalues $\lambda_1,\dots,\lambda_N$ of $\mathfrak{A}(\lambda)$.

Let ρ be a sufficiently large positive number such that no eigenvalues of $\mathfrak{A}(\lambda)$ are contained in the set $\{\lambda \in \mathbb{C} : -\beta_2 < \operatorname{Re}\lambda < -\beta_1,\ |\operatorname{Im}\lambda| > \rho\}$. Using Cauchy's formula, we get

$$\begin{aligned}(5.4.7)\quad (u,\underline{u}) &= \frac{1}{2\pi i}\lim_{\rho\to+\infty}\int\limits_{-\beta_1-i\rho}^{-\beta_1+i\rho} e^{\lambda t}\,\mathfrak{A}(\lambda)^{-1}\,(\check{f}(x,\lambda),\check{\underline{g}}(x,\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\lim_{\rho\to\infty}\Big(\int\limits_{-\beta_2-i\rho}^{-\beta_2+i\rho}\dots d\lambda + \int\limits_{-\beta_1-i\rho}^{-\beta_2-i\rho}\dots d\lambda - \int\limits_{-\beta_1+i\rho}^{-\beta_2+i\rho}\dots d\lambda\Big) \\ &\quad + \sum_{\mu=1}^{N}\operatorname{Res} e^{\lambda t}\mathfrak{A}(\lambda)^{-1}(\check{f}(x,\lambda),\check{\underline{g}}(x,\lambda))\Big|_{\lambda=\lambda_\mu}\end{aligned}$$



The first integral on the right of (5.4.7) tends to the solution $(w, \underline{w})$ from the space (5.4.6) as $\rho \to \infty$. According to Lemma 5.4.1, the second integral and the third integral tend to zero. Hence (5.4.7) yields

$$(u, \underline{u}) = (w, \underline{w}) + \sum_{\mu=1}^{N} \operatorname{Res} e^{\lambda t}\, \mathfrak{A}(\lambda)^{-1} \left(\check{f}(\cdot, \lambda), \underline{\check{g}}(\cdot, \lambda)\right) \Big|_{\lambda=\lambda_\mu}. \tag{5.4.8}$$

It remains to calculate the residues in (5.4.8). First we note that the residues are elements of the kernel of $\mathfrak{A}(\partial_t)$, since

$$\begin{aligned} &\mathfrak{A}(\partial_t) \operatorname{Res} e^{\lambda t}\, \mathfrak{A}(\lambda)^{-1} \left(\check{f}(\cdot, \lambda), \underline{\check{g}}(\cdot, \lambda)\right) \Big|_{\lambda=\lambda_\mu} \\ &= \operatorname{Res} \mathfrak{A}(\partial_t)\, e^{\lambda t} \mathfrak{A}(\lambda)^{-1} \left(\check{f}(\cdot, \lambda), \underline{\check{g}}(\cdot, \lambda)\right) \Big|_{\lambda=\lambda_\mu} = \operatorname{Res} e^{\lambda t} \left(\check{f}(\cdot, \lambda), \underline{\check{g}}(\cdot, \lambda)\right) \Big|_{\lambda=\lambda_\mu} \end{aligned}$$

and $(\check{f}, \underline{\check{g}})$ is holomorphic in a neighbourhood of λ_μ. Using the representation

$$\mathfrak{A}(\lambda)^{-1} = \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} \frac{P_{j,s}^{(\mu)}}{(\lambda - \lambda_\mu)^{\kappa_{\mu,j}-s}} + \Gamma_\mu(\lambda)$$

for the inverse operator pencil $\mathfrak{A}(\lambda)^{-1}$ in a neighbourhood the eigenvalue λ_μ (see Theorem 5.2.1), we obtain

$$\begin{aligned} &\operatorname{Res} e^{\lambda t}\, \mathfrak{A}(\lambda)^{-1} \left(\check{f}(\cdot, \lambda), \underline{\check{g}}(\cdot, \lambda)\right) \Big|_{\lambda=\lambda_\mu} \\ &= \sum_{j=1}^{I_\mu} \frac{1}{(\kappa_{\mu,j} - 1)!} \lim_{\lambda \to \lambda_\mu} \left(\frac{d}{d\lambda}\right)^{\kappa_{\mu,j}-1} \Big(e^{\lambda t} \sum_{s=0}^{\kappa_{\mu,j}-1} (\lambda - \lambda_\mu)^s\, P_{j,s}^{(\mu)} \left(\check{f}(\cdot, \lambda), \underline{\check{g}}(\cdot, \lambda)\right)\Big) \\ &= e^{\lambda_\mu t} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} \frac{1}{s!}\, t^s \left(\varphi^{\mu,j,s}, \underline{\varphi}^{\mu,j,s}\right) \end{aligned}$$

with certain $\varphi^{\mu,j,s} \in C^\infty(\overline{\Omega})$, $\underline{\varphi}^{\mu,j,s} \in C^\infty(\partial\Omega)^J$. Hence the residues in (5.4.8) are elements of the sets $\mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$ of the power-exponential zeros of $\mathfrak{A}(\partial_t)$ corresponding to the eigenvalue λ_μ. By Lemma 5.1.4, the pairs (5.4.3) form a basis in the space $\mathcal{N}(\mathfrak{A}(\partial_t), \lambda_\mu)$. This proves the theorem. ■

By means of Theorem 5.3.2, we can prove the analogous assertion for the generalized solution $(u, \underline{\phi}, \underline{u}) \in \tilde{\mathcal{W}}^{l_1,2m}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$ of the equation

$$\mathfrak{A}(\partial_t)\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g}) \tag{5.4.9}$$

if the right-hand side $(f, \underline{g})$ belongs to the space

$$\bigcap_{i=1}^{2} \Big(\tilde{\mathcal{W}}^{l_i-2m,0}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l_i-\underline{\mu}-1/2}_{2,\beta_i}(\partial\mathcal{C}) \Big) \tag{5.4.10}$$

with arbitrary integer l_1, l_2.

THEOREM 5.4.2. *Suppose the model problem (5.2.14), (5.2.15) is elliptic, no eigenvalues of* $\mathfrak{A}(\lambda)$ *lie on the lines* $\operatorname{Re}\lambda = -\beta_1$, $\operatorname{Re}\lambda = -\beta_2$, *and* $\lambda_1, \dots, \lambda_N$ *are the eigenvalues of* $\mathfrak{A}(\lambda)$ *in the strip* $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$. *If*

$$(u, \underline{\phi}, \underline{u}) \in \tilde{\mathcal{W}}^{l_1,2m}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C}) \tag{5.4.11}$$

is a solution of the equation (5.4.9) and $(f, \underline{g})$ *belongs to the space (5.4.10), then there is the representation*

$$(u, \underline{\phi}, \underline{u}) = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\,(u_{\mu,j,s}\,, \mathcal{D}u_{\mu,j,s}|_{\partial\mathcal{C}}\,, \underline{u}_{\mu,j,s}) + (w, \underline{\psi}, \underline{w})\,. \tag{5.4.12}$$

Here $c_{\mu,j,s}$ *are constants,* $u_{\mu,j,s}$ *and* $\underline{u}_{\mu,j,s}$ *are given by (5.4.3), and* $(w, \underline{\psi}, \underline{w})$ *belongs to the space*

$$\tilde{\mathcal{W}}^{l_2,2m}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C}). \tag{5.4.13}$$

Proof: There exists a sequence $\{(f^{(k)}, \underline{g}^{(k)})\}_{k=1,2,\dots} \subset C_0^\infty(\overline{\mathcal{C}}) \times C_0^\infty(\partial\mathcal{C})^{m+J}$ which converges to $(f, \underline{g})$ in the norm of the space (5.4.10). Let l be the maximum of $l_1, l_2, 2m, \mu_1+1, \dots, \mu_{m+J}+1$ and let

$$(u^{(k)}, \underline{u}^{(k)}) \in \mathcal{W}^{l}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C}), \tag{5.4.14}$$

$$(w^{(k)}, \underline{w}^{(k)}) \in \mathcal{W}^{l}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C}) \tag{5.4.15}$$

be the uniquely determined solutions of the boundary value problem (5.2.14), (5.2.15) with the right hand sides $f^{(k)}$ and $\underline{g}^{(k)}$. From Theorem 5.3.2 it follows that the sequence $\{(u^{(k)}, \mathcal{D}u^{(k)}|_{\partial\mathcal{C}}, \underline{u}^{(k)})\}$ converges to $(u, \underline{\phi}, \underline{u})$ in the space (5.4.11), while the sequence $\{(w^{(k)}, \mathcal{D}w^{(k)}|_{\partial\mathcal{C}}, \underline{w}^{(k)})\}$ converges to the uniquely determined solution $(w, \underline{\psi}, \underline{w})$ of the equation (5.4.9) in the space (5.4.13). Hence the sequence

$$\{(u^{(k)}, \mathcal{D}u^{(k)}|_{\partial\mathcal{C}}, \underline{u}^{(k)}) - (w^{(k)}, \mathcal{D}w^{(k)}|_{\partial\mathcal{C}}, \underline{w}^{(k)})\}_{k=1,2,\dots}$$

converges to $(u, \underline{\phi}, \underline{u}) - (w, \underline{\psi}, \underline{w})$ in the space

$$\Big(\tilde{\mathcal{W}}^{l_1,2m}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})\Big) + \Big(\tilde{\mathcal{W}}^{l_2,2m}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C})\Big).$$

By Theorem 5.4.1, the difference $(u^{(k)}, \underline{u}^{(k)}) - (w^{(k)}, \underline{w}^{(k)})$ lies in the linear span of the functions (5.4.3). Therefore, $(u^{(k)}, \mathcal{D}u^{(k)}|_{\partial\mathcal{C}}, \underline{u}^{(k)}) - (w^{(k)}, \mathcal{D}w^{(k)}|_{\partial\mathcal{C}}, \underline{w}^{(k)})$ and the limit $(u, \underline{\phi}, \underline{u}) - (w, \underline{\psi}, \underline{w})$ lie in the linear span of the triples

$$(u_{\mu,j,s}\,, \mathcal{D}u_{\mu,j,s}|_{\partial\mathcal{C}}\,, \underline{u}_{\mu,j,s})\,, \tag{5.4.16}$$

$\mu = 1, \dots, N$, $j = 1, \dots, I_\mu$, $s = 0, \dots, \kappa_{\mu,j}-1$. This proves the theorem. ■

As immediate consequences of Theorems 5.3.2 and 5.4.2, the following assertions hold.

COROLLARY 5.4.1. *Suppose that the model problem (5.2.14), (5.2.15) is elliptic and there are no eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *in the closed strip between the lines* $\operatorname{Re}\lambda = -\beta_1$ *and* $\operatorname{Re}\lambda = -\beta_2$. *Then every solution* $(u,\underline{\phi},\underline{u}) \in \tilde{\mathcal{W}}^{l_1,2m}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$ *of the equation (5.4.9) with* $(f,\underline{g})$ *from the space (5.4.10) belongs to the space* $\tilde{\mathcal{W}}^{l_2,2m}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C})$.

COROLLARY 5.4.2. *Under the assumptions of Corollary 5.4.1, the operator* $\mathfrak{A}(\partial_t)$ *realizes isomorphisms*

$$\bigcap_{i=1}^{2}\Big(\tilde{\mathcal{W}}^{l_i,2m}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l_i+\underline{\tau}-1/2}_{2,\beta_i}(\partial\mathcal{C})\Big) \to \bigcap_{i=1}^{2}\Big(\tilde{\mathcal{W}}^{l_i-2m,0}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l_i-\underline{\mu}-1/2}_{2,\beta_i}(\partial\mathcal{C})\Big)$$

and

$$\begin{gathered}\Big(\tilde{\mathcal{W}}^{l_1,2m}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})\Big) + \Big(\tilde{\mathcal{W}}^{l_2,2m}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C})\Big)\\ \to \Big(\tilde{\mathcal{W}}^{l_1-2m,0}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1-\underline{\mu}-1/2}_{2,\beta_1}(\partial\mathcal{C})\Big) + \Big(\tilde{\mathcal{W}}^{l_2-2m,0}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l_2-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{C})\Big).\end{gathered}$$

5.4.2. Formulas for the coefficients in the asymptotics. As in the previous subsection let

$$\{(\varphi^{(\mu)}_{j,s}, \underline{\varphi}^{(\mu)}_{j,s})\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

be canonical systems of Jordan chains of $\mathfrak{A}(\lambda)$ corresponding to the eigenvalues λ_μ. Furthermore, let $\{(\psi^{(\mu)}_{j,s}, \underline{\psi}^{(\mu)}_{j,s})\}$ be canonical systems of Jordan chains of $\mathfrak{A}^*(\lambda)$ corresponding to $\overline{\lambda}_\mu$ which satisfy the "biorthonormality condition" (cf. formula (5.1.6))

$$\begin{aligned}(5.4.17)\quad &\sum_{p=0}^{\sigma}\sum_{q=p+1}^{p+s+1}\frac{1}{q!}\,\langle\mathfrak{A}^{(q)}(\lambda_\mu)\,(\varphi^{(\mu)}_{j,p+s+1-q}, \underline{\varphi}^{(\mu)}_{j,p+s+1-q}),\,(\psi^{(\mu)}_{k,\sigma-p}, \underline{\psi}^{(\mu)}_{k,\sigma-p})\rangle_2\\ &= \delta_{j,k}\,\delta_{s,\kappa_{\mu,k}-1-\sigma}\end{aligned}$$

for $j,k = 1,\dots,I_\mu$, $s = 0,\dots,\kappa_{\mu,j}-1$, $\sigma = 0,\dots,\kappa_{\mu,k}-1$. Here $\langle\cdot,\cdot\rangle_2$ denotes the scalar product in $L_2(\Omega) \times L_2(\partial\Omega)^{m+J}$.

We recall some properties of the operator $\mathfrak{A}^*(\lambda) = \mathfrak{A}(\overline{\lambda})^*$ which were given in Section 3.3. As before, we consider $\mathfrak{A}(\lambda)$ as a linear and continuous operator from $\mathcal{X}$ into $\mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are the spaces (5.2.20), (5.2.21), respectively. Then the adjoint operator $\mathfrak{A}^*(\overline{\lambda})$ of $\mathfrak{A}(\lambda)$ continuously maps $\mathcal{Y}^*$ into $\mathcal{X}^*$. Moreover, the restriction of $\mathfrak{A}^*(\lambda)$ to the space

$$(5.4.18)\qquad D^{q,0}_2(\Omega) \times W^{q-2m+\underline{\mu}+1/2}_2(\partial\Omega)$$

with $q \ge 0$ continuously maps the space (5.4.18) into

$$(5.4.19)\qquad D^{q-2m,2m}_2(\Omega) \times W^{q-2m-\underline{\tau}+1/2}_2(\partial\Omega)$$

(see Lemma 3.3.1). If the boundary value problem is elliptic, then every solution $(v,\underline{v}) \in \mathcal{Y}^*$ of the equation

$$\mathfrak{A}^*(\overline{\lambda})\,(v,\underline{v}) = (F,\underline{h})$$

with $F \in D_2^{q-2m,2m}(\Omega)$, $\underline{h} \in W_2^{q-2m-\underline{\tau}+1/2}(\partial\Omega)$, $q \geq 0$, belongs to the space (5.4.18). Furthermore, for $q \geq 2m$ the functional $F \in D_2^{q-2m,2m}(\Omega)$ has a unique representation

$$(5.4.20) \qquad (u, F)_\Omega = (u, f)_\Omega + (\mathcal{D}u, \underline{g})_{\partial\Omega}\,, \qquad u \in W_2^{2m}(\Omega),$$

where $f \in W_2^{q-2m}(\Omega)$, $\underline{g} \in \prod_{j=1}^{2m} W_2^{q-2m+j-1/2}(\partial\Omega)$, and $(v, \underline{v})$ is also a solution of the formally adjoint boundary value problem

$$L^+(x, \partial_x, \lambda)\, v = f \quad \text{in } \Omega,$$
$$P(x, \partial_x, \lambda)\, v + Q^+(x, \partial_x, \lambda)\, \underline{v} = \underline{g}, \quad C^+(x, \partial_x, \lambda)\, \underline{v} = \underline{h} \quad \text{on } \partial\Omega$$

(see Corollary 3.3.1). In particular, the equation $\mathfrak{A}^*(\lambda)\,(v, \underline{v}) = 0$ with $(v, \underline{v}) \in \mathcal{Y}^*$ implies $v \in C^\infty(\overline{\Omega})$, $\underline{v} \in C^\infty(\partial\Omega)^{m+J}$, and $\mathfrak{A}^+(\lambda)\,(v, \underline{v}) = 0$. Conversely, by Lemma 3.3.1, any solution $(v, \underline{v}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^{m+J}$ of the equation $\mathfrak{A}^+(\lambda)\,(v, \underline{v}) = 0$ is also a solution of the equation $\mathfrak{A}^*(\lambda)\,(v, \underline{v}) = 0$. Thus, the eigenvalues and eigenvectors of the pencils $\mathfrak{A}^*$ and $\mathfrak{A}^+$ coincide. We prove the same assertion for the generalized eigenvectors.

LEMMA 5.4.2. *Suppose that the boundary value problem (5.2.18), (5.2.19) is elliptic. Then the eigenvectors and generalized eigenvectors of $\mathfrak{A}^*(\lambda)$ belong to $C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^{m+J}$. Furthermore, $(\psi_0, \underline{\psi}_0), \ldots, (\psi_s, \underline{\psi}_s)$ is a Jordan chain of $\mathfrak{A}^*(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$ if and only if it is a Jordan chain of $\mathfrak{A}^+(\lambda)$ corresponding to the same eigenvalue.*

Proof: Let q be an integer, $q \geq 2m$. We denote by S_q the operator

$$W_2^{q-2m}(\Omega) \times \prod_{j=1}^{2m} W_2^{q-2m+j-1/2}(\partial\Omega) \ni (f, \underline{g}) \to F \in D_2^{q-2m,2m}(\Omega),$$

where the functional F is defined by (5.4.20). The operator S_q is bijective. Furthermore,

$$(5.4.21) \qquad \mathfrak{A}^*(\lambda)\,(v, \underline{v}) = S_q\, \mathfrak{A}^+(\lambda)\,(v, \underline{v})$$

for $(v, \underline{v}) \in W_2^q(\Omega) \times W_2^{q-2m+\underline{\mu}+1/2}(\partial\Omega)$. Let $(\psi_0, \underline{\psi}_0), \ldots, (\psi_s, \underline{\psi}_s)$ be a Jordan chain of $\mathfrak{A}^*(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$. From Corollary 3.3.1 and from the equation

$$(5.4.22) \qquad \mathfrak{A}^*(\overline{\lambda}_\mu)\,(\psi_\sigma, \underline{\psi}_\sigma) = -\sum_{j=1}^{\sigma} \frac{1}{j!}\, \frac{d^j}{d\lambda^j} \mathfrak{A}^*(\lambda)|_{\lambda=\overline{\lambda}_\mu}\, (\psi_{\sigma-j}, \underline{\psi}_{\sigma-j})$$

we conclude by induction that $(\psi_\sigma, \underline{\psi}_\sigma) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^{m+J}$ for $\sigma = 0, 1, \ldots, s$. Furthermore, from (5.4.21), (5.4.22) and from the bijectivity of the operator S_q it follows that

$$\sum_{j=0}^{\sigma} \frac{1}{j!}\, \frac{d^j}{d\lambda^j}\, \mathfrak{A}^+(\lambda)\Big|_{\lambda=\overline{\lambda}_\mu} (\psi_{\sigma-j}, \underline{\psi}_{\sigma-j}) = 0 \quad \text{for } \sigma = 0, 1, \ldots, s.$$

Consequently, $(\psi_0, \underline{\psi}_0), \ldots, (\psi_s, \underline{\psi}_s)$ is a Jordan chain of $\mathfrak{A}^+(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$. Analogously, it can be shown that every Jordan chain of $\mathfrak{A}^+(\lambda)$ is a Jordan chain of $\mathfrak{A}^*(\lambda)$. ■

By Lemma 5.4.2, the given system $\{(\psi^{(\mu)}_{j,s}, \underline{\psi}^{(\mu)}_{j,s})\}$ of Jordan chains of $\mathfrak{A}^*(\lambda)$ is also a canonical system of Jordan chains of $\mathfrak{A}^+(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$. Consequently, the pairs

$$e^{\overline{\lambda}_\mu t} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} t^\sigma \, (\psi^{(\mu)}_{j,s-\sigma}(x) \, , \, \underline{\psi}^{(\mu)}_{j,s-\sigma}(x) \,)$$

are power-exponential zeros of the operator $\mathfrak{A}^+(\partial_t)$ and the vector-functions

$$(5.4.23) \qquad (v_{\mu,j,s}(x,t) \, , \, \underline{v}_{\mu,j,s}(x,t) \,) = e^{-\overline{\lambda}_\mu t} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (-t)^\sigma \, (\psi^{(\mu)}_{j,s-\sigma}(x) \, , \, \underline{\psi}^{(\mu)}_{j,s-\sigma}(x) \,)$$

are power-exponential zeros of the operator $\mathfrak{A}^+(-\partial_t)$ of the formally adjoint boundary value problem (5.3.6), (5.3.7).

THEOREM 5.4.3. *Suppose that the conditions of Theorem 5.4.1 are satisfied. Then the solution $(u, \underline{u})$ admits the decomposition (5.4.5). The coefficients $c_{\mu,j,s}$ in (5.4.5) are determined by the formula*

$$(5.4.24) \qquad c_{\mu,j,s} = (f \, , \, v_{\mu,j,\kappa_{\mu,j}-1-s})_{\mathcal{C}} + (\underline{g} \, , \, \underline{v}_{\mu,j,\kappa_{\mu,j}-1-s})_{\partial\mathcal{C}} \, .$$

Proof: In accordance with the notation in Section 5.1, we introduce the vector-functions

$$\Phi_{\mu,j}(x,t) = \sum_{s=0}^{\kappa_{\mu,j}-1} \frac{1}{s!} t^s \, (\varphi^{(\mu)}_{j,\kappa_{\mu,j}-1-s}(x) \, , \, \underline{\varphi}^{(\mu)}_{j,\kappa_{\mu,j}-1-s}(x) \,) \, ,$$

$$\Psi_{\mu,j}(x,t) = \sum_{s=0}^{\kappa_{\mu,j}-1} \frac{1}{s!} (-t)^s \, (\psi^{(\mu)}_{j,\kappa_{\mu,j}-1-s}(x) \, , \, \underline{\psi}^{(\mu)}_{j,\kappa_{\mu,j}-1-s}(x) \,) \, .$$

Let $\zeta = \zeta(t)$ be a smooth function on $\mathbb{R}^1$ equal to zero in a neighbourhood of $-\infty$ and to one in a neighbourhood of $+\infty$. Then (5.4.5) yields

$$\mathfrak{A}(\partial_t) \left(\zeta(u, \underline{u}) - \zeta(w, \underline{w}) \, \right) = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=1}^{\kappa_{\mu,j}-1} c_{\mu,j,s} \, \mathfrak{A}(\partial_t) \left(\zeta(t) \, e^{\lambda_\mu t} \, \partial_t^{\kappa_{\mu,j}-1-s} \, \Phi_{\mu,j} \right) .$$

Applying Lemma 5.1.5, we get

$$(5.4.25) \qquad \int_{-\infty}^{+\infty} \left\langle \mathfrak{A}(\partial_t) \left(\zeta(u, \underline{u}) - \zeta(w, \underline{w}) \right) , \, e^{-\overline{\lambda}_\nu t} \, (-\partial_t)^\sigma \, \Psi_{\nu,k} \right\rangle_2 dt = c_{\nu,k,\sigma}$$

for $\nu = 1, \dots, N, \; k = 1, \dots, I_\nu, \; \sigma = 0, \dots, \kappa_{\nu,k} - 1$, where again $\langle \cdot, \cdot \rangle_2$ denotes the scalar product in $L_2(\Omega) \times L_2(\partial\Omega)^{m+J}$. Since

$$\zeta(w, \underline{w}) \in \bigcap_{i=1}^{2} \left(\mathcal{W}^{l_i}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l_i + \underline{\tau} - 1/2}_{2,\beta_i}(\partial\mathcal{C}) \right)$$

and $e^{-\overline{\lambda}_\nu t} \, (-\partial_t)^\sigma \, \Psi_{\nu,k}$ belongs to the dual space of

$$\bigcap_{i=1}^{2} \left(\mathcal{W}^{l_i - 2m}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l_i - \underline{\mu} - 1/2}_{2,\beta_i}(\partial\mathcal{C}) \right) ,$$

we have

$$\int_{-\infty}^{+\infty} \Big\langle \mathfrak{A}(\partial_t)\, \zeta(w,\underline{w})\,,\, e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\, \Psi_{\nu,k}\Big\rangle_2\, dt$$

$$= \int_{-\infty}^{+\infty} \Big\langle \zeta(w,\underline{w})\,,\mathfrak{A}^*(-\partial_t)\, e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\, \Psi_{\nu,k}\Big\rangle_2\, dt = 0.$$

For the same reason, we obtain

$$\int_{-\infty}^{+\infty} \Big\langle \mathfrak{A}(\partial_t)\,(1-\zeta)(u,\underline{u})\,,\, e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\, \Psi_{\nu,k}\Big\rangle_2\, dt = 0.$$

Hence (5.4.25) implies

$$\begin{aligned} c_{\nu,k,\sigma} &= \int_{-\infty}^{+\infty} \Big\langle \mathfrak{A}(\partial_t)\,(u,\underline{u})\,,\, e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\, \Psi_{\nu,k}\Big\rangle_2\, dt \\ &= \int_{-\infty}^{+\infty} \Big\langle (f,\underline{g})\,,\, e^{-\overline{\lambda}_\nu t}\,(-\partial_t)^\sigma\, \Psi_{\nu,k}\Big\rangle_2\, dt. \end{aligned}$$

This proves the theorem. ∎

COROLLARY 5.4.3. *The coefficients $c_{\mu,j,s}$ in Theorem 5.4.2 are given by formula (5.4.24).*

Proof: Let $\{(f^{(k)},\underline{g}^{(k)})\}_{k=1,2,\dots} \subset C_0^\infty(\overline{\mathcal{C}}) \times C_0^\infty(\partial\mathcal{C})^{m+J}$ be a sequence which convergences to $(f,\underline{g})$ in the norm of the space (5.4.10). Furthermore, let l be the maximum of $l_1, l_2, 2m, \mu_1+1, \dots, \mu_{m+J}+1$ and let $(u^{(k)},\underline{u}^{(k)})$ and $(w^{(k)},\underline{w}^{(k)})$ be the uniquely determined solutions of the equation $\mathfrak{A}(\partial_t)\,(u,\underline{u}) = (f^{(k)},\underline{g}^{(k)})$ in the spaces (5.4.14) and (5.4.15), respectively. Then Theorem 5.4.1 yields

$$(u^{(k)}, \mathcal{D}u^{(k)}|_{\partial\mathcal{C}}, \underline{u}^{(k)}) = \sum_{\mu,j,s} c^{(k)}_{\mu,j,s}\,(u_{\mu,j,s}, \mathcal{D}u_{\mu,j,s}, \underline{u}_{\mu,j,s}) + (w^{(k)}, \mathcal{D}w^{(k)}|_{\partial\mathcal{C}}, \underline{w}^{(k)}),$$

and by Theorem 5.4.3, the coefficients $c^{(k)}_{\mu,j,s}$ are given by the formula

$$c^{(k)}_{\mu,j,s} = (f^{(k)}\,,\, v_{\mu,j,\kappa_{\mu,j}-1-s})_{\mathcal{C}} + (\underline{g}^{(k)}\,,\, \underline{v}_{\mu,j,\kappa_{\mu,j}-1-s})_{\partial\mathcal{C}}\,.$$

For $k\to\infty$ the triples $(u^{(k)}, \mathcal{D}u^{(k)}|_{\partial\mathcal{C}}, \underline{u}^{(k)})$, $(w^{(k)}, \mathcal{D}w^{(k)}|_{\partial\mathcal{C}}, \underline{w}^{(k)})$ tend to $(u,\underline{\phi},\underline{u})$ and $(w,\underline{\psi},\underline{w})$, respectively, whereas $c^{(k)}_{\mu,j,s}$ tends to the coefficient $c_{\mu,j,s}$ given by formula (5.4.24). This proves our assertion. ∎

5.4.3. A formula for the coefficients in terms of the classical Green formula. Now we consider the boundary value problem

$$L(x,\partial_x,\partial_t)u = f \quad \text{in } \mathcal{C}, \tag{5.4.26}$$

$$B(x,\partial_x,\partial_t)u = \underline{g} \quad \text{on } \partial\mathcal{C}, \tag{5.4.27}$$

where $L(x, \partial_x, \partial_t)$ is an elliptic differential operator of order $2m$ with smooth coefficients depending only on the variable x and $B(x, \partial_x, \partial_t)$ is a vector of differential operators $B_k(x, \partial_x, \partial_t)$, $k = 1, \dots, m$, of order $\mu_k < 2m$ with smooth coefficients depending only on x. Furthermore, we suppose that problem (5.4.26), (5.4.27) is elliptic and the system of the boundary operators $B_1, \dots, B_m$ is normal on $\partial\mathcal{C}$ (see Definition 3.1.4). Then this system can be completed by operators $B_k(x, \partial_x, \partial_t)$, $k = m+1, \dots, 2m$, of order $\mu_k < 2m$ to a Dirichlet system of order $2m$ on $\partial\mathcal{C}$. Obviously, the operators $B_{m+1}, \dots, B_{2m}$ can be chosen in such a way that the coefficients of these operators depend only on the variable x. As it was shown in Section 3.1, the following *classical Green formula* is valid for u, $v \in C_0^\infty(\overline{\mathcal{C}})$:

$$\int_{\mathcal{C}} L(\partial_t)\, u \cdot \overline{v}\, dx\, dt + \sum_{k=1}^{m} \int_{\partial\mathcal{C}} B_k(\partial_t)u \cdot \overline{B'_{k+m}(-\partial_t)v}\, d\sigma\, dt = \int_{\mathcal{C}} u\, \overline{L^+(-\partial_t)v}\, dx\, dt + \sum_{k=1}^{m} \int_{\partial\mathcal{C}} B_{k+m}(\partial_t)u \cdot \overline{B'_k(-\partial_t)v}\, d\sigma\, dt \tag{5.4.28}$$

(for the sake of brevity, we have omitted the arguments x, ∂_x in the operators L, L^+, B_k, and B'_k). Here $B'_1(x, \partial_x, \partial_t), \dots, B'_{2m}(x, \partial_x, \partial_t)$ are differential operators with smooth coefficients depending only on the variable x which form a Dirichlet system of order $2m$ on $\partial\mathcal{C}$. The boundary value problem

$$L^+(x, \partial_x, -\partial_t)v = f \quad \text{in } \mathcal{C},$$
$$B'_k(x, \partial_x, -\partial_t)v = g_k \quad \text{on } \partial\mathcal{C}, \quad k = 1, \dots, m,$$

is said to be formally adjoint to the problem (5.4.26), (5.4.27) with respect to the Green formula (5.4.28). We denote the operator of this problem by $\mathfrak{A}_c^+(-\partial_t)$ and the corresponding operator pencil by $\mathfrak{A}_c^+(-\lambda)$.

Naturally, the boundary value problem (5.4.26), (5.4.27) is contained in the class of boundary value problems considered before. Therefore, additionally to the operator $\mathfrak{A}_c^+(-\partial_t)$, there exists the operator $\mathfrak{A}^+(-\partial_t)$ of the formally adjoint problem with respect to the Green formula (5.3.5). There are the following relations between the kernels of the operators $\mathfrak{A}^+(-\partial_t)$ and $\mathfrak{A}_c^+(-\partial_t)$ (cf. Lemma 3.1.1).

LEMMA 5.4.3. *Let $\mathcal{T}(-\partial_t)$ be the vector $\big(B'_{m+1}(-\partial_t), \dots, B'_{2m}(-\partial_t)\big)$. Then*

$$\mathfrak{A}^+(-\partial_t)\,(v, \underline{v}) = 0$$

if and only if

$$\mathfrak{A}_c^+(-\partial_t)v = 0 \text{ and } \underline{v} = \mathcal{T}(-\partial_t)v|_{\partial\mathcal{C}}\,.$$

As a consequence we obtain the following relation between the eigenvectors and generalized eigenvectors of the operator pencils $\mathfrak{A}^+(\lambda)$ and $\mathfrak{A}_c^+(\lambda)$.

LEMMA 5.4.4. *Let $(\psi_0, \underline{\psi}_0), \dots, (\psi_s, \underline{\psi}_s)$ be a Jordan chain of $\mathfrak{A}^+(\lambda_\mu)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$. Then $\psi_0, \dots, \psi_s$ is a Jordan chain of $\mathfrak{A}_c^+(\lambda)$ corresponding to $\overline{\lambda}_\mu$ and the vector-functions $\underline{\psi}_0, \dots, \underline{\psi}_s$ are determined by the equality*

$$\underline{\psi}_\sigma = \sum_{j=0}^{\sigma} \frac{1}{j!}\, \mathcal{T}^{(j)}(\overline{\lambda}_\mu)\, \psi_{\sigma-j}\,\Big|_{\partial\Omega} \tag{5.4.29}$$

with $\mathcal{T}^{(j)}(\lambda) = d^j\mathcal{T}(\lambda)/d\lambda^j$. *Conversely, every Jordan chain* $\psi_0, \dots, \psi_s$ *of* $\mathfrak{A}_c^+(\lambda)$ *generates a Jordan chain* $(\psi_0, \underline{\psi}_0), \dots, (\psi_s, \underline{\psi}_s)$ *of* $\mathfrak{A}^+(\lambda)$, *where the vector-functions* $\underline{\psi}_0, \dots, \underline{\psi}_s$ *are given by (5.4.29).*

Proof: Suppose that $(\psi_0, \underline{\psi}_0), \dots, (\psi_s, \underline{\psi}_s)$ is a Jordan chain of $\mathfrak{A}^+(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$. Then by Lemma 5.1.3,

$$(v, \underline{v}) = e^{\overline{\lambda}_\mu t} \sum_{\sigma=0}^{s} \frac{1}{\sigma!}\, t^\sigma\, (\psi_{s-\sigma}, \underline{\psi}_{s-\sigma})$$

is a solution of the equation $\mathfrak{A}^+(\partial_t)\,(v, \underline{v}) = 0$ and Lemma 5.4.3 yields $\mathfrak{A}_c^+(\partial_t)\, v = 0$. Consequently, $\psi_0, \dots, \psi_s$ is a Jordan chain of $\mathfrak{A}_c^+(\lambda)$ to the eigenvalue $\overline{\lambda}_\mu$. Moreover, by Lemma 5.4.3, we get

$$\begin{aligned}
\sum_{\sigma=0}^{s} \frac{t^\sigma}{\sigma!}\, \underline{\psi}_{s-\sigma} &= e^{-\overline{\lambda}_\mu t}\, \underline{v} = e^{-\overline{\lambda}_\mu t}\, \mathcal{T}(\partial_t) v\,\big|_{\partial\mathcal{C}} = e^{-\overline{\lambda}_\mu t}\, \mathcal{T}(\partial_t)\, e^{\overline{\lambda}_\mu t} \sum_{\sigma=0}^{s} \frac{t^\sigma}{\sigma!}\, \psi_{s-\sigma}\,\Big|_{\partial\mathcal{C}} \\
&= \mathcal{T}(\overline{\lambda}_\mu + \partial_t) \sum_{\sigma=0}^{s} \frac{t^\sigma}{\sigma!}\, \psi_{s-\sigma}\,\Big|_{\partial\mathcal{C}} = \sum_{j=0}^{s} \frac{1}{j!} \mathcal{T}^{(j)}(\overline{\lambda}_\mu)\, \partial_t^j \sum_{\sigma=0}^{s} \frac{t^\sigma}{\sigma!}\, \psi_{s-\sigma}\,\Big|_{\partial\mathcal{C}} \\
&= \sum_{\sigma=0}^{s} \frac{t^\sigma}{\sigma!} \sum_{j=0}^{s-\sigma} \frac{1}{j!} \mathcal{T}^{(j)}(\overline{\lambda}_\mu)\, \psi_{s-j-\sigma}\,\Big|_{\partial\mathcal{C}}.
\end{aligned}$$

This implies (5.4.29). Analogously, we can prove that $(\psi_0, \underline{\psi}_0), \dots, (\psi_s, \underline{\psi}_s)$ is a Jordan chain of $\mathfrak{A}^+(\lambda)$ if $\psi_0, \dots, \psi_s$ is a Jordan chain of $\mathfrak{A}_c^+(\lambda)$ and $\underline{\psi}_0, \dots, \underline{\psi}_s$ satisfy (5.4.29). ■

Let λ_μ be the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and let

$$\{\varphi_{j,s}^{(\mu)}\}_{j=1,\dots,I_\mu,\, s=0,\dots,\kappa_{\mu,j}-1}\,, \qquad \{(\psi_{j,s}^{(\mu)}, \underline{\psi}_{j,s}^{(\mu)})\}_{j=1,\dots,I_\mu,\, s=0,\dots,\kappa_{\mu,j}-1}$$

be the canonical systems of Jordan chains of $\mathfrak{A}(\lambda)$ and $\mathfrak{A}^*(\lambda)$ introduced in the previous subsections. Then by Lemmas 5.4.2 and 5.4.4, the functions $\psi_{j,s}^{(\mu)}$ form a canonical system of Jordan chains of $\mathfrak{A}_c^+(\lambda)$ to the eigenvalue $\overline{\lambda}_\mu$ and the biorthonormality condition (5.4.17) takes the form

$$\begin{aligned}
&\sum_{p=0}^{\sigma} \sum_{q=p+1}^{p+s+1} \frac{1}{q!} \Big((L^{(q)}(\lambda_\mu)\, \varphi_{j,p+s+1-q}^{(\mu)}, \psi_{k,\sigma-p}^{(\mu)})_{\mathcal{C}} \\
&+ \Big(B^{(q)}(\lambda_\mu)\, \varphi_{j,p+s+1-q}^{(\mu)}, \sum_{\varsigma=0}^{\sigma-p} \frac{1}{\varsigma!}\, \mathcal{T}^{(q)}(\overline{\lambda}_\mu)\, \psi_{k,\sigma-p-\varsigma}^{(\mu)} \Big)_{\partial\mathcal{C}} \Big) = \delta_{j,k}\, \delta_{s,\kappa_{\mu,k}-1-\sigma}\,.
\end{aligned}$$

Furthermore, by Lemma 5.4.3, the vector-functions $\underline{v}_{\mu,j,s}$ in (5.4.23) are given by the equalities

$$\underline{v}_{\mu,j,s} = \mathcal{T}(-\partial_t)\, v_{\mu,j,s}\,|_{\partial\mathcal{C}}\,.$$

Thus, according to Theorem 5.4.1 and Theorem 5.4.3, we get the following assertion.

THEOREM 5.4.4. *Suppose that the boundary value problem (5.4.26), (5.4.27) is elliptic and* $B_1, \dots, B_m$ *form a normal system of boundary operators on* $\partial\mathcal{C}$ *of order less than* $2m$. *Furthermore, we assume that* $\lambda_1, \dots, \lambda_N$ *are the eigenvalues of* $\mathfrak{A}(\lambda)$ *in the strip* $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$, *while the lines* $\operatorname{Re}\lambda = -\beta_1$ *and* $\operatorname{Re}\lambda = -\beta_2$ *do not contain eigenvalues of* $\mathfrak{A}(\lambda)$.

If $u \in \mathcal{W}^{l_1}_{2,\beta_1}(\mathcal{C})$ is a solution of the boundary value problem (5.4.26), (5.4.27) and $(f, \underline{g})$ belongs to the space

$$\bigcap_{i=1}^{2} \Big(\mathcal{W}^{l_i - 2m}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l_i - \underline{\mu} - 1/2}_{2,\beta_i}(\partial\mathcal{C}) \Big)$$

$(l_1,\, l_2 \geq 2m)$, then there is the representation

$$u = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\, u_{\mu,j,s} + w\,,$$

where the functions $u_{\mu,j,s}$ are given by (5.4.3) and $w \in W^{l_2}_{2,\beta_2}(\mathcal{C})$. The coefficients $c_{\mu,j,s}$ are determined by the formula

$$c_{\mu,j,s} = \big(f, v_{\mu,j,\kappa_{\mu,j}-1-s}\big)_{\mathcal{C}} + \sum_{k=1}^{m} \big(g_k\,,\ B'_{m+k}(-\partial_t) v_{\mu,j,\kappa_{\mu,j}-1-s}\big)_{\partial\mathcal{C}}\,,$$

where $v_{\mu,j,s}$ is given by (5.4.23).

5.5. The boundary value problem with coefficients which stabilize at infinity

The last section of this chapter is dedicated to elliptic boundary value problems in the cylinder $\mathcal{C} = \Omega \times \mathbb{R}$ with coefficients depending both on $x \in \Omega$ and $t \in \mathbb{R}$. We assume that the coefficients of the differential operators stabilize at infinity. This means, in particular, that there exist the limits of the coefficients for $t \to \pm\infty$. Replacing the coefficients by their limits, we obtain a model problem in the cylinder $\mathcal{C}$. If this model problem is uniquely solvable in a weighted Sobolev space (i.e., if no eigenvalues of the corresponding operator pencil lie on a line parallel to the imaginary axis), we obtain the Fredholm property of the operator of the given boundary value problem.

Furthermore, we describe the behaviour of the solutions at infinity. In particular, we show that the solution is the sum of a linear combination of finitely many "singular functions" and a regular remainder if the functions f, g_k on the right-hand sides of the boundary value problem belong to weighted Sobolev spaces with another exponent β in the weight function. Under certain additional conditions on the coefficients of the differential operators, we obtain a representation of these singular functions.

5.5.1. The stabilization condition. We consider the problem

$$L(x,t,\partial_x,\partial_t)\, u = f \quad \text{in } \mathcal{C}, \tag{5.5.1}$$

$$B(x,t,\partial_x,\partial_t)\, u + C(x,t,\partial_x,\partial_t)\, \underline{u} = \underline{g} \quad \text{on } \partial\mathcal{C}, \tag{5.5.2}$$

where

$$L(x,t,\partial_x,\partial_t) = \sum_{|\alpha|+j \leq 2m} a_{\alpha,j}(x,t)\, \partial_x^\alpha\, \partial_t^j$$

is a differential operator of order $2m$ with infinitely differentiable coefficients $a_{\alpha,j}$ on $\overline{\Omega} \times \mathbb{R}$, $B(x,t,\partial_x,\partial_t)$ is a vector of differential operators

$$B_k(x,t,\partial_x,\partial_t) = \sum_{|\alpha|+j \leq \mu_k} b_{k;\alpha,j}(x,t)\, \partial_x^\alpha\, \partial_t^j$$

with infinitely differentiable coefficients $b_{k;\alpha,j}$, and $C(x,t,\partial_x,\partial_t)$ is a matrix of tangential differential operators $C_{k,j}(x,t,\partial_x,\partial_t)$ of order $\le \mu_k+\tau_j$ on $\partial\mathcal{C}$ with smooth coefficients $c_{k,j;\alpha,s}$. We assume again that $\operatorname{ord} B_k < 2m$ for $k=1,\dots,m+J$. Then the vector B admits the representation

$$B(x,t,\partial_x,\partial_t)\,u\big|_{\partial\mathcal{C}} = Q(x,t,\partial_x,\partial_t)\cdot \mathcal{D}u\big|_{\partial\mathcal{C}}\,, \tag{5.5.3}$$

where Q is a $(m+J)\times J$-matrix of tangential differential operators $Q_{k,j}(x,t,\partial_x,\partial_t)$, $\operatorname{ord} Q_{k,j} \le \mu_k+1-j$, $Q_{k,j}\equiv 0$ if $\mu_k+1-j<0$.

Moreover, we suppose that the coefficients of L, B, and C *stabilize for* $t\to\pm\infty$, i.e., there exist smooth functions $a^{(0)}_{\alpha,j}$, $b^{(0)}_{k;\alpha,j}$, $c^{(0)}_{k,j;\alpha,s}$ on $\overline{\Omega}$ and in a neighbourhood of $\partial\Omega$, respectively, such that

$$\begin{aligned}
&\partial_x^\gamma\,\partial_t^\mu\,(a_{\alpha,j}(x,t)-a^{(0)}_{\alpha,j}(x)) && \to 0 \quad \text{as } t\to\pm\infty,\\
&\partial_x^\gamma\,\partial_t^\mu\,(b_{k;\alpha,j}(x,t)-b^{(0)}_{k;\alpha,j}(x)) && \to 0 \quad \text{as } t\to\pm\infty\\
&\partial_x^\gamma\,\partial_t^\mu\,(c_{k,j;\alpha,s}(x,t)-c^{(0)}_{k,j;\alpha,s}(x)) && \to 0 \quad \text{as } t\to\pm\infty
\end{aligned}$$

uniformly with respect to x for all nonnegative integer μ and all multi-indices γ.

We denote the operator of the problem (5.5.1), (5.5.2) by $\mathfrak{A}(t,\partial_t)$, while $\mathfrak{A}_0(\partial_t)$ denotes the operator of the model problem

$$L^{(0)}(x,\partial_x,\partial_t)\,u = f \quad \text{in } \mathcal{C}, \tag{5.5.4}$$

$$B^{(0)}(x,\partial_x,\partial_t)\,u + C^{(0)}(x,\partial_x,\partial_t)\,\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{C} \tag{5.5.5}$$

which arises from (5.5.1), (5.5.2) if we replace the coefficients $a_{\alpha,j}(x,t)$, $b_{k;\alpha,j}(x,t)$, $c_{k,j;\alpha,s}(x,t)$ by $a^{(0)}_{\alpha,j}(x,t)$, $b^{(0)}_{k;\alpha,j}(x,t)$, and $c^{(0)}_{k,j;\alpha,s}(x,t)$, respectively. Both operators $\mathfrak{A}(t,\partial_t)$ and $\mathfrak{A}_0(\partial_t)$ continuously map the space (5.2.16) into (5.2.17) for arbitrary integer $l\ge 2m$ and real β. Furthermore, as an immediate consequence of the stabilization condition, we get the following lemma.

LEMMA 5.5.1. *Suppose that the coefficients of L stabilize for $t\to\pm\infty$. Then there exists a constant c_T such that*

$$\|(L(x,t,\partial_x,\partial_t)-L^{(0)}(x,\partial_x,\partial_t))\,u\|_{\mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C})} \le c_T\,\|u\|_{\mathcal{W}^{l}_{2,\beta}(\mathcal{C})}$$

for all $u\in\mathcal{W}^l_{2,\beta}(\mathcal{C})$ equal to zero in $\Omega\times(-T,+T)$, $l\ge 2m$. The factor c_T tends to zero as $T\to+\infty$.

Analogous assertions are valid for the operators B_k and $C_{k,j}$.

5.5.2. Extension of the operator corresponding to the boundary value problem. Analogously to the case when L, B, and C are model operators, the following Green formula is valid for all u, $v\in C_0^\infty(\overline{\mathcal{C}})$, $\underline{u}\in C_0^\infty(\partial\mathcal{C})^J$, $\underline{v}\in C_0^\infty(\partial\mathcal{C})^{m+J}$:

$$\begin{aligned}
&\big(L(t,\partial_t)u,\,v\big)_{\mathcal{C}} + \big(B(t,\partial_t)u + C(t,\partial_t)\underline{u},\,\underline{v}\big)_{\partial\mathcal{C}}\\
&= \big(u,L^+(t,-\partial_t)v\big)_{\mathcal{C}} + \big(\mathcal{D}u,P(t,-\partial_t)v+Q^+(t,-\partial_t)\underline{v}\big)_{\partial\mathcal{C}} + \big(\underline{u},C^+(t,-\partial_t)\underline{v}\big)_{\partial\mathcal{C}}
\end{aligned}$$

(for the sake of brevity, we have omitted the arguments x, ∂_x in the differential operators). Obviously, the coefficients of the formally adjoint operators $L^+(t,-\partial_t)$, $Q^+(t,-\partial_t)$, $C^+(t,-\partial_t)$ to $L(t,\partial_t)$, $Q(t,\partial_t)$, $C(t,\partial_t)$ stabilize at infinity. Furthermore, it can be easily shown that the coefficients of $P(t,-\partial_t)$ stabilize at infinity. This follows from the explicit formula (2.3.12) for the operator P in the Green formula for the half-space. In the same way as it was carried out for the operator $\mathfrak{A}_0(\partial_t)$ (see Lemma 5.3.3, Theorem 5.3.1), the operator $\mathfrak{A}(t,\partial_t)$ can be extended to the space (5.3.11) with arbitrary integer l. This leads to the following results.

LEMMA 5.5.2. *The operator*

$$\tilde{\mathcal{W}}_{2,\beta}^{2m,2m}(\mathcal{C}) \ni \big(u, \mathcal{D}u|_{\partial\mathcal{C}}\big) \to Lu = f \in \mathcal{W}_{2,\beta}^{0}(\mathcal{C}) \tag{5.5.6}$$

can be uniquely extended to a continuous operator

$$\tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \ni (u, \underline{\phi}) \to f \in \mathcal{W}_{2,-\beta}^{2m-l}(\mathcal{C})^*, \quad l < 2m. \tag{5.5.7}$$

The functional $f = L(u,\underline{\phi}) \in \mathcal{W}_{2,-\beta}^{2m-l}(\mathcal{C})^*$ in (5.5.7) is given by the same formulas as in the case of t-independent coefficients (cf. Lemma 5.3.3). In the case $l \le 0$ we have

$$(f,v)_{\mathcal{C}} = \big(u, L^+(t,-\partial_t)v\big)_{\mathcal{C}} + \sum_{j=1}^{2m} \big(\phi_j, P_j(t,-\partial_t)\,v\big)_{\partial\mathcal{C}} \tag{5.5.8}$$

for $v \in \mathcal{W}_{2,-\beta}^{2m-l}(\mathcal{C})$, while in the case $0 < l < 2m$ the formula (5.3.15) is valid, where $L_{\alpha,j}$, P_j and $P_{l,j}$ are differential operators with t-dependent coefficients stabilizing at infinity.

THEOREM 5.5.1. *Suppose that the coefficients of L, B, and C stabilize at infinity. Then the operator*

$$\begin{aligned}\tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{C}) &\ni \big(u, \mathcal{D}u|_{\partial\mathcal{C}}, \underline{u}\big)\\ &\to \big(Lu,\ Bu|_{\partial\mathcal{C}} + C\underline{u}\big) \in \mathcal{W}_{2,\beta}^{l-2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{C})\end{aligned} \tag{5.5.9}$$

with $l \ge 2m$ can be uniquely extended to a linear and continuous operator

$$\mathfrak{A}(t,\partial_t) \,:\, \tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{C}) \to \tilde{\mathcal{W}}_{2,\beta}^{l-2m,0}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{C})$$

with $l < 2m$. This extension has the form

$$(u,\underline{\phi},\underline{u}) \to \big(L(u,\underline{\phi}),\ Q\underline{\phi} + C\underline{u}\big)\,,$$

where L is the operator (5.5.7) and Q is the matrix in (5.5.3).

Due to the stabilization condition on the coefficients of the operators L, B, and C we can generalize the regularity assertion of Lemma 5.3.5.

LEMMA 5.5.3. *Suppose that the coefficients of L, B, C stabilize at infinity and the boundary value problem (5.5.1), (5.5.2) is elliptic. If $(u,\underline{\phi},\underline{u}) \in \tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{C})$ is a solution of the equation $\mathfrak{A}(t,\partial_t)\,(u,\underline{\phi},\underline{u}) = (f,\underline{g})$, where $f \in \tilde{\mathcal{W}}_{2,\beta}^{l-2m+1,0}(\mathcal{C})$ and $\underline{g} \in \mathcal{W}_{2,\beta}^{l-\underline{\mu}+1/2}(\partial\mathcal{C})$, then $(u,\underline{\phi},\underline{u}) \in \tilde{\mathcal{W}}_{2,\beta}^{l+1,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}+1/2}(\partial\mathcal{C})$. Furthermore, the inequality*

$$\|(u,\underline{\phi},\underline{u})\|_{l+1,\beta} \le c\,\Big(\|f\|_{\tilde{\mathcal{W}}_{2,\beta}^{l-2m+1,0}(\mathcal{C})} + \|\underline{g}\|_{\mathcal{W}_{2,\beta}^{l-\underline{\mu}+1/2}(\partial\mathcal{C})} + \|(u,\underline{\phi},\underline{u})\|_{l,\beta}\Big) \tag{5.5.10}$$

is satisfied with a constant c independent of $(u,\underline{\phi},\underline{u})$. Here $\|\cdot\|_{l,\beta}$ denotes the norm in $\tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{C})$.

Proof: Let $(u,\underline{\phi},\underline{u})$ be an element of the space $\tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{C})$ and let ζ_k, η_k be the same functions as in the proof of Lemma 5.3.4. From Lemma 3.2.4 it follows that $\zeta_k(u,\underline{\phi},\underline{u}) \in \tilde{\mathcal{W}}_{2,\beta}^{l+1,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}+1/2}(\partial\mathcal{C})$ and

$$\begin{aligned}\|\zeta_k(u,\underline{\phi},\underline{u})\|_{l+1,\beta}^2 \;\le\; c_k\Big(&\|\zeta_k f\|^2_{\tilde{\mathcal{W}}_{2,\beta}^{l-2m+1,0}(\mathcal{C})} + \|\zeta_k\underline{g}\|^2_{\mathcal{W}_{2,\beta}^{l-\underline{\mu}+1/2}(\partial\mathcal{C})}\\ &+ \|\eta_k(u,\underline{\phi},\underline{u})\|_{l,\beta}^2\Big)\end{aligned} \tag{5.5.11}$$

with certain constants c_k independent of $(u, \underline{\phi}, \underline{u})$. By (5.3.21), we have

$$\begin{aligned} \|\zeta_k(u,\underline{\phi},\underline{u})\|^2_{l+1,\beta} \;\le\; & c\Big(\|\zeta_k\mathfrak{A}_0(\partial_t)\,(u,\underline{\phi},\underline{u})\|^2_{\tilde{\mathcal{W}}^{l-2m+1,0}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l-\underline{\mu}+1/2}_{2,\beta}(\partial\mathcal{C})} \\ & + \|\eta_k(u,\underline{\phi},\underline{u})\|^2_{l,\beta}\Big) \end{aligned}$$

with a constant c independent of k. Furthermore, as a consequence of Lemma 5.5.1, the inequality

$$\begin{aligned} &\big\|\zeta_k\big(\mathfrak{A}(t,\partial_t)-\mathfrak{A}_0(\partial_t)\big)\,(u,\underline{\phi},\underline{u})\big\|_{\tilde{\mathcal{W}}^{l-2m+1,0}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l-\underline{\mu}+1/2}_{2,\beta}(\partial\mathcal{C})} \\ &\le \varepsilon_k\Big(\|\zeta_k(u,\underline{\phi},\underline{u})\|_{l+1,\beta}+\|\eta_k(u,\underline{\phi},\underline{u})\|_{l,\beta}\Big) \end{aligned}$$

holds, where the factor ε_k tends to zero as $k\to\pm\infty$. From the last two inequalities it follows that the constants c_k in (5.5.11) can be chosen independent of k. Summing up over all integer k in (5.5.11) and using Lemma 5.3.1, we get (5.5.10). ■

REMARK 5.5.1. Analogously to Lemma 5.3.4, there is the estimate

$$\sum_{j=1}^{2m}\|\zeta_k\phi_j\|^2_{\mathcal{W}^{l-j+1/2}_{2,\beta}(\partial\mathcal{C})}\le c\Big(\|\eta_k u\|^2_{\tilde{\mathcal{W}}^{l,0}_{2,\beta}(\mathcal{C})}+\|\eta_k L(u,\underline{\phi})\|^2_{\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C})}\Big)$$

for each $(u,\underline{\phi})\in\tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C})$. Hence the term $\|(u,\underline{\phi},\underline{u})\|_{l,\beta}$ in (5.5.10) can be replaced by

$$\|u\|_{\tilde{\mathcal{W}}^{l,0}_{2,\beta}(\mathcal{C})}+\|\underline{u}\|_{\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})}\,.$$

5.5.3. The Fredholm property for the operator of the boundary value problem. Since the space $\mathcal{W}^{l+1}_{2,\beta}(\mathcal{C})$ is not compactly imbedded into the space $\mathcal{W}^{l}_{2,\beta}(\mathcal{C})$, we can not conclude from Lemma 3.4.1 and from the a priori estimate (5.5.10) in Lemma 5.5.3 that $\dim\ker\mathfrak{A}(t,\partial_t)<\infty$. However, under the additional assumption that the line $\operatorname{Re}\lambda=-\beta$ does not contain eigenvalues of $\mathfrak{A}_0(\lambda)$, the estimate (5.5.10) can be sharpened.

LEMMA 5.5.4. *Suppose that additionally to the assumptions of Lemma 5.5.3 the line* $\operatorname{Re}\lambda=-\beta$ *does not contain eigenvalues of* $\mathfrak{A}_0(\lambda)$. *Then every solution* $(u,\underline{\phi},\underline{u})\in\tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$ *of the equation* $\mathfrak{A}(t,\partial_t)\,(u,\underline{\phi},\underline{u})=(f,\underline{g},\underline{h})$ *satisfies the estimate*

$$\|(u,\underline{\phi},\underline{u})\|_{l,\beta}\le c\Big(\|f\|_{\mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C})}+\|\underline{g}\|_{\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})}+\|\zeta(u,\underline{\phi},\underline{u})\|_{l-1,\beta}\Big), \tag{5.5.12}$$

where $\|\cdot\|_{l,\beta}$ *denotes the norm in* $\tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$ *and* ζ *is a smooth function on* $\overline{\mathcal{C}}$ *with compact support.*

Proof: Let η, χ be infinitely differentiable functions on $\mathcal{C}$ depending only on the variable t such that

$$\begin{aligned} &\eta(t)=1 \quad\text{for } |t|\le T, \quad \eta(t)=0 \quad\text{for } |t|>T+1,\\ &\chi(t)=1 \quad\text{for } T\le|t|\le T+1, \quad \chi(t)=0 \quad\text{for } |t|>T+2, \end{aligned}$$

where T is a sufficiently large number. Then Lemma 5.5.3 yields

$$\|\eta(u,\underline{\phi},\underline{u})\|_{l,\beta}\le c\Big(\|\mathfrak{A}(t,\partial_t)\eta(u,\underline{\phi},\underline{u})\|_{\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})}+\|\eta(u,\underline{\phi},\underline{u})\|_{l-1,\beta}\Big).$$

By Theorem 5.3.2, the operator $\mathfrak{A}_0(\partial_t)$ realizes an isomorphism

$$\tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C}) \to \tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C}).$$

Hence Lemma 5.5.1 implies

$$\|(1-\eta)\,(u,\underline{\phi},\underline{u})\|_{l,\beta} \le c\|\mathfrak{A}(t,\partial_t)\,(1-\eta)(u,\underline{\phi},\underline{u})\|_{\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})}\,.$$

Let $[\mathfrak{A}(t,\partial_t),\eta] = \mathfrak{A}(t,\partial_t)\,\eta - \eta\,\mathfrak{A}(t,\partial_t)$ be the commutator of $\mathfrak{A}(t,\partial_t)$ and η. Since $\chi = 1$ on the support of $[\mathfrak{A}(t,\partial_t),\eta]\,(u,\underline{\phi},\underline{u})$, we have

$$\|[\mathfrak{A}(t,\partial_t),\eta]\,(u,\underline{\phi},\underline{u})\|_{\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})} \le c\,\|\chi(u,\underline{\phi},\underline{u})\|_{l-1,\beta}\,.$$

Consequently, we get

$$\begin{aligned}\|(u,\underline{\phi},\underline{u})\|_{l,\beta} \;&\le\; \|\eta(u,\underline{\phi},\underline{u})\|_{l,\beta} + \|(1-\eta)\,(u,\underline{\phi},\underline{u})\|_{l,\beta}\\ &\le c\left(\|\mathfrak{A}(t,\partial_t)\,(u,\underline{\phi},\underline{u})\|_{\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})} + \|\zeta(u,\underline{\phi},\underline{u})\|_{l-1,\beta}\right),\end{aligned}$$

where ζ is an arbitrary smooth function equal to one on the supports of η and χ. The lemma is proved. ■

Obviously, the term $\|\zeta(u,\underline{\phi},\underline{u})\|_{l-1,\beta}$ in (5.5.12) can be replaced by the norm of $(u,\underline{\phi},\underline{u})$ in the space

$$\left(\tilde{\mathcal{W}}^{l-1,2m}_{2,\beta_1}(\mathcal{C}) + \tilde{\mathcal{W}}^{l-1,2m}_{2,\beta_2}(\mathcal{C})\right) \times \left(\mathcal{W}^{l+\underline{\tau}-3/2}_{2,\beta_1}(\partial\mathcal{C}) + \mathcal{W}^{l+\underline{\tau}-3/2}_{2,\beta_2}(\partial\mathcal{C})\right)$$

where β_1, β_2 are arbitrary real numbers such that $\beta_1 < \beta < \beta_2$. Since the space $\tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$ is compactly imbedded into this space, we can conclude from Lemma 5.5.4 and Lemma 3.4.1 that the kernel of $\mathfrak{A}(t,\partial_t)$ has finite dimension and the range of $\mathfrak{A}(t,\partial_t)$ is closed.

In the same way as in Section 3.3 (cf. Theorem 3.3.1), an estimate analogous to (5.5.12) can be proved for the adjoint operator. We do not give a detailed proof for this assertion here, since we will be confronted with the same considerations in Section 6.3, where the Fredholm property for the operator of the boundary value problem in a domain with conical points will be proved. The estimate for the adjoint operator implies $\dim\operatorname{coker}\mathfrak{A}(t,\partial_t) < \infty$. Therefore, we obtain the following theorem.

THEOREM 5.5.2. *Suppose that the coefficients of* L, B, C *stabilize at infinity and the boundary value problem (5.5.1), (5.5.2) is elliptic. Furthermore, we assume that no eigenvalues of* $\mathfrak{A}_0(\lambda)$ *lie on the line* $\operatorname{Re}\lambda = -\beta$. *Then* $\mathfrak{A}(t,\partial_t)$ *is a Fredholm operator from* $\tilde{\mathcal{W}}^{l,2m}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{C})$ *into the space* $\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta}(\mathcal{C})\times\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})$.

5.5.4. Asymptotics at infinity. Now we study the asymptotics of the solutions of problem (5.5.1), (5.5.2) at infinity. For the sake of simplicity, we consider only solutions which belong to the space $\mathcal{W}^{l}_{2,\beta}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{C})$ with $l \ge 2m$. All results can be easily extended to generalized solutions of problem (5.5.1), (5.5.2).

The structure of the solution.

THEOREM 5.5.3. *Suppose that the coefficients of* L, B, C *stabilize for* $t \to +\infty$ *and the boundary value problem (5.5.1), (5.5.2) is elliptic. Furthermore, we assume that the lines* $\operatorname{Re}\lambda = -\beta_1$, $\operatorname{Re}\lambda = -\beta_2$, *where* $\beta_1 < \beta_2$, *do not contain eigenvalues* *of $\mathfrak{A}_0(\lambda)$ and that the sum of the algebraic multiplicities of all eigenvalues in the strip $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$ is equal to κ. Let l_1, l_2 be integer numbers not less than $2m$ and let*

$$(u, \underline{u}) \in \mathcal{W}^{l_1}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$$

be a solution of the boundary value problem (5.5.1), (5.5.2). If $(f, \underline{g})$ belongs to the intersection of the spaces

$$\mathcal{W}^{l_i-2m}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l_i-\underline{\mu}-1/2}_{2,\beta_i}(\mathcal{C}) \tag{5.5.13}$$

$(i = 1, 2)$, then there exists a real number T such that

$$(u(\cdot, t), \underline{u}(\cdot, t)) = \sum_{j=1}^{\kappa} c_j\, U_j(\cdot, t) + (w(\cdot, t), \underline{w}(\cdot, t)) \qquad \text{for } t > T.$$

Here $(w, \underline{w})$ is a solution of the equation $\mathfrak{A}(t, \partial_t)\,(w, \underline{w}) = (f, \underline{g})$ for $t > T$ which belongs to the space

$$\mathcal{W}^{l_2}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C}), \tag{5.5.14}$$

c_j are constants, and $U_j = (u^{(j)}, \underline{u}^{(j)}) \in \mathcal{W}^{l_1}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$ are linearly independent modulo the space (5.5.14) and satisfy the equation $\mathfrak{A}(t, \partial_t)U_j = 0$ for $t > T$.

Proof: Due to Lemma 5.5.3, we may assume, without loss of generality, that $l_1 = l_2 = l$. Let $\{\zeta_T\}_{T>0}$ be a set of functions on the t-axis such that $\zeta_T(t) = 0$ for $|t| < T$, $\zeta_T(t) = 1$ for $|t| > T + 1$, and $|D^j_t\, \zeta_T(t)| < c_j$ for $j = 0, 1, \dots$, where the constants c_j are independent of T. We introduce the operator

$$\mathfrak{A}_{(T)}(t, \partial_t) \stackrel{def}{=} \mathfrak{A}_0(\partial_t) + \zeta_T(t)\,\big(\mathfrak{A}(t, \partial_t) - \mathfrak{A}_0(\partial_t)\big).$$

Since $\mathfrak{A}_0(\partial_t)$ is an isomorphism from

$$\mathcal{W}^{l}_{2,\beta_i}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_i}(\mathcal{C}) \tag{5.5.15}$$

onto the space (5.5.13) for $i = 1, 2$ and the norm of $\mathfrak{A}_{(T)}(t, \partial_t) - \mathfrak{A}_0(\partial_t)$ is small for large T, the operator $\mathfrak{A}_{(T)}(t, \partial_t)$ is also an isomorphism from (5.5.15) onto (5.5.13) if T is sufficiently large.

Furthermore, $\mathfrak{A}_0(\partial_t)$ and $\mathfrak{A}_{(T)}(t, \partial_t)$ (for large T) are continuous and surjective operators from

$$\Big(\mathcal{W}^{l}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})\Big) + \Big(\mathcal{W}^{l}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C})\Big) \tag{5.5.16}$$

onto

$$\Big(\mathcal{W}^{l-2m}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_1}(\partial\mathcal{C})\Big) + \Big(\mathcal{W}^{l-2m}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{C})\Big). \tag{5.5.17}$$

According to Theorem 5.4.1, the kernel of $\mathfrak{A}_0(\partial_t)$ in (5.5.16) is spanned by the vector-functions $(u_{\mu,j,s}, \underline{u}_{\mu,j,s})$ defined by (5.4.3). Hence for the index of the operator $\mathfrak{A}_0(\partial_t)$ considered as a mapping from (5.5.16) into (5.5.17) the equality $\operatorname{ind}\mathfrak{A}_0(\partial_t) = \dim\ker\mathfrak{A}_0(\partial_t) = \kappa$ holds. By the invariance of the index under small perturbation of the operator (se, e.g., [**166**], Ch.1, Theorem 3.5), we get $\operatorname{ind}\mathfrak{A}_{(T)}(t, \partial_t) = \dim\ker\mathfrak{A}_{(T)}(t, \partial_t) = \kappa$, i.e., there exist exactly κ linear independent solutions U_j of the equation $\mathfrak{A}_{(T)}(t, \partial_t)\, U = 0$ which belong to the space (5.5.16). Since the kernel of the operator $\mathfrak{A}_{(T)}(t, \partial_t)$ in the space (5.5.14) is trivial, these functions are linearly independent modulo the space (5.5.14). Let $\eta_T = \eta_T(t)$ be a smooth function equal to zero for $t < T+1$ and to one for $t > T+2$. Then

$$\begin{aligned}\mathfrak{A}_{(T)}(t,\partial_t)\,\eta_T(u,\underline{u}) &= \mathfrak{A}(t,\partial_t)\,\eta_T(u,\underline{u}) \\ &= \eta_T(f,\underline{g}) + (\mathfrak{A}(t,\partial_t)\eta_T - \eta_T\mathfrak{A}(t,\partial_t))\,(u,\underline{u})\end{aligned}$$

belongs to the intersection of the spaces (5.5.13). Consequently, for sufficiently large T there exists an element $(w,\underline{w})$ of the space (5.5.14) satisfying the equation $\mathfrak{A}_{(T)}(t,\partial_t)\,(w,\underline{w}) = \mathfrak{A}_{(T)}(t,\partial_t)\,\eta_T(u,\underline{u})$. Thus, $\eta_T(u,\underline{u}) - (w,\underline{w})$ is an element of the kernel of $\mathfrak{A}_{(T)}(t,\partial_t)$ in (5.5.16) and therefore,

$$\eta_T(u,\underline{u}) - (w,\underline{w}) = \sum_{j=1}^{\kappa} c_j\,U_j\,.$$

The proof is complete. ■

Asymptotic expansion for boundary value problems with coefficients having exponentially decaying t-derivatives. Theorem 5.5.3 does not contain an explicit formula for the vector-functions U_j in the asymptotics. In order to get a more precise description of the behaviour of the solution at infinity, we have to impose more restrictive conditions on the coefficients. We suppose in the sequel that the coefficients of L, B_k, and $C_{k,j}$ have the form

$$\text{(5.5.18)}\qquad \begin{cases} a_{\alpha,j}(x,t) = a^{(0)}_{\alpha,j}(x) + e^{(\beta_1-\beta_2)t}\,a^{(1)}_{\alpha,j}(x,t), \\ b_{k;\alpha,j}(x,t) = b^{(0)}_{k;\alpha,j}(x) + e^{(\beta_1-\beta_2)t}\,b^{(1)}_{k;\alpha,j}(x,t), \\ c_{k,j;\alpha,\mu}(x,t) = c^{(0)}_{k,j;\alpha,\mu}(x) + e^{(\beta_1-\beta_2)t}\,c^{(1)}_{k,j;\alpha,\mu}(x,t) \end{cases}$$

for $t > T$, where $a^{(1)}_{\alpha,j}$, $b^{(1)}_{k;\alpha,j}$, and $c^{(1)}_{k,j;\alpha,\mu}$ have bounded derivatives of arbitrary order.

LEMMA 5.5.5. *Suppose that the coefficients of L, B, C satisfy the conditions (5.5.18) and the boundary value problem (5.5.1), (5.5.2) is elliptic. Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}_0(\lambda)$ on the lines* $\operatorname{Re}\lambda = -\beta_1$ *and* $\operatorname{Re}\lambda = -\beta_2$, *where $\beta_1 < \beta_2$. If l_1, l_2 are integer numbers not less than $2m$,*

$$(u,\underline{u}) \in \mathcal{W}^{l_1}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$$

is a solution of the boundary value problem (5.5.1), (5.5.2), and $(f,\underline{g})$ belongs to the intersection of the spaces (5.5.13) ($i = 1$, 2), then $(u,\underline{u})$ admits the decomposition

$$(u,\underline{u}) = \sum_{\mu=1}^{N}\sum_{j=1}^{I_\mu}\sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\,(u_{\mu,j,s}\,,\,\underline{u}_{\mu,j,s}) + (w,\underline{w}) \qquad \textit{for } t > T,$$

where $(w,\underline{w})$ belongs to the space (5.5.14) and $u_{\mu,j,s}$, $\underline{u}_{\mu,j,s}$ are defined by (5.4.3).

Proof: Under the assumptions of the theorem, $(\mathfrak{A}(t,\partial_t) - \mathfrak{A}_0(\partial_t))\,(u,\underline{u})$ belongs to the space (5.5.13). Hence it follows from the equation

$$\mathfrak{A}_0(\partial_t)\,(u,\underline{u}) = (f,\underline{g}) - \big(\mathfrak{A}(t,\partial_t) - \mathfrak{A}_0(\partial_t)\big)\,(u,\underline{u})$$

and Theorem 5.4.1 that $(u,\underline{u})$ has the representation given in the lemma. ■

Asymptotics of the solution for special right-hand sides. Now we assume that the coefficients of L, B_k, and $C_{k,j}$ have the representations

(5.5.19)
$$\begin{cases} a_{\alpha,j}(x,t) = a^{(0)}_{\alpha,j}(x) + \sum_{\iota=1}^{s} e^{-\delta_\iota t}\, a^{(\iota)}_{\alpha,j}(x,t) + e^{-\delta_{s+1} t}\, a^{(s+1)}_{\alpha,j}(x,t), \\ b_{k;\alpha,j}(x,t) = b^{(0)}_{k;\alpha,j}(x) + \sum_{\iota=1}^{s} e^{-\delta_\iota t}\, b^{(\iota)}_{k;\alpha,j}(x,t) + e^{-\delta_{s+1} t}\, b^{(s+1)}_{k;\alpha,j}(x,t), \\ c_{k,j;\alpha,\mu}(x,t) = c^{(0)}_{k,j;\alpha,\mu}(x) + \sum_{\iota=1}^{s} e^{-\delta_\iota t}\, c^{(\iota)}_{k,j;\alpha,\mu}(x,t) + e^{-\delta_{s+1} t}\, c^{(s+1)}_{k,j;\alpha,\mu}(x,t) \end{cases}$$

for $t > T$, where $\delta_1, \dots, \delta_{s+1}$ are given complex numbers such that

$$0 < \operatorname{Re}\delta_1 \le \dots \le \operatorname{Re}\delta_s \le \beta_2 - \beta_1 \le \operatorname{Re}\delta_{s+1}$$

$a^{(\iota)}_{\alpha,j}$, $b^{(\iota)}_{k;\alpha,j}$, $c^{(\iota)}_{k,j;\alpha,\mu}$ ($\iota = 1, \dots, s$) are polynomials with respect to the variable t with coefficients which are infinitely differentiable with respect to x, and $a^{(s+1)}_{\alpha,j}$, $b^{(s+1)}_{k;\alpha,j}$, $c^{(s+1)}_{k,j;\alpha,\mu}$ are smooth functions which have bounded derivatives of arbitrary order for $t > T$.

Furthermore, we suppose that f and $\underline{g}$ admit the decompositions

$$f(x,t) = \sum_{\iota=1}^{q} e^{-\sigma_\iota t}\, f^{(\iota)}(x,t) + f^{(q+1)}(x,t), \tag{5.5.20}$$

$$\underline{g}(x,t) = \sum_{\iota=1}^{q} e^{-\sigma_\iota t}\, \underline{g}^{(\iota)}(x,t) + \underline{g}^{(q+1)}(x,t), \tag{5.5.21}$$

where $f^{(q+1)} \in \mathcal{W}^{l-2m}_{2,\beta_2}(\mathcal{C})$, $\underline{g}^{(q+1)} \in \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{C})$, $\sigma_1, \dots, \sigma_q$ are complex numbers such that

$$\beta_1 < \operatorname{Re}\sigma_1 \le \dots \le \operatorname{Re}\sigma_q \le \beta_2\,,$$

$f^{(\iota)}$, $\underline{g}^{(\iota)}$ are polynomials with respect to the varable t with coefficients from the spaces $W^{l-2m}_2(\Omega)$ and $W^{l-\underline{\mu}-1/2}_2(\partial\Omega)$, respectively.

We denote by $\Lambda(\lambda_0)$ the set of all sums

$$\delta = \delta_{\iota_1} + \dots + \delta_{\iota_k}$$

formed by the numbers δ_ι in (5.5.19) such that $\operatorname{Re}\delta \le \beta_2 - \operatorname{Re}\lambda_0$. Here the number $\delta = 0$ is included if $\operatorname{Re}\lambda_0 \le \beta_2$.

THEOREM 5.5.4. *Suppose that the coefficients of L, B, C have the representation (5.5.19) for $t > T$ and the boundary value problem (5.5.1), (5.5.2) is elliptic. Furthermore, we assume that the lines* $\operatorname{Re}\lambda = -\beta_1$, $\operatorname{Re}\lambda = -\beta_2$, *where* $\beta_1 < \beta_2$, *do not contain eigenvalues of* $\mathfrak{A}_0(\lambda)$ *and denote by* $\lambda_1, \dots, \lambda_N$ *the eigenvalues of* $\mathfrak{A}_0(\lambda)$ *lying in the strip* $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$. *If*

$$(u, \underline{u}) \in \mathcal{W}^{l}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$$

is a solution of the boundary value problem (5.5.1), (5.5.2) with functions f, $\underline{g}$ on the right–hand side having the form (5.5.20), (5.5.21), then there is the representation

$$(5.5.22)\qquad (u,\underline{u}) = \sum_{\mu=1}^{N} e^{\lambda_\mu t} \sum_{\delta\in\Lambda(-\lambda_\mu)} e^{-\delta t}\,(u^{(1)}_{\mu,\delta}\,,\,\underline{u}^{(1)}_{\mu,\delta}) + \sum_{\iota=1}^{q} e^{-\sigma_\iota t} \sum_{\delta\in\Lambda(\sigma_\iota)} e^{-\delta t}\,(u^{(2)}_{\mu,\delta}\,,\,\underline{u}^{(2)}_{\mu,\delta}) + (w,\underline{w})$$

for $t>T$, where $w\in\mathcal{W}^{l}_{2,\beta_2}(\mathcal{C})$, $\underline{w}\in\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{C})$, and $u^{(i)}_{\mu,\delta}$, $\underline{u}^{(i)}_{\mu,\delta}$ $(i=1,\,2)$ are polynomials in t with coefficients from $W^l_2(\Omega)$ and $W_2^{l-\underline{\mu}-1/2}(\partial\Omega)$, respectively.

Proof: Let ζ be a infinitely differentiable function on the t-axis equal to zero for $t<T$ and to one for $t>T+c$. By Lemma 5.1.6, there exists a function

$$(v,\underline{v}) = \sum_{\iota=1}^{q} e^{-\sigma_\iota t}\,(v^{(\iota)},\underline{v}^{(\iota)}),$$

where $v^{(\iota)}$, $\underline{v}^{(\iota)}$ are polynomials of the variable t with coefficients in $W^l_2(\Omega)$ and $W_2^{l+\underline{\tau}-1/2}(\partial\Omega)$, respectively, such that

$$\mathfrak{A}_0(\partial_t)\,(v,\underline{v}) = (f,\underline{g}) - (f^{(q+1)},\underline{g}^{(q+1)}).$$

Then $\zeta((u,\underline{u})-(v,\underline{v}))$ satisfies the equation

$$(5.5.23)\quad \mathfrak{A}_0(\partial_t)\,\zeta\big((u,\underline{u})-(v,\underline{v})\big) = \zeta(f^{(q+1)},\underline{g}^{(q+1)}) + \zeta\big(\mathfrak{A}_0(\partial_t)-\mathfrak{A}(t,\partial_t)\big)\,(u,\underline{u}) + [\mathfrak{A}_0(\partial_t),\zeta]\,\big((u,\underline{u})-(v,\underline{v})\big),$$

where $[\mathfrak{A}_0(\partial_t),\zeta] = \mathfrak{A}_0(\partial_t)\zeta - \zeta\mathfrak{A}_0(\partial_t)$ denotes the commutator of $\mathfrak{A}_0(\partial_t)$ and ζ. By our assumptions on the coefficients of L, B_k, and $C_{k,j}$, the right-hand side of (5.5.23) belongs to the space

$$\mathcal{W}^{l-2m}_{2,\beta_1+Re\,\delta_1-\varepsilon}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_1+Re\,\delta_1-\varepsilon}(\partial\mathcal{C})$$

with arbitrary small $\varepsilon>0$. We can choose the number ε such that the line $\mathrm{Re}\,\lambda = -(\beta_1+\mathrm{Re}\,\delta_1-\varepsilon)$ does not contain eigenvalues of $\mathfrak{A}_0(\lambda)$. Hence by Theorem 5.4.1, there is the representation

$$\zeta\big((u,\underline{u})-(v,\underline{v})\big) = \sum_\mu e^{\lambda_\mu t}(w^{(\mu)},\underline{w}^{(\mu)}) + (w,\underline{w}),$$

where λ_μ are the eigenvalues of $\mathfrak{A}_0(\lambda)$ in the strip $-(\beta_1+\mathrm{Re}\,\delta_1-\varepsilon) < \mathrm{Re}\,\lambda < -\beta_1$, $(w^{(\mu)},\underline{w}^{(\mu)})$ are polynomials in t with coefficients in $C^\infty(\overline{\Omega})\times C^\infty(\partial\Omega)$, and

$$(w,\underline{w}) \in \mathcal{W}^{l}_{2,\beta_1+Re\,\delta_1-\varepsilon}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_1+Re\,\delta_1-\varepsilon}(\partial\mathcal{C}).$$

Since $e^{\lambda_\mu t}(w^{(\mu)},\underline{w}^{(\mu)})$ are solutions of the equation $\mathfrak{A}_0(\partial_t)\,(u,\underline{u})=0$, we get

$$\mathfrak{A}(t,\partial_t)\,(w,\underline{w}) = (f^{(q+1)},\underline{g}^{(q+1)}) - \big(\mathfrak{A}(t,\partial_t)-\mathfrak{A}_0(\partial_t)\big)\Big((v,\underline{v}) + \sum_\mu e^{\lambda_\mu t}\,(w^{(\mu)},\underline{w}^{(\mu)})\Big)$$

for $t > T + c$. Here $(\mathfrak{A}(t, \partial_t) - \mathfrak{A}_0(\partial_t))\,(v, \underline{v})$ is a sum of a vector-function from $\mathcal{W}^{l-2m}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{C})$ and finitely many vector-functions of the form

$$e^{-(\sigma_k+\delta_\iota)t}\,(z(x,t), \underline{z}(x,t)),$$

where z, $\underline{z}$ are polynomials of the variable t with coefficients in $W^{l-2m}_2(\Omega)$ and $W^{l-\underline{\mu}-1/2}_2(\partial\Omega)$, respectively. Analogously, $(\mathfrak{A}(t, \partial_t) - \mathfrak{A}_0(\partial_t))\, e^{\lambda_\mu t}(w^{(\mu)}, \underline{w}^{(\mu)})$ is a sum of a vector-function from $\mathcal{W}^{l-2m}_{2,\beta_2}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{C})$ and vector-functions of the form

$$e^{(\lambda_\mu-\delta_\iota)t}\,(z(x,t), \underline{z}(x,t)),$$

where $(z, \underline{z})$ are polynomials with respect to the variable t.

Repeating this procedure for $(w, \underline{w})$, we arrive at (5.5.22) after a finite number of steps. ■

REMARK 5.5.2. Theorem 5.5.3 and Theorem 5.5.4 describe the behaviour of the solution for $t \to +\infty$. Analogous results can be obtained for $t \to -\infty$ by means of the transformation $t \to -t$. We suppose, e.g., that the coefficients of L, B_k, $C_{k,j}$ have the representation (5.5.19) for $t < -T$ with δ_ι instead of $-\delta_\iota$ ($\iota = 1, \dots, s+1$) and

$$0 < \operatorname{Re}\delta_1 \le \dots \le \operatorname{Re}\delta_s \le \beta_1 - \beta_2 \le \operatorname{Re}\delta_{s+1}\,.$$

Furthermore, let f, $\underline{g}$ be functions which admit the decompositions (5.5.20), (5.5.21) with σ_ι insted of $-\sigma_\iota$ ($\iota = 1, \dots, q$) and

$$-\beta_1 \le \operatorname{Re}\sigma_1 \le \dots \le \operatorname{Re}\sigma_q \le -\beta_2\,.$$

As in Theorem 5.5.4, we further assume that the boundary value problem (5.5.1), (5.5.2) is elliptic and the lines $\operatorname{Re}\lambda = -\beta_1$, $\operatorname{Re}\lambda = -\beta_2$ ($\beta_1 > \beta_2$) do not contain eigenvalues of $\mathfrak{A}_0(\lambda)$. Then the solution $(u, \underline{u}) \in \mathcal{W}^{l}_{2,\beta_1}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{C})$ of the boundary value problem (5.5.1), (5.5.2) has the representation

$$(u, \underline{u}) = \sum_{\mu=1}^{N} e^{\lambda_\mu t} \sum_{\delta\in\Lambda(\lambda_\mu)} e^{\delta t}\,(u^{(1)}_{\mu,\delta}\,, \underline{u}^{(1)}_{\mu,\delta}) + \sum_{\iota=1}^{q} e^{\sigma_\iota t} \sum_{\delta\in\Lambda(\sigma_\iota)} e^{\delta t}\,(u^{(2)}_{\mu,\delta}\,, \underline{u}^{(2)}_{\mu,\delta}) + (w, \underline{w})$$

for $t < -T$, where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of $\mathfrak{A}_0(\lambda)$ lying in the strip $-\beta_1 < \operatorname{Re}\lambda < -\beta_2$, $u^{(i)}_{\mu,\delta}$, $\underline{u}^{(i)}_{\mu,\delta}$ ($i = 1, 2$) are polynomials with respect to the variable t, and $(w, \underline{w})$ belongs to the space (5.5.14). In contrast to Theorem 5.5.4, here $\Lambda(\lambda_0)$ denotes the set of all sums

$$\delta = \delta_{\iota_1} + \dots + \delta_{\iota_k}$$

such that $\operatorname{Re}\delta \le -\beta_2 - \operatorname{Re}\lambda_0$.

CHAPTER 6

Elliptic boundary value problems in domains with conical points

This chapter is concerned with boundary value problems in domains with conical points (angular for $n = 2$) on the boundary. We investigate the solvability of elliptic boundary value problems in weighted Sobolev spaces $V^l_{2,\beta}$, where the index β characterizes the powerlike growth of the solution near the conical points. Here both Sobolev spaces of positive and negative orders are considered. Furthermore, we describe the behaviour of the solutions near the conical points. We show that, under additional conditions on the right-hand sides of the boundary value problem, the solution is the sum of finitely many singular functions and a "regular" remainder.

6.1. The model problem in an infinite cone

First we consider boundary value problems in an infinite cone $\mathcal{K}$, where the differential operators L, B_k, and $C_{k,j}$ are so-called model operators. For example, homogeneous operators with constant coefficients are model operators in the domain $\mathcal{K}$. By means of the change of coordinates $x \to (\omega, t)$, where $t = \log r$ and r, ω are spherical coordinates, the model problem in $\mathcal{K}$ can be reduced to a model problem in a cylinder. In this way, the results of this section are immediate consequences of the corresponding assertions in Sections 5.2–5.4. In particular, we obtain necessary and sufficient conditions for the unique solvability of the model problem in weighted Sobolev spaces. Another goal of this section is the description of the behaviour of the solutions near the vertex of the cone.

6.1.1. Weighted Sobolev spaces in a cone. Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone with vertex at the origin, i.e.,

$$\mathcal{K} = \{\, x \in \mathbb{R}^n \, : \, 0 < r < \infty, \, \omega \in \Omega\},$$

where Ω is a domain on the unit sphere S^{n-1} with smooth boundary $\partial\Omega$. Here and in the sequel, $r = |x|$ and $\omega = x/|x|$. For integer $l \ge 0$ and real β we define the space $V^l_{2,\beta}(\mathcal{K})$ as the closure of $C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ with respect to the norm

$$\|u\|_{V^l_{2,\beta}(\mathcal{K})} = \Big(\int\limits_{\mathcal{K}} \sum_{|\alpha|\le l} r^{2(\beta-l+|\alpha|)}\, |D^\alpha_x u|^2 \, dx\Big)^{1/2}. \tag{6.1.1}$$

If $l \ge 1$, then $V^{l-1/2}_{2,\beta}(\partial\mathcal{K})$ denotes the space of traces of functions from $V^l_{2,\beta}(\mathcal{K})$ on the boundary $\partial\mathcal{K}$ equipped with the norm

$$\|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K})} = \inf\Big\{ \|v\|_{V^l_{2,\beta}(\mathcal{K})} \, : v \in V^l_{2,\beta}(\mathcal{K}),\ v|_{\partial\mathcal{K}} = u \Big\}. \tag{6.1.2}$$

Obviously, the space $V^l_{2,\beta}(\mathcal{K})$ is continuously imbedded into the space $V^{l_1}_{2,\beta_1}(\mathcal{K})$ if $l \ge l_1$ and $\beta - l = \beta_1 - l_1$. An analogous assertion holds for the space $V^{l-1/2}_{2,\beta}(\partial\mathcal{K})$.

Equivalent norms. Let ζ_k be infinitely differentiable functions on $\overline{\mathcal{K}}$ such that

(6.1.3)
$$\operatorname{supp}\zeta_k \subset \{x \in \overline{\mathcal{K}} : 2^{k-1} < |x| < 2^{k+1}\}, \qquad \sum_{k=-\infty}^{+\infty} \zeta_k = 1, \quad |D^\alpha_x \zeta_k| < c_\alpha\, 2^{-k|\alpha|}\,,$$

where the constants c_α are independent of k.

Repeating the proof of Lemma 5.2.2, we obtain the following lemma.

LEMMA 6.1.1. *The norm in the space $V^l_{2,\beta}(\mathcal{K})$ is equivalent to the norm*

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{V^l_{2,\beta}(\mathcal{K})} \Big)^{1/2}.$$

An analogous assertion holds for the norm in the space $V^{l-1/2}_{2,\beta}(\partial\mathcal{K})$, $l \ge 1$.

Passing to spherical coordinates ω, r, the cone $\mathcal{K}$ can be represented as $\Omega\times\mathbb{R}_+ = \{(\omega, r) : \omega \in \Omega,\ 0 < r < \infty\}$. The scalar products in $L_2(\mathcal{K})$ and $L_2(\partial\mathcal{K})$ generate scalar products in $L_2(\Omega)$ and $L_2(\partial\Omega)$ such that

$$(u,v)_{\mathcal{K}} = \int_0^\infty r^{n-1} \big(u(\cdot,r), v(\cdot,r)\big)_\Omega\, dr \quad \text{for each } u,v \in L_2(\mathcal{K}),$$

$$(u,v)_{\partial\mathcal{K}} = \int_0^\infty r^{n-2} \big(u(\cdot,r), v(\cdot,r)\big)_{\partial\Omega}\, dr \quad \text{for each } u,v \in L_2(\partial\mathcal{K}),$$

where $u(\omega, r) = u(x)$. It can be easliy verified that the derivative ∂^α_x has the form

$$\partial^\alpha_x = r^{-|\alpha|} \sum_{j=0}^{|\alpha|} p_{\alpha,j}(\omega, \partial_\omega)\, (r\partial_r)^j\,,$$

where $p_{\alpha,j}(\omega,\partial_\omega)$ are differential operators of order $\le |\alpha| - j$ with smooth coefficients on the sphere, and $\partial_r = r^{-1}\sum_{j=1}^n x_j\, \partial/\partial x_j$. On the other hand, for every differential operator $p(\omega,\partial_\omega)$ of order k with smooth coefficients on the sphere and every $j \ge 0$ there is a representation

$$p(\omega,\partial_\omega)\,(r\partial_r)^j = \sum_{|\alpha|\le k+j} a_\alpha(x/r)\, r^{|\alpha|}\, \partial^\alpha_x\,,$$

with smooth coefficients a_α on the sphere S^{n-1}. Hence the norm in $V^l_{2,\beta}(\mathcal{K})$ is equivalent to

(6.1.4)
$$\|u\| = \Big(\int_0^\infty r^{2(\beta-l)+n-1} \sum_{j=0}^{l} \|(r\partial_r)^j\, u(\cdot,r)\|^2_{W_2^{l-j}(\Omega)}\, dr \Big)^{1/2},$$

where $u(\omega, r) = u(x)$.

The coordinate transformation $t = \log r$ takes the half-cylinder $\Omega \times \mathbb{R}_+$ onto the cylinder $\mathcal{C} = \Omega \times \mathbb{R}$. For arbitrary functions u on $\Omega \times \mathbb{R}_+$ or $\partial\Omega \times \mathbb{R}_+$ let $\mathcal{E}u$ be the function

$$(\mathcal{E}\, u)(\omega, t) = u(\omega, e^t)\,. \tag{6.1.5}$$

Since $r\partial_r u = \partial_t \mathcal{E}u$, the norm (6.1.4) is equal to

$$\Big(\int\limits_{-\infty}^{+\infty} e^{2(\beta-l)t+nt} \sum_{j=0}^{l} \|\partial_t^j (\mathcal{E}u)(\cdot,t)\|^2_{W_2^{l-j}(\Omega)}\, dt \Big)^{1/2}.$$

Therefore,

$$\|u\| = \|e^{(\beta-l+n/2)t}\, \mathcal{E}u\|_{W_2^l(\mathcal{C})} = \|\mathcal{E}u\|_{\mathcal{W}^l_{2,\beta-l+n/2}(\mathcal{C})} \tag{6.1.6}$$

is another equivalent norm to (6.1.1). Analogously, the norm

$$\|u\| = \|e^{(\beta-l+n/2)t}\, \mathcal{E}u\|_{W_2^{l-1/2}(\partial\mathcal{C})} = \|\mathcal{E}u\|_{\mathcal{W}^{l-1/2}_{2,\beta-l+n/2}(\partial\mathcal{C})} \tag{6.1.7}$$

is equivalent to (6.1.2). This is used in the proof of the following lemma.

LEMMA 6.1.2. *The norm in $V_{2,\beta}^{l-1/2}(\partial\mathcal{K})$ is equivalent to*

$$\begin{aligned} \|u\| \;=\; & \Big(\sum_{j=0}^{l-1} \int\limits_0^\infty r^{2(\beta-l)+n-1}\, \|(r\partial_r)^j u(\cdot,r)\|^2_{W_2^{l-j-1/2}(\partial\Omega)}\, dr \\ & + \sum_{j=0}^{l-1} \int\limits_0^\infty \int\limits_{r/2}^{2r} r^{2(\beta-l)+n} \frac{\|(r\partial_r)^j u(\cdot,r) - (\rho\partial_\rho)^j u(\cdot,\rho)\|^2_{W_2^{l-j-1}(\partial\Omega)}}{|r-\rho|^2}\, d\rho\, dr \Big)^{1/2} \end{aligned}$$

for arbitrary integer $l \ge 1$.

Proof: Using the equivalence of the norm in $W_2^{l-1/2}(\partial\mathcal{C})$ to the norm

$$\begin{aligned} \|u\| \;=\; & \Big(\sum_{j=0}^{l-1} \int\limits_{\mathbb{R}} \|D_t^j u(\cdot,t)\|^2_{W_2^{l-j-1/2}(\partial\Omega)}\, dt \\ & + \sum_{j=0}^{l-1} \int\limits_{-\infty}^{+\infty} \int\limits_{t-\log 2}^{t+\log 2} \frac{\|D_t^j u(\cdot,t) - D_\tau^j u(\cdot,\tau)\|^2_{W_2^{l-j-1}(\partial\Omega)}}{|t-\tau|^2}\, d\tau\, dt \Big)^{1/2} \end{aligned}$$

(see [**126**, Ch.1, §10]), we obtain that the norm (6.1.7) is equivalent to

$$\begin{aligned} & \Big(\sum_{j=0}^{l-1} \int\limits_{\mathbb{R}} e^{2(\beta-l+n/2)t}\, \|D_t^j \mathcal{E}u(\cdot,t)\|^2_{W_2^{l-j-1/2}(\partial\Omega)}\, dt \\ & + \sum_{j=0}^{l-1} \int\limits_{-\infty}^{+\infty} \int\limits_{t-\log 2}^{t+\log 2} e^{2(\beta-l+n/2)t}\, \frac{\|D_t^j \mathcal{E}u(\cdot,t) - D_\tau^j \mathcal{E}u(\cdot,\tau)\|^2_{W_2^{l-j-1}(\partial\Omega)}}{|t-\tau|^2}\, d\tau\, dt \Big)^{1/2}. \end{aligned}$$

Substituting $t = \log r$, $\tau = \log\rho$ and using the inequalities

$$\frac{1}{2r} < \frac{|\log r - \log\rho|}{|r-\rho|} < \frac{2}{r} \quad \text{for } r/2 < \rho < 2r,$$

we get the assertion of the lemma. ■

Furthermore, equivalent norms to (6.1.1), (6.1.2) can be given by means of the *Mellin transformation*

(6.1.8) $$\tilde{u}(\lambda) = (\mathcal{M}_{r\to\lambda}u)(\lambda) = \int\limits_0^\infty r^{-\lambda-1}\, u(r)\, dr\,.$$

We recall some basic properties of this transformation which follow from Lemma 5.2.3 and from the equality $(\mathcal{M}_{r\to\lambda}u)(\lambda) = \mathcal{L}_{t\to\lambda}\, u(e^t)$.

LEMMA 6.1.3. 1) *The transformation (6.1.8) realizes a linear and continuous mapping from $C_0^\infty(\mathbb{R}_+)$ into the space $\mathcal{A}(\mathbb{C})$ of analytic functions on $\mathbb{C}$.*

2) *Every $u \in C_0^\infty(\mathbb{R}_+)$ satisfies the equality*

$$\mathcal{M}_{r\to\lambda}(r\partial_r)u = \lambda\mathcal{M}_{r\to\lambda}u.$$

Furthermore, for all $u, v \in C_0^\infty(\mathbb{R}_+)$ the Parseval equality

(6.1.9) $$\int\limits_0^\infty r^{2\beta-1}\, u(r)\, \overline{v(r)}\, dr = \frac{1}{2\pi i} \int\limits_{Re\,\lambda=-\beta} \tilde{u}(\lambda)\, \overline{\tilde{v}(\lambda)}\, d\lambda$$

is valid. Hence the transformation (6.1.8) can be continuously extended to the isomorphism

$$\big\{u\,:\, r^{\beta-1/2}\, u \in L_2(\mathbb{R}_+)\big\} \to L_2(\ell_{-\beta}),$$

where $\ell_{-\beta}$ denotes the line $\mathrm{Re}\,\lambda = -\beta$ in the complex plane.

3) *The inverse Mellin transformation is given by the formula*

$$u(r) = (\mathcal{M}^{-1}_{\lambda\to r}\tilde{u})(r) = \frac{1}{2\pi i} \int\limits_{Re\,\lambda=-\beta} r^\lambda\, \tilde{u}(\lambda)\, d\lambda.$$

4) *If $r^{\beta_i-1/2}u \in L_2(\mathbb{R}_+)$ for $i = 1, 2$, where β_1, β_2 are arbitrary real numbers, $\beta_1 < \beta_2$, then $\tilde{u}$ is holomorphic in the strip $-\beta_2 < \mathrm{Re}\,\lambda < -\beta_1$.*

Using the Parseval equality, we obtain the following assertions.

LEMMA 6.1.4. *The norm (6.1.1) is equivalent to*

$$\|u\| = \Big(\frac{1}{2\pi i} \int\limits_{Re\,\lambda=-\beta+l-n/2} \sum_{j=0}^{l} |\lambda|^{2j}\, \|\tilde{u}(\cdot,\lambda)\|^2_{W_2^{l-j}(\Omega)}\, d\lambda\Big)^{1/2},$$

where $\tilde{u}(\omega, \lambda) = (\mathcal{M}_{r\to\lambda}u)(\omega, \lambda)$ and $u(\omega, r) = u(x)$. Analogously, the norm (6.1.2) is equivalent to

$$\|u\| = \Big(\frac{1}{2\pi i} \int\limits_{Re\,\lambda=-\beta+l-n/2} \big(\|\tilde{u}(\cdot,\lambda)\|^2_{W_2^{l-1/2}(\partial\Omega)} + |\lambda|^{2l-1}\, \|\tilde{u}(\cdot,\lambda)\|^2_{L_2(\partial\Omega)}\,\big)\, d\lambda\Big)^{1/2}.$$

Proof: The first assertion follows from the equivalence of the norms (6.1.1), (6.1.4) and from the Parseval equality (6.1.9), while the second assertion is a consequence of Lemma 5.2.4. ∎

Weighted Sobolev spaces of negative order. Let l be a nonnegative integer. Then $V^l_{2,\beta}(\mathcal{K})^*$ denotes the dual space of $V^l_{2,\beta}(\mathcal{K})$ equipped with the norm

$$\|u\|_{V^l_{2,\beta}(\mathcal{K})^*} = \sup\Big\{|(u,v)_{\mathcal{K}}|\,:\, v \in V^l_{2,\beta}(\mathcal{K}),\ \|v\|_{V^l_{2,\beta}(\mathcal{K})} = 1\Big\},$$

where $(\cdot,\cdot)_{\mathcal{K}}$ is the extension of the scalar product in $L_2(\mathcal{K})$ to pairs $(u,v) \in V^l_{2,\beta}(\mathcal{K})^* \times V^l_{2,\beta}(\mathcal{K})$. Obviously, for $l = 0$ we have $V^0_{2,\beta}(\mathcal{K})^* = V^0_{2,-\beta}(\mathcal{K})$. Analogously, $V^{-l+1/2}_{2,\beta}(\partial\mathcal{K})$ is defined as the dual space of $V^{l-1/2}_{2,-\beta}(\partial\mathcal{C})$.

For integer k and l, $k \ge 0$, let $\tilde{V}^{l,k}_{2,\beta}(\mathcal{K})$ be the set of all pairs $(u, \underline{\phi})$, where

$$u \in \begin{cases} V^l_{2,\beta}(\mathcal{K}) & \text{if } l \ge 0, \\ V^{-l}_{2,-\beta}(\mathcal{K})^* & \text{if } l < 0, \end{cases}$$

and

$$\underline{\phi} = (\phi_1, \dots, \phi_k) \in \prod_{j=1}^{k} V^{l-j+1/2}_{2,\beta}(\partial\mathcal{K}), \qquad \phi_j = D^{j-1}_\nu u|_{\partial\mathcal{K}\backslash\{0\}} \text{ for } j \le \min(k,l).$$

In particular, $\tilde{V}^{l,0}_{2,\beta}(\mathcal{K})$ coincides with $V^l_{2,\beta}(\mathcal{K})$ if $l \ge 0$ and with $V^{-l}_{2,-\beta}(\mathcal{K})^*$ if $l < 0$. The norm in $\tilde{V}^{l,k}_{2,\beta}(\mathcal{K})$ is defined as

$$\|(u,\underline{\phi})\|_{\tilde{V}^{l,k}_{2,\beta}(\mathcal{K})} = \|u\|_{\tilde{V}^{l,0}_{2,\beta}(\mathcal{K})} + \sum_{j=1}^{k} \|\phi_j\|_{V^{l-j+1/2}_{2,\beta}(\partial\mathcal{K})}\,.$$

The mapping $\mathcal{E}$ defined by (6.1.5) is an isomorphism from $\tilde{V}^{l,k}_{2,\beta}(\mathcal{K})$ onto the space $\tilde{\mathcal{W}}^{l,k}_{2,\beta-l+n/2}(\mathcal{C})$. Furthermore, the norm in $\tilde{V}^{l,k}_{2,\beta}(\mathcal{K})$ is equivalent to

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{\tilde{V}^{l,k}_{2,\beta}(\mathcal{K})} \Big)^{1/2},$$

where ζ_k are infinitely differentiable functions satisfying the conditions (6.1.3) (cf. Lemma 5.3.1).

From the definition of the space $V^l_{2,\beta}(\mathcal{K})$ it follows immediately that

$$V^{l_1}_{2,\beta_1}(\mathcal{K}) \subset V^{l_2}_{2,\beta_2}(\mathcal{K}) \quad \text{if } l_1 \ge l_2 \ge 0,\ \beta_1 - l_1 = \beta_2 - l_2.$$

This imbedding is continuous. Analogously, there are the imbeddings

$$V^{l_1-1/2}_{2,\beta_1}(\mathcal{K}) \subset V^{l_2-1/2}_{2,\beta_2}(\mathcal{K}), \quad \tilde{V}^{l_1,k}_{2,\beta_1}(\mathcal{K}) \subset \tilde{V}^{l_1,k}_{2,\beta_2}(\mathcal{K}) \quad \text{if } l_1 \ge l_2,\ \beta_1 - l_1 = \beta_2 - l_2.$$

In all these imbeddings the first space is dense in the second one.

6.1.2. Solvability of the model problem in a cone. A differential operator $P(x,\partial_x)$ is said to be a *model operator* of order k in the cone $\mathcal{K}$ if P has the form

$$P(x,\partial_x) = r^{-k}\,\mathcal{P}(\omega,\partial_\omega, r\partial_r) = r^{-k} \sum_{j=0}^{k} p_j(\omega,\partial_\omega)\,(r\partial_r)^j\,, \tag{6.1.10}$$

where $p_j(\omega,\partial_\omega)$ are differential operators of order $\le k-j$ with smooth coefficients on $\overline{\Omega}$. Analogously, the differential operator $P(x,\partial_x)$ is said to be a *tangential model operator* of order k on $\partial\mathcal{K}\backslash\{0\}$ if P has the form (6.1.10), where $p_j(\omega,\partial_\omega)$ are tangential differential operators of order $\le k-j$ on $\partial\Omega$ with infinitely differentiable coefficients.

Note that the order with respect to differentiation of a model operator of order k can be strictly less than k.

Examples. 1) Every homogeneous operator $P(\partial_x) = \sum\limits_{|\alpha|=k} a_\alpha\, \partial_x^\alpha$ of order k with constant coefficients is a model operator of order k.

2) The operator $u \to \dfrac{\partial u}{\partial \nu}\big|_{\partial\mathcal{K}\backslash\{0\}}$ is a model operator of first order, since $\dfrac{\partial u}{\partial \nu_x}(x) = \dfrac{1}{r}\,\dfrac{\partial u}{\partial \nu_\omega}(\omega, r)$.

Obviously, every model operator of order k continuously maps $V^l_{2,\beta}(\mathcal{K})$ into the space $V^{l-k}_{2,\beta}(\mathcal{K})$ for arbitrary integer $l \geq k$, $\beta \in \mathbb{R}$. We consider the boundary value problem

$$Lu = f \quad \text{in } \mathcal{K}, \tag{6.1.11}$$

$$Bu + C\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{K}\backslash\{0\}. \tag{6.1.12}$$

Here

$$L = L(x, \partial_x) = r^{-2m}\, \mathcal{L}(\omega, \partial_\omega, r\partial_r) \tag{6.1.13}$$

is a model operator of order $2m$, B is a vector of model operators

$$B_k(x, \partial_x) = r^{-\mu_k}\, \mathcal{B}_k(\omega, \partial_\omega, r\partial_r), \quad k = 1, \dots, m+J, \tag{6.1.14}$$

of order μ_k, and C is a matrix of model operators

$$C_{k,j}(x, \partial_x) = r^{-\mu_k - \tau_j}\, \mathcal{C}_{k,j}(\omega, \partial_\omega, r\partial_r), \quad k = 1, \dots, m+J,\ j = 1, \dots, J, \tag{6.1.15}$$

of order $\mu_k + \tau_j$ which are tangential on $\partial\mathcal{K}\backslash\{0\}$. We call the boundary value problem (6.1.11), (6.1.12) *model problem* in the cone $\mathcal{K}$. The model problem is said to be *elliptic* if the differential operator L is elliptic in $\overline{\mathcal{K}}\backslash\{0\}$ and condition (ii) of Definition 3.1.2 is satisfied for all $x^{(0)} \in \partial\mathcal{K}\backslash\{0\}$.

We suppose in the sequel that the orders (with respect to differentiation) of the operators B_k, $k = 1, \dots, m+J$, are less than $2m$. Then

$$B(x, \partial_x)u|_{\partial\mathcal{K}\backslash\{0\}} = Q(x, \partial_x)\, \mathcal{D}u|_{\partial\mathcal{K}\backslash\{0\}} \tag{6.1.16}$$

for $u \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$, where Q is a $(m+J) \times 2m$-matrix of tangential differential operators $Q_{k,j}(x, \partial_x)$, $\operatorname{ord} Q_{k,j} \leq \mu_k - j + 1$, and $\mathcal{D}$ denotes the column vector with the components $1, D_\nu, \dots, D_\nu^{2m-1}$.

The operator $\mathcal{A}$ of the boundary value problem (6.1.11), (6.1.12) continuously maps

$$V^l_{2,\beta}(\mathcal{K}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K}) \tag{6.1.17}$$

into the space

$$V^{l-2m}_{2,\beta}(\mathcal{K}) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K}) \tag{6.1.18}$$

for arbitrary integer $l \geq 2m$ and real β. Here $V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$, $V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$ denote the products of the spaces $V^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{K})$ and $V^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{K})$, respectively.

Applying the coordinate transformation $x \to (\omega, r)$ to (6.1.11), (6.1.12), we get the problem

$$\text{(6.1.19)}\quad \mathcal{L}(\omega, \partial_\omega, r\partial_r)u = r^{2m} f \qquad \text{in } \Omega \times \mathbb{R}_+ ,$$

$$\text{(6.1.20)}\quad \mathcal{B}_k(\omega, \partial_\omega, r\partial_r)\, u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(\omega, \partial_\omega, r\partial_r + \tau_j)\, r^{-\tau_j} u_j = r^{\mu_k} g_k \text{ on } \partial\Omega \times \mathbb{R}_+ ,$$

$k = 1, \dots, m+J$, with the unknowns u, $r^{-\tau_1}u_1, \dots, r^{-\tau_J}u_J$. Here we have used the equality $(r\partial_r + \tau_j)\,(r^{-\tau_j}u_j) = r^{-\tau_j}\, r\partial_r\, u_j$. We denote the operator of this problem by $\mathfrak{A}(r\partial_r)$ and the operator of the corresponding parameter-depending boundary value problem

$$\text{(6.1.21)}\quad \mathcal{L}(\omega, \partial_\omega, \lambda)u = f \quad \text{in } \Omega,$$

$$\text{(6.1.22)}\quad \mathcal{B}_k(\omega, \partial_\omega, \lambda)u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(\omega, \partial_\omega, \lambda + \tau_j)u_j = g_j \quad \text{on } \partial\Omega,\ k = 1, \dots, m+J,$$

by $\mathfrak{A}(\lambda)$.

LEMMA 6.1.5. *If the boundary value problem (6.1.11), (6.1.12) is elliptic in* $\overline{\mathcal{K}}\backslash\{0\}$, *then problem (6.1.21), (6.1.22) is elliptic with parameter (see Definition 3.6.1).*

Proof: Applying the transformation $r = e^t$ to problem (6.1.19), (6.1.20), we get the boundary value problem

$$\text{(6.1.23)}\quad \mathcal{L}(\omega, \partial_\omega, \partial_t)u = e^{2mt} f \qquad \text{in } \Omega \times \mathbb{R},$$

$$\text{(6.1.24)}\quad \mathcal{B}_k(\omega, \partial_\omega, \partial_t)u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(\omega, \partial_\omega, \partial_t + \tau_j)\, e^{-\tau_j t}u_j = e^{\mu_k t} g_j \text{ on } \partial\Omega \times \mathbb{R} ,$$

$k = 1, \dots, m+J$, with the unknowns $u, e^{-\tau_1 t}u_1, \dots, e^{-\tau_J t}u_J$. Since the ellipticity is invariant with respect to diffeomorphisms, the boundary value problems (6.1.19), (6.1.20) and (6.1.23), (6.1.24) are elliptic in $\Omega \times \mathbb{R}_+$ and $\Omega \times \mathbb{R}$, respectively. Consequently, problem (6.1.21), (6.1.22) is elliptic with parameter. ∎

Thus, the operator pencil $\mathfrak{A}(\lambda)$ has the same properties as in Section 5.2 (cf. Theorem 5.2.1) if problem (6.1.11), (6.1.12) is elliptic. In particular, the spectrum of $\mathfrak{A}(\lambda)$ is an enumerable set of eigenvalues.

THEOREM 6.1.1. *Suppose the model problem (6.1.11), (6.1.12) is elliptic and no eigenvalues of* $\mathfrak{A}(\lambda)$ *lie on the line* $\operatorname{Re}\lambda = -\beta + l - n/2$, *where* l *is an integer,* $l \geq 2m$. *Then the boundary value problem (6.1.11), (6.1.12) is uniquely solvable in the space (6.1.17) for all* $f \in V^l_{2,\beta}(\mathcal{K})$, $\underline{g} \in V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$ *and the solution* $(u, \underline{u})$ *satisfies the estimate*

$$\text{(6.1.25)}\quad \|u\|_{V^l_{2,\beta}(\mathcal{K})} + \|\underline{u}\|_{V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \leq c\left(\|f\|_{V^{l-2m}_{2,\beta}(\mathcal{K})} + \|\underline{g}\|_{V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})}\right)$$

with a constant c *independent of* f *and* $\underline{g}$.

Proof: Let $\mathcal{E}$ be the operator defined by (6.1.5). As we have seen in the previous subsection, this operator realizes isomorphisms

$$V^l_{2,\beta}(\mathcal{K}) \to \mathcal{W}^l_{2,\beta-l+n/2}(\mathcal{C}) \quad \text{and} \quad V^{l-1/2}_{2,\beta}(\partial\mathcal{K}) \to \mathcal{W}^{l-1/2}_{2,\beta-l+n/2}(\partial\mathcal{C}),$$

where $\mathcal{C}$ denotes the cylinder $\Omega \times \mathbb{R}$. By Theorem 5.2.2, the operator $\mathfrak{A}(\partial_t)$ of the boundary value problem (6.1.23), (6.1.24) is an isomorphism

$$\mathcal{W}^l_{2,\beta-l+n/2}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta-l+n/2}(\partial\mathcal{C}) \to \mathcal{W}^{l-2m}_{2,\beta-l+n/2}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta-l+n/2}(\partial\mathcal{C})$$

if no eigenvalues of $\mathfrak{A}(\lambda)$ lie on the line $\operatorname{Re}\lambda = -\beta + l - n/2$. Hence the operator $\mathfrak{A}(r\partial_r) = \mathcal{E}^{-1}\,\mathfrak{A}(\partial_t)\,\mathcal{E}$ is an isomorphism

$$V^l_{2,\beta}(\mathcal{K}) \times \prod_{j=1}^{J} V^{l+\tau_j-1/2}_{2,\beta+\tau_j}(\partial\mathcal{K}) \to V^{l-2m}_{2,\beta-2m}(\mathcal{K}) \times \prod_{k=1}^{m+J} V^{l-\mu_k-1/2}_{2,\beta-\mu_k}(\partial\mathcal{K}).$$

Since the mapping $u \to r^\alpha u$ realizes isomorphisms from $V^l_{2,\beta}(\mathcal{K})$ onto $V^l_{2,\beta-\alpha}(\mathcal{K})$ and from $V^{l-1/2}_{2,\beta}(\partial\mathcal{K})$ onto $V^{l-1/2}_{2,\beta-\alpha}(\partial\mathcal{K})$, we conclude that $\mathcal{A}$ is an isomorphism from (6.1.17) onto (6.1.18). ■

REMARK 6.1.1. By Lemma 5.2.5, the condition on the eigenvalues of $\mathfrak{A}(\lambda)$ is necessary for the validity of the estimate (6.1.25).

Note that the following formula for the solution $(u, \underline{u})$ of problem (6.1.11), (6.1.12) in the cone $\mathcal{K}$ is valid (cf. formula (5.2.25)):

$$(u, r^{-\underline{\tau}}\underline{u}) = \frac{1}{2\pi i} \int\limits_{Re\,\lambda=-\beta+l-n/2} r^\lambda\, \mathfrak{A}(\lambda)^{-1}\, \mathcal{M}_{r\to\lambda}\,(r^{2m}f, r^{\underline{\mu}}\underline{g})\, d\lambda.$$

Here $r^{-\underline{\tau}}\underline{u}$ denotes the vector $(r^{-\tau_1}u_1, \dots, r^{-\tau_J}u_J)$, while $r^{\underline{\mu}}\underline{g}$ denotes the vector $(r^{\mu_1}g_1, \dots, r^{\mu_{m+J}}g_{m+J})$.

Solvability of the formally adjoint problem. Analogously to Theorem 3.1.1, the following Green formula holds for all $u, v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$, $\underline{u} \in C_0^\infty(\partial\mathcal{K}\backslash\{0\})^J$, $\underline{v} \in C_0^\infty(\partial\mathcal{K}\backslash\{0\})^{m+J}$:

(6.1.26)
$$\int\limits_{\mathcal{K}} Lu \cdot \overline{v}\, dx + \int\limits_{\partial\mathcal{K}} \big(Bu + C\underline{u},\, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma$$
$$= \int\limits_{\mathcal{K}} u \cdot \overline{L^+ v}\, dx + \int\limits_{\partial\mathcal{K}} \big(\mathcal{D}u,\, Pv + Q^+\underline{v}\big)_{\mathbb{C}^{2m}}\, d\sigma + \int\limits_{\partial\mathcal{K}} \big(\underline{u}, C^+\underline{v}\big)_{\mathbb{C}^J}\, d\sigma$$

Here P is a vector of differential operators P_j, $j = 1, \dots, 2m$, $\operatorname{ord} P_j = 2m - j$, which are uniquely determined by L.

LEMMA 6.1.6. *If the operators L, B_k, $C_{k,j}$ are model operators of order $2m$, μ_k, and $\mu_k + \tau_j$, respectively, then all operators in the Green formula (6.1.26) are model operators. More precisely,*

(6.1.27)
$$L^+(x, \partial_x) = r^{-2m}\, \mathcal{L}^+(\omega, \partial_\omega, r\partial_r)$$

is a model operator of order $2m$, P is a vector of model operators

(6.1.28)
$$P_j(x, \partial_x) = r^{-2m+j}\, \mathcal{P}_j(\omega, \partial_\omega, r\partial_r)$$

of order $2m - j$, Q^+ is a matrix of tangential model operators

(6.1.29)
$$Q^+_{k,j}(x, \partial_x) = r^{-\mu_k+j-1}\, \mathcal{Q}^+_{k,j}(\omega, \partial_\omega, r\partial_r)$$

of order $\mu_k - j + 1$ on $\partial\mathcal{K}\backslash\{0\}$, and C^+ is a matrix of tangential model operators

(6.1.30)
$$C^+_{k,j}(x, \partial_x) = r^{-\mu_k-\tau_j}\, \mathcal{C}^+_{k,j}(\omega, \partial_\omega, r\partial_r)$$

of order $\mu_k + \tau_j$ on $\partial\mathcal{K}\backslash\{0\}$. Furthermore, the Green formula

$$(6.1.31)\qquad \int_\Omega \mathcal{L}(\lambda)\tilde u \cdot \bar{\tilde v}\, d\omega + \sum_{k=1}^{m+J} \int_{\partial\Omega} \Big(\mathcal{B}_k(\lambda)\tilde u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(\lambda+\tau_j)\tilde u_j\Big) \cdot \overline{\tilde v_k}\, d\varsigma$$
$$= \int_\Omega \tilde u \cdot \overline{\mathcal{L}^+(-\overline{\lambda}+2m-n)\tilde v}\, d\omega + \sum_{j=1}^{J}\int_{\partial\Omega} \tilde u_j \cdot \sum_{k=1}^{m+J} \overline{\mathcal{C}^+_{k,j}(-\overline{\lambda}+\mu_k+1-n)\tilde v_k}\, d\varsigma$$
$$+ \sum_{j=1}^{2m}\int_{\partial\Omega} D_\nu^{j-1}\tilde u \cdot \Big(\overline{\mathcal{P}_j(-\overline{\lambda}+2m-n)\tilde v} + \sum_{k=1}^{m+J} \overline{\mathcal{Q}^+_{k,j}(-\overline{\lambda}+\mu_k+1-n)\tilde v_k}\Big)\, d\varsigma$$

is satisfied for all $\tilde u, \tilde v \in C^\infty(\overline{\Omega})$, $\underline{\tilde u} \in C^\infty(\partial\Omega)^J$, $\underline{\tilde v} \in C^\infty(\partial\Omega)^{m+J}$. (For the sake of brevity, we have omitted the arguments ω and ∂_ω in the operators of this formula.)

Proof: Let $\tilde u(\omega,\lambda)$, $\tilde u_j(\omega,\lambda)$, $\tilde v(\omega,\lambda)$, $\tilde v_k(\omega,\lambda)$ be the Mellin transforms of the functions $u(\omega,r)$, $r^{-\tau_j}u_j(\omega,r)$, $r^{n-2m}v(\omega,r)$, and $r^{n-1-\mu_k}v_k(\omega,r)$ with respect to the variable r. Then by Theorem 3.1.1, the Green formula (6.1.31) is satisfied for each λ, where $\mathcal{L}$, $\mathcal{P}_j$, $\mathcal{Q}^+_{k,j}$, and $\mathcal{C}^+_{k,j}$ are certain differential operators polynomially depending on λ, which are uniquely determined by $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{C}$. In particular, $\mathcal{L}^+(\omega,\partial_\omega,-\overline{\lambda}+2m-n)$ is the formally adjoint operator to $\mathcal{L}(\omega,\partial_\omega,\lambda)$ on Ω. We integrate (6.1.31) with respect to λ over the line $\lambda = i\tau$, $-\infty < \tau < +\infty$. Since $-\overline{\lambda} = \lambda$ on this line, the Parseval equality (6.1.9) with $\beta = 0$ and the equality

$$(r\partial_r + \alpha)^j\, r^{-\alpha}u = r^{-\alpha}\,(r\partial_r)^j u$$

yield

$$\int_{\mathcal{C}_+} r^{n-1-2m}\mathcal{L}u \cdot \overline{v}\, d\omega\, dr + \sum_{k=1}^{m+J}\int_{\partial\mathcal{C}_+} r^{n-2-\mu_k}\Big(\mathcal{B}_k u + \sum_{j=1}^{J} r^{-\tau_j}\,\mathcal{C}_{k,j}u_j\Big)\overline{v_k}\, d\varsigma\, dr$$
$$= \int_{\mathcal{C}_+} r^{n-1}\, u \cdot \overline{r^{-2m}\,\mathcal{L}^+ v}\, d\omega\, dr + \sum_{j=1}^{J}\int_{\partial\mathcal{C}_+} r^{n-2}\, u_j \cdot \sum_{k=1}^{m+J}\overline{r^{-\mu_k-\tau_j}\,\mathcal{C}^+_{k,j}v_k}\, d\varsigma\, dr$$
$$+ \sum_{j=1}^{2m}\int_{\partial\mathcal{C}_+} r^{n-2}\,(r^{-1}D_\nu)^{j-1}u \cdot \Big(\overline{r^{-2m+j}\mathcal{P}_j v} + \sum_{k=1}^{m+J}\overline{r^{-\mu_k+j-1}\mathcal{Q}^+_{k,j}v_k}\Big)\, d\varsigma\, dr,$$

where $\mathcal{C}_+ = \Omega\times\mathbb{R}_+$, by $\mathcal{L}$ we mean the operator $\mathcal{L}(\omega,\partial_\omega,r\partial_r)$, and the same abbreviation was used for the operators $\mathcal{L}^+(\omega,\partial_\omega,r\partial_r)$, $\mathcal{B}_k(\omega,\partial_\omega,r\partial_r)$, $\mathcal{C}_{k,j}(\omega,\partial_\omega,r\partial_r)$, $\mathcal{P}_j(\omega,\partial_\omega,r\partial_r)$, $\mathcal{C}^+_{k,j}(\omega,\partial_\omega,r\partial_r)$, $\mathcal{Q}_{k,j}(\omega,\partial_\omega,r\partial_r)$. The change of coordinates $(\omega,r)\to x$ leads to formula (6.1.26), where the operators L^+, P, Q^+ and C^+ are defined by (6.1.27)–(6.1.30). This proves the lemma. ∎

The boundary value problem

$$(6.1.32)\qquad L^+v = f \quad \text{in } \mathcal{K},$$
$$(6.1.33)\qquad Pv + Q^+\underline{v} = \underline{g}, \quad C^+\underline{v} = \underline{h} \quad \text{on } \partial\mathcal{K}$$

is called *formally adjoint* to the problem (6.1.11), (6.1.12). The operator $\mathcal{A}^+$ of this problem continuously maps

$$V^l_{2,\beta}(\mathcal{K}) \times \prod_{k=1}^{m+J} V^{l-2m+\mu_k+1/2}_{2,\beta}(\partial\mathcal{K}) \tag{6.1.34}$$

into the space

$$V^{l-2m}_{2,\beta}(\mathcal{K}) \times \prod_{j=1}^{2m} V^{l-2m+j-1/2}_{2,\beta}(\partial\mathcal{K}) \times \prod_{j=1}^{J} V^{l-2m-\tau_j+1/2}_{2,\beta}(\mathcal{K}). \tag{6.1.35}$$

We denote the operator of the parameter-depending problem

$$\mathcal{L}^+(\omega, \partial_\omega, -\lambda + 2m - n)\, v = f \quad \text{in } \Omega, \tag{6.1.36}$$

$$\mathcal{P}_j(\omega, \partial_\omega, -\lambda + 2m - n)\, v + \sum_{k=1}^{m+J} \mathcal{Q}^+_{k,j}(\omega, \partial_\omega, -\lambda + \mu_k + 1 - n)\, v_k = g_j \quad \text{on } \partial\Omega,\ j = 1, \dots, 2m, \tag{6.1.37}$$

$$\sum_{k=1}^{m+J} \mathcal{C}^+_{k,j}(\omega, \partial_\omega, -\lambda + \mu_k + 1 - n)\, v_k = h_j \quad \text{on } \partial\Omega,\ j = 1, \dots, J, \tag{6.1.38}$$

by $\mathfrak{A}^+(\lambda)$. According to the Green formula (6.1.31), the operator $\mathfrak{A}^+(\overline{\lambda})$ is formally adjoint to $\mathfrak{A}(\lambda)$. Hence the assertions of Lemma 5.3.2 are valid for $\mathfrak{A}^+(\lambda)$ if the boundary value problem (6.1.11), (6.1.12) is elliptic in $\overline{\mathcal{K}}\backslash\{0\}$. In particular, any number λ_0 is an eigenvalue of $\mathfrak{A}^+(\lambda)$ if and only if $\overline{\lambda}_0$ is an eigenvalue of the pencil $\mathfrak{A}(\lambda)$.

By Theorem 6.1.1, the operator $\mathcal{A}^+$ realizes an isomorphism from (6.1.34) onto (6.1.35) for arbitrary integer $l \geq 2m$ if there are no eigenvalues of the operator pencil $\mathfrak{A}^+(-\lambda + 2m - n)$ on the line $\operatorname{Re}\lambda = -\beta + l - n/2$, i.e., if the line $\operatorname{Re}\lambda = \beta - l + 2m - n/2$ does not contain eigenvalues of $\mathfrak{A}^+(\lambda)$. Using the above mentioned relation between the eigenvalues of $\mathfrak{A}(\lambda)$ and $\mathfrak{A}^+(\lambda)$, we get the following assertion.

LEMMA 6.1.7. *Suppose that the boundary value (6.1.11), (6.1.12) is elliptic and there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the line $\operatorname{Re}\lambda = \beta - l + 2m - n/2$, where l is an integer, $l \geq 2m$. Then the operator $\mathcal{A}^+$ is an isomorphism from (6.1.34) onto (6.1.35).*

6.1.3. Extension of the operator of the model problem. Our goal is to construct an extension of the operator of the boundary value problem (6.1.11), (6.1.12) to the space $\tilde{V}^{l,2m}_{2,\beta}(\mathcal{K}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ with integer $l < 2m$.

We start with the model operator L. Let l be an integer, $0 < l < 2m$. Then we write L in the form

$$L(x, \partial_x) = \sum_{|\alpha| \leq 2m-l} \partial_x^\alpha\, L_\alpha(x, \partial_x)\,, \tag{6.1.39}$$

where L_α are model operators of order $\leq l$. Analogously to Lemma 3.2.1, we obtain the following result.

LEMMA 6.1.8. *Let L be the operator (6.1.39). Then the formula*

$$\int_{\mathcal{K}} Lu \cdot \overline{v}\, dx = \sum_{|\alpha|\le 2m-l} \int_{\mathcal{K}} L_\alpha(x,\partial_x)u \cdot \overline{(-\partial_x)^\alpha v}\, dx + \sum_{j=l+1}^{2m} \int_{\partial\mathcal{K}} D_\nu^{j-1}u \cdot \overline{P_j v}\, d\sigma + \sum_{j=1}^{l} \int_{\partial\mathcal{K}} D_\nu^{j-1}u \cdot \overline{P_{l,j} v}\, d\sigma \tag{6.1.40}$$

is valid for $u, v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$. Here P_j are the same operators as in the Green formula (6.1.26) and $P_{l,j}$ are model operators of order $\le 2m-j$ such that the functional

$$v \to \sum_{j=1}^{l} \int_{\partial\mathcal{K}} D_\nu^{j-1}u \cdot \overline{P_{l,j} v}\, d\sigma$$

is continuous on $V_{2,-\beta}^{2m-l}(\mathcal{K})$ for arbitrary $u \in V_{2,\beta}^{l}(\mathcal{K})$ and arbitrary real β.

Using formulas (6.1.26) and (6.1.40), we can construct an extension of the operator L to the space $\tilde{V}_{2,\beta}^{l,2m}(\mathcal{K})$, $l < 2m$, as follows.

LEMMA 6.1.9. *The operator*

$$\tilde{V}_{2,\beta-l+2m}^{2m,2m}(\mathcal{K}) \ni \big(u, \mathcal{D}u|_{\partial\mathcal{K}\backslash\{0\}}\big) \to Lu \in V_{2,\beta-l+2m}^{0}(\mathcal{K}) \tag{6.1.41}$$

can be uniquely extended to a continuous operator

$$\tilde{V}_{2,\beta}^{l,2m}(\mathcal{K}) \ni (u, \underline{\phi}) \to f \in V_{2,-\beta}^{2m-l}(\mathcal{K})^*, \quad l < 2m. \tag{6.1.42}$$

The functional $f = L(u,\underline{\phi})$ in (6.1.42) is given by the equality

$$(f,v)_{\mathcal{K}} = (u, L^+v)_{\mathcal{K}} + \sum_{j=1}^{2m} \left(\phi_j, P_j v\right)_{\partial\mathcal{K}}, \quad v \in V_{2,-\beta}^{2m-l}(\mathcal{K}), \tag{6.1.43}$$

if $l \le 0$ and by the equality

$$(f,v)_{\mathcal{K}} = \sum_{|\alpha|\le 2m-l} \int_{\mathcal{K}} L_\alpha(x,\partial_x)u \cdot \overline{(-\partial_x)^\alpha v}\, dx + \sum_{j=l+1}^{2m} (\phi_j\,, P_j v)_{\partial\mathcal{K}} + \sum_{j=1}^{l} \left(D_\nu^{j-1}u\,, P_{l,j}v\right)_{\partial\mathcal{K}}, \quad v \in V_{2,-\beta}^{2m-l}(\mathcal{K}), \tag{6.1.44}$$

if $0 < l < 2m$.

Proof: According to (6.1.26) and (6.1.40), the operator (6.1.42) is an extension of the operator (6.1.41). The uniqueness of this extension follows from the density of $\tilde{V}_{2,\beta-l+2m}^{2m,2m}(\mathcal{K})$ in $\tilde{V}_{2,\beta}^{l,2m}(\mathcal{K})$. ■

Furthermore, by (6.1.16), the mapping

$$\tilde{V}_{2,\beta}^{l,2m}(\mathcal{K}) \ni (u, \underline{\phi}) \to Q\underline{\phi} \in V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{K}), \quad l < 2m,$$

is an extension of the operator

$$\tilde{V}_{2,\beta-l+2m}^{2m,2m}(\mathcal{K}) \ni (u, \mathcal{D}u|_{\partial\mathcal{K}\backslash\{0\}}) \to Bu|_{\partial\mathcal{K}\backslash\{0\}} \in V_{2,\beta-l+2m}^{2m-\underline{\mu}-1/2}(\partial\mathcal{K}).$$

Thus, we get the following theorem.

THEOREM 6.1.2. *The operator*

$$\tilde V^{2m,2m}_{2,\beta-l+2m}(\mathcal{K}) \times V^{2m+\underline{\tau}-1/2}_{2,\beta-l+2m}(\partial\mathcal{K}) \ni \big(u, \mathcal{D}u|_{\partial\mathcal{K}\setminus\{0\}}\,,\, \underline{u}\big) \\ \to \big(Lu,\, Bu|_{\partial\mathcal{K}\setminus\{0\}} + C\underline{u}\big) \in V^0_{2,\beta-l+2m}(\mathcal{K}) \times V^{2m-\underline{\mu}-1/2}_{2,\beta-l+2m}(\partial\mathcal{K}) \tag{6.1.45}$$

can be uniquely extended to a continuous operator

$$\mathcal{A}\,:\, \tilde V^{l,2m}_{2,\beta}(\mathcal{K}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K}) \to \tilde V^{l-2m,0}_{2,\beta}(\mathcal{K}) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K}) \tag{6.1.46}$$

with $l < 2m$. This extension has the form

$$(u, \underline{\phi}, \underline{u}) \to \big(L(u,\underline{\phi}),\, Q\underline{\phi} + C\underline{u}\big)\,, \tag{6.1.47}$$

where L is the operator (6.1.42) described in Lemma 6.1.9 and Q is given by (6.1.16).

REMARK 6.1.2. In the case $l \le 0$ the functional $f = L(u,\underline{\phi}) \in V^{2m-l}_{2,-\beta}(\mathcal{K})^*$ and the vector-function $\underline{g} = Q\underline{\phi} + C\underline{u} \in V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$ in (6.1.47) satisfy the equality

$$(f,v)_{\mathcal{K}} + (\underline{g},\underline{v})_{\partial\mathcal{K}} = (u, L^+v)_{\mathcal{K}} + (\underline{\phi}, Pv + Q^+\underline{v})_{\partial\mathcal{K}} + (\underline{u}, C^+\underline{v})_{\partial\mathcal{K}} \tag{6.1.48}$$

for all $v \in V^{2m-l}_{2,-\beta}(\mathcal{K})$ and $\underline{v} \in V^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{K})$. This means that the operator (6.1.46) is adjoint to the operator

$$V^{2m-l}_{2,-\beta}(\mathcal{K}) \times V^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{K}) \ni (v,\underline{v}) \to \big(L^+v,\, Pv|_{\partial\mathcal{K}\setminus\{0\}} + Q^+\underline{v},\, C^+\underline{v}\big) \\ \in V^{-l}_{2,-\beta}(\mathcal{K}) \times \Big(\prod_{j=1}^{2m} V^{-l+j+1/2}_{2,-\beta}(\partial\mathcal{K})\Big) \times V^{-l-\underline{\tau}+1/2}_{2,-\beta}(\partial\mathcal{K})$$

of the formally adjoint problem (6.1.32), (6.1.33) if $l \le 0$.

Analogously to Theorem 6.1.1, we obtain the following statement.

THEOREM 6.1.3. *If the boundary value problem (6.1.11), (6.1.12) is elliptic and no eigenvalues of $\mathfrak{A}(\lambda)$ are situated on the line $\operatorname{Re}\lambda = -\beta + l - n/2$, then the operator (6.1.46) is an isomorphism for arbitrary integer l.*

Proof: The equation

$$\mathcal{A}\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$$

is equivalent to

$$\mathfrak{A}(\partial_t)\,(u, \underline{\phi}, e^{-\tau_1 t}u_1, \dots, e^{-\tau_J t}u_J) = (e^{2mt}f, e^{\mu_1 t}g_1, \dots, e^{\mu_{m+J} t}g_{m+J})\,,$$

where $\mathfrak{A}(\partial_t)$ is the operator of problem (6.1.23), (6.1.24) in the cylinder $\mathcal{C}$ which arises from $\mathcal{K}$ via the coordinate change $x \to (\omega, t)$. By Theorem 5.3.2, the operator $\mathfrak{A}(\partial_t)$ is an isomorphism

$$\tilde{\mathcal{W}}^{l,2m}_{2,\beta-l+n/2}(\mathcal{C}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta-l+n/2}(\partial\mathcal{C}) \to \tilde{\mathcal{W}}^{l-2m,0}_{2,\beta-l+n/2}(\mathcal{C}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta-l+n/2}(\partial\mathcal{C})$$

for arbitrary integer l. From this we conclude that (6.1.46) is an isomorphism for arbitrary integer l. ■

The following regularity assertion for the operator $\mathcal{A}$ can be easily deduced from Lemma 5.3.5.

LEMMA 6.1.10. *Suppose the boundary value problem (6.1.11), (6.1.12) is elliptic. If* $(u, \underline{\phi}, \underline{u})$ *is an element of the space*

$$\tilde{V}^{l,2m}_{2,\beta}(\mathcal{K}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K}), \tag{6.1.49}$$

and

$$(f, \underline{g}) = \mathcal{A}(u, \underline{\phi}, \underline{u}) \in \tilde{V}^{l-2m+1,0}_{2,\beta+1}(\mathcal{K}) \times V^{l-\underline{\mu}+1/2}_{2,\beta+1}(\partial\mathcal{K}),$$

then $(u, \underline{\phi}, \underline{u}) \in \tilde{V}^{l+1,2m}_{2,\beta+1}(\mathcal{K}) \times V^{l+\underline{\tau}+1/2}_{2,\beta+1}(\partial\mathcal{K})$. *Furthermore,*

$$\|(u, \underline{\phi}, \underline{u})\|_{l+1,\beta+1} \le c \Big(\|f\|_{\tilde{V}^{l-2m+1,0}_{2,\beta+1}(\mathcal{K})} + \|\underline{g}\|_{V^{l-\underline{\mu}+1/2}_{2,\beta+1}(\partial\mathcal{K})} + \|(u, \underline{\phi}, \underline{u})\|_{l,\beta} \Big), \tag{6.1.50}$$

where $\|\cdot\|_{l,\beta}$ *denotes the norm in (6.1.49).*

REMARK 6.1.3. Analogously to Lemma 5.3.4, there exists a constant c such that

$$\sum_{j=1}^{2m} \|\phi_j\|_{V^{l-j+1/2}_{2,\beta}(\partial\mathcal{K})} \le c \Big(\|u\|_{\tilde{V}^{l,0}_{2,\beta}(\mathcal{K})} + \|Lu\|_{\tilde{V}^{l-2m,0}_{2,\beta}(\mathcal{K})} \Big) \tag{6.1.51}$$

for each $(u, \underline{\phi}) \in \tilde{V}^{l,2m}_{2,\beta}(\mathcal{K})$. Therefore, the term $\|(u, \underline{\phi}, \underline{u})\|_{l,\beta}$ on the right of (6.1.50) can be replaced by the sum

$$\|u\|_{\tilde{V}^{l,0}_{2,\beta}(\mathcal{K})} + \|\underline{u}\|_{V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} .$$

6.1.4. Asymptotics of the solution. Let λ_μ be the eigenvalues of the operator pencil $\mathfrak{A}(\lambda)$ with the geometrical multiplicities I_μ and the partial multiplicities $\kappa_{\mu,1}, \dots, \kappa_{\mu,I_\mu}$. Furthermore, let

$$\big\{(\varphi^{(\mu)}_{j,s}, \underline{\varphi}^{(\mu)}_{j,s})\big\} = \big\{(\varphi^{(\mu)}_{j,s}, \varphi^{(\mu)}_{1;j,s}, \dots, \varphi^{(\mu)}_{J;j,s})\big\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

be canonical systems of Jordan chains of $\mathfrak{A}(\lambda)$ corresponding to λ_μ. As noted in Section 5.2, the functions $\varphi^{(\mu)}_{j,s}$ and the vector-functions $\underline{\varphi}^{(\mu)}_{j,s}$ are infinitely differentiable.

We introduce the functions

$$u_{\mu,j,s} = r^{\lambda_\mu} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma\, \varphi^{(\mu)}_{j,s-\sigma}(\omega), \tag{6.1.52}$$

and the vector-functions $\underline{u}_{\mu,j,s}$ with the components

$$u_{q;\mu,j,s} = r^{\lambda_\mu+\tau_q} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma\, \varphi^{(\mu)}_{q;j,s-\sigma} \qquad q = 1, \dots, J.$$

From Lemma 5.1.3 it follows that $\big(u_{\mu,j,s}, r^{-\tau_1} u_{1;\mu,j,s}, \dots, r^{-\tau_J} u_{J;\mu,j,s}\big)$ are solutions of the equation

$$\mathfrak{A}(r\partial_r)(u, \underline{u}) = 0.$$

Consequently, the pairs $(u_{\mu,j,s}, \underline{u}_{\mu,j,s})$ are solutions of the homogeneous boundary value problem (6.1.11), (6.1.12).

By means of Theorem 5.4.1, we obtain the following result.

THEOREM 6.1.4. *Suppose that the boundary value problem (6.1.11), (6.1.12) is elliptic, the lines* $\mathrm{Re}\,\lambda = -\beta_i + l_i - n/2$ $(i = 1, 2)$ *do not contain eigenvalues of the pencil* $\mathfrak{A}(\lambda)$, *and the strip* $-\beta_1 + l_1 - n/2 < \mathrm{Re}\,\lambda < -\beta_2 + l_2 - n/2$ *contains the eigenvalues* $\lambda_1, \dots, \lambda_N$. *If*

$$(u, \underline{u}) \in V^{l_1}_{2,\beta_1}(\mathcal{K}) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K}) \tag{6.1.53}$$

is a solution of problem (6.1.11), (6.1.12) and

$$(f, \underline{g}) \in \bigcap_{i=1}^{2} \Big(V^{l_i-2m}_{2,\beta_i}(\mathcal{K}) \times V^{l_i-\underline{\mu}-1/2}_{2,\beta_i}(\partial\mathcal{K}) \Big),$$

where $l_1 \geq 2m$, $l_2 \geq 2m$, $\beta_1 - l_1 > \beta_2 - l_2$, *then*

$$(u, \underline{u}) = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\, (u_{\mu,j,s}\, , \underline{u}_{\mu,j,s}) + (w, \underline{w}), \tag{6.1.54}$$

where $c_{\mu,j,s}$ *are constants and* $(w, \underline{w}) \in V^{l_2}_{2,\beta_2}(\mathcal{K}) \times V^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{K})$.

Proof: We consider u and $\underline{u}$ as functions of the variables ω, t, where $t = \log r$ and ω are coordinates on the unit sphere. Then $(u, e^{-\tau_1 t}u_1, \dots, e^{-\tau_J t}u_J)$ is a solution of problem (6.1.23), (6.1.24), i.e.,

$$\mathfrak{A}(\partial_t)\, (u, e^{-\tau_1 t}u_1, \dots, e^{-\tau_J t}u_J) = (e^{2mt}f, e^{\mu_1 t}g_1, \dots, e^{\mu_{m+J} t}g_{m+J})\,. \tag{6.1.55}$$

Under our assumptions on $(u, \underline{u})$ and $(f, \underline{g})$, the function u and the vector-function $(e^{-\tau_1 t}u_1, \dots, e^{-\tau_J t}u_J)$ are elements of $\mathcal{W}^{l_1}_{2,\beta_1-l_1+n/2}(\mathcal{C})$ and $\mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1-l_1+n/2}(\partial\mathcal{C})$, respectively, while the right-hand side of (6.1.55) belongs to the space

$$\bigcap_{i=1}^{2} \Big(\mathcal{W}^{l_i-2m}_{2,\beta_i-l_i+n/2}(\mathcal{C}) \times \mathcal{W}^{l_i-\underline{\mu}-1/2}_{2,\beta_i-l_i+n/2}(\partial\mathcal{C}) \Big).$$

Applying Theorem 5.4.1, we get the representation (6.1.54). ∎

By the same arguments as in the proof of Theorem 5.4.2, we obtain the following generalization of Theorem 6.1.4.

THEOREM 6.1.5. *We assume that the model problem (6.1.11), (6.1.12) is elliptic and there are no eigenvalues of the operator pencil* $\mathfrak{A}(\lambda)$ *on the lines* $\mathrm{Re}\,\lambda = -\beta_i + l_i - n/2$ $(i = 1, 2)$, *where* l_1, l_2 *are arbitrary integers and* $\beta_1 - l_1 > \beta_2 - l_2$. *If*

$$(u, \underline{\phi}, \underline{u}) \in \tilde{V}^{l_1,2m}_{2,\beta_1}(\mathcal{K}) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K}),$$

is a solution of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$, *where*

$$(f, \underline{g}) \in \bigcap_{i=1}^{2} \Big(\tilde{V}^{l_i-2m,0}_{2,\beta_i}(\mathcal{K}) \times V^{l_i-\underline{\mu}-1/2}_{2,\beta_i}(\partial\mathcal{K}) \Big),$$

then there is the representation

$$(u, \underline{\phi}, \underline{u}) = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\, (u_{\mu,j,s}\, , \mathcal{D}u_{\mu,j,s}\, , \underline{u}_{\mu,j,s}) + (w, \underline{\psi}, \underline{w})\,,$$

where $(w, \underline{\psi}, \underline{w}) \in \tilde{V}^{l_2,2m}_{2,\beta_2}(\mathcal{K}) \times V^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{K})$.

6.1.5. Formulas for the coefficients in the asymptotics. Using the coefficients formula for the model problem in the cylinder (see Theorem 5.4.3), we can derive an analogous formula for the coefficients in the asymptotics of the solution of the model problem in the cone. Let λ_μ be the eigenvalues of $\mathfrak{A}(\lambda)$ and let $\{(\varphi^{(\mu)}_{j,s}, \underline{\varphi}^{(\mu)}_{j,s})\}$ be the canonical systems of Jordan chains introduced in the previous subsection. Furthermore, let

$$\{(\psi^{(\mu)}_{j,s}, \underline{\psi}^{(\mu)}_{j,s})\}_{j=1,\dots,I_\mu,\, s=0,\dots,\kappa_{\mu,j}-1}$$

be canonical systems of Jordan chains of the pencil $\mathfrak{A}^+(\lambda)$ corresponding to the eigenvalues $\overline{\lambda}_\mu$ such that the biorthonormality condition (5.4.17) is satisfied. Here $\mathfrak{A}^+(\lambda)$ denotes the operator of problem (6.1.36)–(6.1.38). Since $\mathfrak{A}^+(\lambda)$ is the operator of an elliptic boundary value problem in Ω for every fixed λ, both the functions $\psi^{(\mu)}_{j,s}$ and the vector-functions $\underline{\psi}^{(\mu)}_{j,s} = (\psi^{(\mu)}_{1;j,s}, \dots, \psi^{(\mu)}_{m+J,j,s})$ are infinitely differentiable (cf. Lemma 5.4.2).

We introduce the functions

$$v_{\mu,j,s} = -r^{-\overline{\lambda}_\mu+2m-n} \sum_{\sigma=0}^{s} \frac{1}{\sigma!}(-\log r)^\sigma\, \psi^{(\mu)}_{j,s-\sigma}(\omega) \tag{6.1.56}$$

and the vector-functions $\underline{v}_{\mu,j,s}$ with the components

(6.1.57)

$$v_{k;\mu,j,s} = -r^{-\overline{\lambda}_\mu+\mu_k+1-n} \sum_{\sigma=0}^{s} \frac{1}{\sigma!}(-\log r)^\sigma\, \psi^{(\mu)}_{k;j,s-\sigma}(\omega), \quad k = 1,\dots,m+J.$$

LEMMA 6.1.11. *The pairs $(v_{\mu,j,s}, \underline{v}_{\mu,j,s})$ defined by (6.1.56), (6.1.57) are solutions of the homogeneous formally adjoint problem (6.1.32), (6.1.33).*

Proof: We set $w_{\mu,j,s} = r^{n-2m}\, v_{\mu,j,s}$ and denote by $\underline{w}_{\mu,j,s}$ the vector-function with the components $w_{k;\mu,j,s} = r^{n-1-\mu_k}\, v_{k;\mu,j,s}$, $k = 1,\dots,m+J$. From the assumption that $(\psi^{(\mu)}_{j,0}, \underline{\psi}^{(\mu)}_{j,0}), \dots, (\psi^{(\mu)}_{j,s}, \underline{\psi}^{(\mu)}_{j,s})$ is a Jordan chain of $\mathfrak{A}^+(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$ it follows that $(w_{\mu,j,s}, \underline{w}_{\mu,j,s})$ is a solution of the equation

$$\mathfrak{A}^+(-r\partial_r)\,(w, \underline{w}) = 0$$

(see Lemma 5.1.3). This means that $w_{\mu,j,s}$, $\underline{w}_{\mu,j,s}$ satisfy the equations

$$\mathcal{L}^+(r\partial_r + 2m - n)\, w_{\mu,j,s} = 0 \qquad \text{in } \Omega \times \mathbb{R}_+,$$

$$\mathcal{P}_j(r\partial_r + 2m - n)\, w_{\mu,j,s} + \sum_{k=1}^{m+J} \mathcal{Q}^+_{k,j}(r\partial_r + \mu_k + 1 - n)\, w_{k;\mu,j,s} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+,$$

$$\sum_{k=1}^{m+J} \mathcal{C}^+_{k,j}(r\partial_r + \mu_k + 1 - n)\, w_{k;\mu,j,s} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+$$

(for the sake of brevity, we have omitted the arguments ω, ∂_ω in the operators $\mathcal{L}$, $\mathcal{P}_j$, $\mathcal{Q}^+_{k,j}$, and $\mathcal{C}^+_{k,j}$). Using the representations (6.1.27)–(6.1.30) for the operators L^+, P, Q^+, C^+ and the equality

$$(r\partial_r + \alpha)^j r^{-\alpha} u = r^{-\alpha}\,(r\partial_r)^j u\,,$$

we obtain $\mathcal{A}^+(v_{\mu,j,s}, \underline{v}_{\mu,j,s}) = 0$. ■

THEOREM 6.1.6. *Let the conditions of Theorem 6.1.4 be satisfied. Then the coefficients in (6.1.54) are given by the formula*

$$c_{\mu,j,s} = (f,\, v_{\mu,j,\kappa_{\mu,j}-1-s})_{\mathcal{K}} + (\underline{g},\, \underline{v}_{\mu,j,\kappa_{\mu,j}-1-s})_{\partial\mathcal{K}}\,. \tag{6.1.58}$$

Proof: If $(u, \underline{u})$ is a solution of problem (6.1.11), (6.1.12), then

$$(u, r^{-\tau_1}u_1, \dots, r^{-\tau_J}u_J)$$

is a solution of the equation (6.1.55) in the coordinates ω, t. Applying Theorem 5.4.1, we get the representation (6.1.54). Here, by Theorem 5.4.3, the coefficients $c_{\mu,j,s}$ satisfy the equality

$$c_{\mu,j,s} = (e^{2mt} f,\, w_{\mu,j,\kappa_{\mu,j}-1-s})_{\mathcal{C}} + \sum_{k=1}^{m+J} (e^{\mu_k t} g_k,\, \underline{w}_{k;\mu,j,\kappa_{\mu,j}-1-s})_{\partial\mathcal{C}}\,,$$

where $w_{\mu,j,s} = e^{(n-2m)t} v_{\mu,j,s}$ and $w_{k;\mu,j,s} = e^{(n-1-\mu_k)t} v_{k;\mu,j,s}$. (Due to the different assumptions in this theorem and in Theorem 5.4.3, the functions $w_{\mu,j,s}$, $w_{k;\mu,j,s}$ here have another sign as the functions $v_{\mu,j,s}$, $v_{k;\mu,j,s}$ in Theorem 5.4.3. In Theorem 5.4.3 it was assumed that $\beta_1 < \beta_2$, while now $\beta_1 - l_1$ is greater than $\beta_2 - l_2$.)

The change of coordinates $(\omega, t) \to x$ yields (6.1.58). ∎

REMARK 6.1.4. Formula (6.1.58) is also valid for the coefficients $c_{\mu,j,s}$ of the solution $(u, \underline{\phi}, \underline{u})$ in Theorem 6.1.5.

6.1.6. A formula for the coefficients in terms of the classical Green formula. Now we consider the boundary value problem

$$Lu = f \quad \text{in } \mathcal{K}, \tag{6.1.59}$$

$$B_k u = g_k \quad \text{on } \partial\mathcal{K}\backslash\{0\},\ k = 1, \dots, m, \tag{6.1.60}$$

where L is an elliptic model operator of order $2m$ and B_k are model operators of order $\mu_k < 2m$ which form a normal system on $\partial\mathcal{K}\backslash\{0\}$. This system can be completed by model operators B_k ($k = m+1, \dots, 2m$) of order $\mu_k < 2m$ to a Dirichlet system of order $2m$ on $\partial\mathcal{K}\backslash\{0\}$. Then the *classical Green formula*

$$\int_{\mathcal{K}} Lu \cdot \overline{v}\, dx + \sum_{k=1}^{m} \int_{\partial\mathcal{K}} B_k u \cdot \overline{B'_{k+m} v}\, d\sigma = \int_{\mathcal{K}} u \cdot \overline{L^+ v}\, dx + \sum_{k=1}^{m} \int_{\partial\mathcal{K}} B_{k+m} u \cdot \overline{B'_k v}\, d\sigma$$

is valid for all u, $v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$. Here B'_k are model operators of order $\mu'_k = 2m - 1 - \mu_{k+m}$ if $k \le m$, and of order $\mu'_k = 2m - 1 - \mu_{k-m}$, if $k \ge m+1$, i.e.,

$$B'_k(x, \partial_x) = r^{-\mu'_k}\, \mathcal{B}'_k(\omega, \partial_\omega, r\partial_r).$$

Let the operators $\mathcal{L}$, $\mathcal{B}_k$, and $\mathcal{L}^+$ be defined by (6.1.13), (6.1.14), and (6.1.27). Then analogously to Lemma 6.1.6, the Green formula

$$\int_{\Omega} \mathcal{L}(\lambda)\tilde{u} \cdot \overline{\tilde{v}}\, d\omega + \sum_{k=1}^{m} \int_{\partial\Omega} \mathcal{B}_k(\lambda)\tilde{u} \cdot \overline{\mathcal{B}'_{k+m}(-\overline{\lambda} + 2m - n)\tilde{v}}\, d\varsigma$$

$$= \int_{\Omega} \tilde{u} \cdot \overline{\mathcal{L}^+(-\overline{\lambda} + 2m - n)\tilde{v}}\, d\omega + \sum_{k=1}^{m} \int_{\partial\Omega} \mathcal{B}_{k+m}(\lambda)u \cdot \overline{\mathcal{B}'_k(-\overline{\lambda} + 2m - n)\tilde{v}}\, d\varsigma$$

holds for all $\tilde{u}$, $\tilde{v} \in C^\infty(\overline{\Omega})$. We denote the operator of the boundary value problem

$$\mathcal{L}^+(\omega, \partial_\omega, -\lambda + 2m - n)\, v = f \quad \text{in } \Omega,$$
$$\mathcal{B}'_k(\omega, \partial_\omega, -\lambda + 2m - n)\, v = g_k \quad \text{on } \partial\Omega, \quad k = 1, \dots, m,$$

by $\mathfrak{A}^+_c(\lambda)$. Furthermore, let $\mathfrak{A}^+(\lambda)$ be the operator of problem (6.1.36), (6.1.37). According to Lemma 5.4.4, there are the following relations between the eigenvalues and eigenvectors of the operator pencils $\mathfrak{A}^+(\lambda)$ and $\mathfrak{A}^+_c(\lambda)$.

LEMMA 6.1.12. *The pencils $\mathfrak{A}^+(\lambda)$ and $\mathfrak{A}^+_c(\lambda)$ have the same eigenvalues. Furthermore, the elements $(\psi_0, \underline{\psi}_0), \dots, (\psi_s, \underline{\psi}_s)$ form a Jordan chain of $\mathfrak{A}^+(\lambda)$ corresponding to the eigenvalue λ_μ if and only if $\psi_0, \dots, \psi_s$ is a Jordan chain of $\mathfrak{A}^+_c(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$ and*

$$\underline{\psi}_\sigma = \sum_{j=0}^{\sigma} \frac{1}{j!} \mathcal{T}^{(j)}(-\overline{\lambda}_\mu + 2m - n)\, \psi_{\sigma-j} \Big|_{\partial\Omega},$$

where $\mathcal{T}(\lambda)$ is the vector of the operators $\mathcal{T}_k(\lambda) = \mathcal{B}'_{k+m}(\omega, \partial_\omega, \lambda)$, $k = 1, \dots, m$.

Let

$$\{\varphi^{(\mu)}_{j,s}\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1} \quad \text{and} \quad \{(\psi^{(\mu)}_{j,s}, \underline{\psi}^{(\mu)}_{j,s})\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

be the canonical systems of Jordan chains of the pencils $\mathfrak{A}(\lambda)$ and $\mathfrak{A}^+(\lambda)$ introduced in the previous subsections. By Lemma 6.1.12, $\{\psi^{(\mu)}_{j,s}\}$ is a canonical systems of Jordan chains of $\mathfrak{A}^+_c(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$ and the biorthonormality condition (5.4.17) takes the form

$$\sum_{p=0}^{\sigma} \sum_{q=p+1}^{p+s+1} \frac{1}{q!} \Big(\big(\mathcal{L}^{(q)}(\lambda_\mu)\, \varphi_{j,p+s+1-q}\,,\, \psi_{l,\sigma-p}\big)_\Omega$$
$$+ \sum_{k=1}^{m} \Big(\mathcal{B}^{(q)}_k(\lambda_\mu)\, \varphi^{(\mu)}_{j,p+s+1-q}\,,\, \sum_{\nu=0}^{\sigma-p} \frac{1}{\nu!} (\mathcal{B}'_{k+m})^{(\nu)}(-\overline{\lambda}_\mu + 2m - n)\, \psi^{(\mu)}_{l,\sigma-p-\nu}\Big)_{\partial\Omega}\Big)$$
$$= \delta_{j,l}\, \delta_{s,\kappa_{\mu,j}-1-\sigma} \tag{6.1.61}$$

for $j, l = 1, \dots, I_\mu$, $s = 0, \dots, \kappa_{\mu,j} - 1$, $\sigma = 0, \dots, \kappa_{\mu,l} - 1$.

Let $u_{\mu,j,s}$, $v_{\mu,j,s}$ be the functions (6.1.52) and (6.1.56), respectively. From Lemma 5.1.3 it follows that the functions $u_{\mu,j,s}$ are solutions of the homogeneous boundary value problem (6.1.59), (6.1.60), while $v_{\mu,j,s}$ are solutions of the formally adjoint problem

$$L^+ u = 0 \ \text{ in } \mathcal{K}, \qquad B' u = 0 \ \text{ on } \partial\mathcal{K}\backslash\{0\}.$$

Analogously to Theorem 5.4.4, the following assertion holds.

THEOREM 6.1.7. *Suppose that the boundary value problem (6.1.59), (6.1.60) is elliptic and $B_1, \dots, B_m$ form a normal system of model operators on $\partial\mathcal{K}\backslash\{0\}$. Furthermore, we assume that the lines $\operatorname{Re}\lambda = -\beta_i + l_i - n/2$ $(i = 1, 2)$ do not contain eigenvalues of $\mathfrak{A}(\lambda)$. If $u \in V^{l_1}_{2,\beta_1}(\mathcal{K})$ is a solution of problem (6.1.59), (6.1.60), where*

$$f \in V^{l_1-2m}_{2,\beta_1}(\mathcal{K}) \cap V^{l_2-2m}_{2,\beta_2}(\mathcal{K}), \qquad \underline{g} \in V^{l_1-\underline{\mu}-1/2}_{2,\beta_1}(\partial\mathcal{K}) \cap V^{l_2-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{K}),$$

$l_1 \ge 2m$, $l_2 \ge 2m$, then there is the representation

$$u = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\, u_{\mu,j,s} + w,$$

where $w \in V^{l_2}_{2,\beta_2}(\mathcal{K})$. The coefficients $c_{\mu,j,s}$ are given by the equality

$$c_{\mu,j,s} = (f,\, v_{\mu,j,\kappa_{\mu,j}-1-s})_{\mathcal{K}} + \sum_{k=1}^{m} (g_k,\, B'_{k+m} v_{\mu,j,\kappa_{\mu,j}-1-s})_{\partial\mathcal{K}}\,.$$

6.1.7. Solutions of the model problem for special right-hand sides. Now we consider the model problem (6.1.11), (6.1.12) for functions f, g_k of the form

$$(6.1.62) \qquad f = r^{\lambda_0-2m} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma f^{(\sigma)}(\omega),$$

$$(6.1.63) \qquad g_k = r^{\lambda_0-\mu_k} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma g_k^{(\sigma)}(\omega), \quad k = 1, \dots, m+J,$$

where $f^{(\sigma)} \in W_2^{l-2m}(\Omega)$, $g_k^{(\sigma)} \in W_2^{l-\mu_k-1/2}(\partial\Omega)$.

LEMMA 6.1.13. *Suppose that the model problem (6.1.11), (6.1.12) is elliptic in $\overline{\mathcal{K}}\backslash\{0\}$ and f, g_k are functions of the form (6.1.62) and (6.1.63), respectively. Then there exists a solution $(u, \underline{u}) = (u, u_1, \dots, u_J)$ of this problem which has the form*

$$u = r^{\lambda_0} \sum_{\sigma=0}^{s+\kappa_1} \frac{1}{\sigma!} (\log r)^\sigma u^{(\sigma)}(\omega),$$

$$u_j = r^{\lambda_0+\tau_j} \sum_{\sigma=0}^{s+\kappa_1} \frac{1}{\sigma!} (\log r)^\sigma u_j^{(\sigma)}(\omega),$$

where $u^{(\sigma)} \in W_2^l(\Omega)$, $u_j^{(\sigma)} \in W_2^{l+\tau_j-1/2}(\partial\Omega)$. Here $\kappa_1 = 0$ if λ_0 is a regular point of the pencil $\mathfrak{A}(\lambda)$ and κ_1 is the maximal partial multiplicity of λ_0 if λ_0 is an eigenvalue.

Proof: By Lemma 5.1.6, there exists a solution

$$\big(w(\omega,t), \underline{w}(\omega,t)\big) = e^{\lambda_0 t} \sum_{\sigma=0}^{s+\kappa_1} \frac{1}{\sigma!} t^\sigma \left((u^{(\sigma)}(\omega), \underline{u}^{(\sigma)}(\omega)\right)$$

of the equation

$$\mathfrak{A}(\partial_t)\,(w, \underline{w}) = e^{\lambda_0 t} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} t^\sigma \left((f^{(\sigma)}(\omega), \underline{g}^{(\sigma)}(\omega)\right).$$

Applying the coordinate transformations $t = \log r$ and $(\omega, r) \to x$, we get the assertion of the lemma. ∎

6.1.8. Examples.

The Dirichlet problem for the Laplace operator in a plane angle. Let

$$\mathcal{K} = \{(x_1, x_2) \in \mathbb{R}^2 : r > 0,\ 0 < \theta < \alpha\}$$

be a plane angle with the sides $\gamma_- : \theta = 0$, $\gamma_+ : \theta = \alpha$. Here r, θ are the polar coordinates of the point $x = (x_1, x_2)$ and α is a positive number less than 2π. We consider the problem

$$-\Delta u = f \quad \text{in } \mathcal{K}, \qquad u = g_\pm \quad \text{on } \gamma_\pm. \tag{6.1.64}$$

In polar coordinates this problem has the form

$$-(r\partial_r)^2 u(\theta, r) - \partial_\theta^2 u(\theta, r) = F \overset{def}{=} r^2 f \quad \text{for } 0 < \theta < \alpha,$$
$$u(0, r) = g_-(r),\ u(\alpha, r) = g_+(r).$$

Applying the Mellin transformation $r \to \lambda$, we get the boundary value problem

$$-\tilde{u}''(\theta) - \lambda^2 \tilde{u}(\theta) = \tilde{F} \quad \text{for } 0 < \theta < \alpha, \qquad \tilde{u}(0) = \tilde{g}_-\,,\ \tilde{u}(\alpha) = \tilde{g}_+.$$

The operator of this problem is denoted by $\mathfrak{A}(\lambda)$. It can be easily verified that the spectrum of the pencil $\mathfrak{A}(\lambda)$ consists of the eigenvalue $\lambda_j = j\pi/\alpha$, $j = \pm 1, \pm 2, \dots$. All of them are simple. The corresponding eigenfunctions are

$$\varphi_j(\theta) = \sin \frac{j\pi\theta}{\alpha}\,.$$

Thus, as a consequence of Theorems 6.1.1, 6.1.4, the following statements hold.

1) *The operator of the boundary value problem (6.1.64) realizes an isomorphism*

$$V_{2,\beta}^l(\mathcal{K}) \to V_{2,\beta}^{l-2}(\mathcal{K}) \times V_{2,\beta}^{l-1/2}(\gamma_-) \times V_{2,\beta}^{l-1/2}(\gamma_+)$$

for $l \geq 2$ if and only if $(l - 1 - \beta)\alpha/\pi$ is noninteger or equal to zero.

2) *Let $u \in V_{2,\beta_1}^{l_1}(\mathcal{K})$ be a solution of problem (6.1.64), where $f \in V_{2,\beta_2}^{l_2-2}(\mathcal{K})$, $g_\pm \in V_{2,\beta_2}^{l_2-1/2}(\gamma_\pm)$, $l_1 - \beta_1 < l_2 - \beta_2$. If the numbers $(l_i - 1 - \beta_i)\alpha/\pi$ are noninteger or equal to zero for $i = 1, 2$, then the solution u admits the decomposition*

$$u = \sum_j c_j\, r^{j\pi/\alpha} \sin \frac{j\pi\theta}{\alpha} + w, \tag{6.1.65}$$

where $w \in V_{2,\beta_2}^{l_2}(\mathcal{K})$ and the summation is extended over all integer $j \neq 0$ in the interval $\big((l_1 - 1 - \beta_1)\alpha/\pi\,,\ (l_2 - 1 - \beta_2)\alpha/\pi\big)$.

We derive a formula for the coefficients c_j in (6.1.65). The Dirichlet problem (6.1.64) is formally adjoint to itself with respect to Green's formula

$$\int_{\mathcal{K}} \Delta u \cdot \overline{v}\, dx + \int_{\partial\mathcal{K}} u \cdot \frac{\partial \overline{v}}{\partial \nu}\, d\sigma = \int_{\mathcal{K}} u \cdot \Delta \overline{v}\, dx + \int_{\partial\mathcal{K}} \frac{\partial u}{\partial \nu} \cdot \overline{v}\, d\sigma.$$

Therefore, the coefficients c_j can be calculated by means of solutions

$$v_j = -r^{-j\pi/\alpha}\, \psi_j(\theta)$$

of the homogeneous Dirichlet problem (6.1.64). Here $\psi_j = C_j \sin(j\pi\theta/\alpha)$ are eigenfunctions of the operator pencil $\mathfrak{A}_c^+(\lambda) = \mathfrak{A}(\lambda)$ to the eigenvalue $-j\pi/\alpha$ satisfying the biorthonormality condition (cf. (6.1.61))

$$1 = \int_0^\alpha \mathcal{L}'(\lambda_j) \sin\frac{j\pi\theta}{\alpha} \cdot \overline{\psi_j}\, d\theta = -2C_j\, j\pi/\alpha \int_0^\alpha \sin^2\frac{j\pi\theta}{\alpha}\, d\theta = -j\pi\, C_j\,.$$

Hence the coefficients c_j in (6.1.65) are determined by the formula

$$c_j = (f, v_j)_{\mathcal{K}} - \sum_\pm \big(g_\pm\,,\, \frac{\partial v_j}{\partial \nu_\pm}\big)_{\gamma_\pm}\,,$$

where

$$v_j = \frac{1}{j\pi}\, r^{-j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha}\,.$$

Here $\nu_\pm$ denotes the exterior normal to $\gamma_\pm$.

The Neumann problem for the Laplace operator in a plane angle. We consider the Neumann problem

$$(6.1.66) \qquad -\Delta u = f \quad \text{in } \mathcal{K}, \qquad \frac{\partial u}{\partial \nu_\pm} = g_\pm \quad \text{on } \gamma_\pm\,,$$

where $\mathcal{K}$ is the same plane angle as above. Passing to polar coordinates r, θ, we obtain

$$-(r\partial_r)^2\, u(\theta, r) - \partial_\theta^2 u(\theta, r) = F \stackrel{def}{=} r^2 f \quad \text{for } 0 < \theta < \alpha,\ r > 0,$$
$$(\partial_\theta u)(0, r) = G_-(r) \stackrel{def}{=} -r\, g_-(r), \quad (\partial_\theta u)(\alpha, r) = G_+(r) \stackrel{def}{=} r\, g_+(r) \quad \text{for } r > 0,$$

and the Mellin transformation $r \to \lambda$ leads to the boundary value problem

$$-\tilde{u}''(\theta) - \lambda^2\, \tilde{u}(\theta) = \tilde{F} \quad \text{for } 0 < \theta < \alpha, \qquad \tilde{u}'(0) = \tilde{G}_-\,, \tilde{u}'(\alpha) = \tilde{G}_+\,.$$

We denote the operator of this boundary value problem by $\mathfrak{A}(\lambda)$. The spectrum of the operator pencil $\mathfrak{A}$ consists of the eigenvalues $\lambda_j = j\pi/\alpha$, $j = 0, \pm1, \pm2, \dots$ which are simple for $j \neq 0$. The corresponding eigenfunctions are

$$\varphi_j(\theta) = \cos\frac{j\pi\theta}{\alpha}\,.$$

For $j = 0$ the eigenfunction $\varphi_0 = 1$ has the generalized eigenfunction $\varphi_{0,1} = 1$. Thus the following assertions are true.

1) *The operator of the boundary value problem (6.1.66) realizes an isomorphism*

$$V^l_{2,\beta}(\mathcal{K}) \to V^{l-2}_{2,\beta}(\mathcal{K}) \times V^{l-1/2}_{2,\beta}(\gamma_-) \times V^{l-1/2}_{2,\beta}(\gamma_+)$$

for $l \geq 2$ if and only if $(l - 1 - \beta)\alpha/\pi$ is noninteger.

2) *Let $u \in V^{l_1}_{2,\beta_1}(\mathcal{K})$ be a solution of problem (6.1.66), where $f \in V^{l_2-2}_{2,\beta_2}(\mathcal{K})$, $g_\pm \in V^{l_2-1/2}_{2,\beta_2}(\gamma_\pm)$, $l_1 - \beta_1 < l_2 - \beta_2$. If the numbers $(l_i - 1 - \beta_i)\alpha/\pi$ are noninteger for $i = 1, 2$ and the interval*

$$(6.1.67) \qquad (l_1 - 1 - \beta_1)\alpha/\pi < j < (l_2 - 1 - \beta_2)\alpha/\pi$$

does not contain the number $j = 0$, then the solution u admits the decomposition

$$u = \sum_j c_j \, r^{j\pi/\alpha} \cos \frac{j\pi\theta}{\alpha} + w, \tag{6.1.68}$$

where $w \in V^{l_2}_{2,\beta_2}(\mathcal{K})$ and the summation is extended over all integer j in the interval (6.1.67). If, however, $j = 0$ is contained in the interval (6.1.67), then the term c_0 in (6.1.68) has to be replaced by the expression

$$c_{0,0} + c_{0,1}\,(1 + \log r)\,. \tag{6.1.69}$$

The coefficients c_j in (6.1.68) can be calculated for $j \neq 0$ by means of solutions $v_j = -r^{-j\pi/\alpha}\,\psi_j(\theta)$ of the homogeneous Neumann problem (6.1.66), where $\psi_j = C_j \cos(j\pi\theta/\alpha)$ are eigenfunctions of the operator pencil $\mathfrak{A}(\lambda)$ satisfying the biorthonormality condition

$$1 = \int_0^\alpha \mathcal{L}'(\lambda_j) \cos \frac{j\pi\theta}{\alpha} \cdot \overline{\psi}_j \, d\theta = -2C_j \, j\pi/\alpha \int_0^\alpha \cos^2 \frac{j\pi\theta}{\alpha} \, d\theta = -j\pi C_j\,.$$

Consequently, as for the Dirichlet problem, we obtain the coefficients formula

$$c_j = (f, v_j)_{\mathcal{K}} + \sum_{\pm} (g_\pm \, , v_j)_{\gamma_\pm}$$

for $j \neq 0$, where

$$v_j = \frac{1}{j\pi}\, r^{-j\pi/\alpha} \cos \frac{j\pi\theta}{\alpha}\,.$$

We calculate the coefficients $c_{0,0}$ and $c_{0,1}$ in (6.1.69). For this we have to find a Jordan chain $\psi_{0,0}, \psi_{0,1}$ of the operator pencil $\mathfrak{A}(\lambda)$ corresponding to the eigenvalue $\lambda_0 = 0$ such that the biorthonormality conditions

$$\sum_{p=0}^{\sigma} \int_0^\alpha \sum_{q=p+1}^{p+s+1} \frac{1}{q!} \mathcal{L}^{(q)}(0)\, 1 \cdot \overline{\psi}_{0,\sigma-p} \, d\theta = \delta_{s,1-\sigma} \qquad \text{for } s, \sigma = 0,\, 1$$

are satisfied. Here $\mathcal{L}'(0) = \mathcal{L}'''(0) = 0$, $\mathcal{L}''(0) = 2$. Since $\psi_{0,0}$, $\psi_{0,1}$ are constants, we obtain $\psi_{0,0} = 1/\alpha$ and $\psi_{0,1} = -1/\alpha$. Consequently, we get

$$\begin{aligned} c_{0,0} &= \big(f, v_{0,1}\big)_{\mathcal{K}} + \sum_{\pm} \big(g_\pm \, , v_{0,1}\big)_{\gamma_\pm}\,, \\ c_{0,1} &= \big(f, v_{0,0}\big)_{\mathcal{K}} + \sum_{\pm} \big(g_\pm \, , v_{0,0}\big)_{\gamma_\pm}\,, \end{aligned}$$

where

$$\begin{aligned} v_{0,0} &= -\psi_{0,0} = -\frac{1}{\alpha}\,, \\ v_{0,1} &= -\big(\psi_{0,1} - \psi_{0,0}\,\log r\big) = \frac{1}{\alpha}\,(1 + \log r)\,. \end{aligned}$$

6.2. Elliptic boundary value problems in a bounded domain with conical points

This and the following sections are concerned with boundary value problems in a bounded domain with angular or conical points on the boundary. Here we suppose that the coefficients of the differential operators are smooth outside the conical points and satisfy certain stabilization conditions at the conical points. We call such operators *admissible.* In this section we show that the operator of the boundary value problem is continuous in weighted Sobolev spaces both of positive and negative orders.

6.2.1. Weighted Sobolev spaces in bounded domains with conical points. Let $\mathcal{G}$ be a bounded domain in the Euclidean spaces $\mathbb{R}^n$. We suppose that there exists a finite set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$ of points on the boundary $\partial\mathcal{G}$ such that $\partial\mathcal{G}\backslash\mathcal{S}$ is smooth (belongs to the class C^∞). Moreover, we assume that for each of the points $x^{(\tau)}$, $\tau = 1, \dots, d$, there exists a neighbourhood $\mathcal{U}_\tau$ such that $\mathcal{G} \cap \mathcal{U}_\tau = \mathcal{K}_\tau \cap \mathcal{U}_\tau$, where $\mathcal{K}_\tau$ is an infinite cone with the vertex $x^{(\tau)}$. The cone $\mathcal{K}_\tau$ cuts out a set Ω_τ on the unit sphere with center in $x^{(\tau)}$. Here the boundary $\partial\Omega_\tau$ of Ω_τ is a smooth manifold.

Let ζ_τ, $\tau = 1, \dots, d$, be infinitely differentiable functions in $\overline{\mathcal{G}}$ equal to one in a neighbourhood of $x^{(\tau)}$ and to zero in $\overline{\mathcal{G}}\backslash\mathcal{U}_\tau$. We set $\zeta_0 = 1 - \zeta_1 - \cdots - \zeta_d$ and define the space $V^l_{2,\beta}(\mathcal{G})$, where $\beta = (\beta_1, \dots, \beta_d)$ is a vector of real numbers and l is a nonnegative integer, as the set of all functions on $\mathcal{G}$ such that $\zeta_0 u \in W^l_2(\mathcal{G})$ and $\zeta_\tau u \in V^l_{2,\beta_\tau}(\mathcal{K}_\tau)$, $\tau = 1, \dots, d$. Obviously the space $V^l_{2,\beta}(\mathcal{G})$ does not depend on the choice of the cut-off functions ζ_τ. Equipped with the norm

$$\|u\|_{V^l_{2,\beta}(\mathcal{G})} = \|\zeta_0 u\|_{W^l_2(\mathcal{G})} + \sum_{\tau=1}^{d} \|\zeta_\tau u\|_{V^l_{2,\beta_\tau}(\mathcal{K}_\tau)},$$

the space $V^l_{2,\beta}(\mathcal{G})$ is complete.

Furthermore, we define $V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ for $l \geq 1$ as the space of traces of functions from $V^l_{2,\beta}(\mathcal{G})$ on $\partial\mathcal{G}\backslash\mathcal{S}$. The norm in $V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ is

$$\|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} = \inf\left\{\|v\|_{V^l_{2,\beta}(\mathcal{G})} \,:\, v \in V^l_{2,\beta}(\mathcal{G}),\ v|_{\partial\mathcal{G}\backslash\mathcal{S}} = u\right\}.$$

If $l \leq 0$, then let $V^{-l}_{2,-\beta}(\mathcal{G})^*$, $V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ be the dual spaces of $V^{-l}_{2,-\beta}(\mathcal{G})$ and $V^{-l+1/2}_{2,-\beta}(\mathcal{G})$ equipped with the norms

$$\|u\|_{V^{-l}_{2,-\beta}(\mathcal{G})^*} = \sup\left\{|(u,v)_\mathcal{G}| \,:\, v \in V^{-l}_{2,-\beta}(\mathcal{G}),\ \|v\|_{V^{-l}_{2,-\beta}(\mathcal{G})} = 1\right\}$$

and

$$\|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} = \sup\left\{|(u,v)_{\partial\mathcal{G}}| \,:\, v \in V^{-l+1/2}_{2,-\beta}(\partial\mathcal{G}),\ \|v\|_{V^{-l+1/2}_{2,-\beta}(\partial\mathcal{G})} = 1\right\},$$

respectively.

Analogously to the space $\tilde{V}^{l,k}_{2,\beta}$ in the cone $\mathcal{K}$, we define the space $\tilde{V}^{l,k}_{2,\beta}(\mathcal{G})$ for integer l and nonnegative integer k as the set of all pairs $(u, \underline{\phi})$, where

$$u \in \begin{cases} V^l_{2,\beta}(\mathcal{G}) & \text{if } l \geq 0, \\ V^{-l}_{2,-\beta}(\mathcal{G})^* & \text{if } l < 0, \end{cases}$$

and

$$\underline{\phi} = (\phi_1, \dots, \phi_k) \in \prod_{j=1}^{k} V_{2,\beta}^{l-j+1/2}(\partial\mathcal{G}), \quad \phi_j = D_\nu^{j-1} u|_{\partial\mathcal{G}\setminus\mathcal{S}} \text{ for } j \le \min(k,l).$$

The norm in $\tilde{V}_{2,\beta}^{l,k}(\mathcal{G})$ is

$$\|(u,\underline{\phi})\|_{\tilde{V}_{2,\beta}^{l,k}(\mathcal{G})} = \|u\|_{\tilde{V}_{2,\beta}^{l,0}(\mathcal{G})} + \sum_{j=1}^{k} \|\phi_j\|_{V_{2,\beta}^{l-j+1/2}(\partial\mathcal{G})},$$

where the norm in $\tilde{V}_{2,\beta}^{l,0}(\mathcal{G})$ coincides with the $V_{2,\beta}^{l}(\mathcal{G})$-norm if $l \ge 0$ and with the $V_{2,-\beta}^{-l}(\mathcal{G})^*$-norm if $l < 0$.

LEMMA 6.2.1. *Let $\beta = (\beta_1, \dots, \beta_d)$, $\gamma = (\gamma_1, \dots, \gamma_d)$ be real d-tuples. If $l \ge l_1 \ge 0$ and $\beta_\tau - l \le \gamma_\tau - l_1$ for $\tau = 1, \dots, d$, then the space $V_{2,\beta}^{l}(\mathcal{G})$ is continuously imbedded into $V_{2,\gamma}^{l_1}(\mathcal{G})$. If, moreover, $l > l_1 \ge 0$, $\beta_\tau - l < \gamma_\tau - l_1$ for $\tau = 1, \dots, d$, then this imbedding is compact.*

Proof: The continuity of the imbedding $V_{2,\beta}^{l}(\mathcal{G}) \subset V_{2,\gamma}^{l_1}(\mathcal{G})$ is obvious. We prove the compactness of the imbedding.

For the sake of simplicity, we assume that the set $\mathcal{S}$ consists of one conical point $x^{(1)}$ only. Then the norm in $V_{2,\beta}^{l}(\mathcal{G})$ is equivalent to

$$\|u\| = \Big(\int_{\mathcal{G}} \sum_{|\alpha|\le l} r^{2(\beta-l+|\alpha|} \, |D_x^\alpha(x)|^2 \, dx \Big)^{1/2},$$

where $r = |x - x^{(1)}|$. Let $\mathfrak{M} \subset \{u \in V_{2,\beta}^{l}(\mathcal{G}) : \|u\| \le c_0\}$ be a bounded subset of $V_{2,\beta}^{l}(\mathcal{G})$. We have to show that $\mathfrak{M}$ is precompact in $V_{2,\gamma}^{l_1}(\mathcal{G})$, i.e., that for arbitrary positive ε there exists a finite ε-net $\{u_1, \dots, u_{N(\varepsilon)}\}$ in $\mathfrak{M}$ such that

$$\min_{1\le j\le N(\varepsilon)} \|u - u_j\|_{V_{2,\gamma}^{l_1}(\mathcal{G})} \le \varepsilon \tag{6.2.1}$$

for all $u \in \mathfrak{M}$.

Let δ be a sufficiently small positive real number and

$$\mathcal{G}_\delta = \{x \in \mathcal{G} : |x - x^{(1)}| > \delta\}.$$

Furthermore, we set $\chi_\delta(r) = \chi(\delta^{-1}r)$, where χ is an infinitely differentiable function on $\mathbb{R}_+$ equal to one in the interval $(0,1)$ and to zero in $(2,+\infty)$. Since the support of $(1-\chi_\delta)$ is contained in $\overline{\mathcal{G}_\delta}$, the set $\mathfrak{M}_\delta = \{(1-\chi_\delta)u : u \in \mathfrak{M}\}$ can be considered as a bounded subset of the space $W_2^l(\mathcal{G}_\delta)$ which is compactly imbedded into $W_2^{l_1}(\mathcal{G}_\delta)$. Consequently, for each $\varepsilon' > 0$ there exists a finite ε'-net $\{(1-\chi_\delta)u_1, \dots, (1-\chi_\delta)u_N\}$ in $\mathfrak{M}_\delta$, where $u_j \in \mathfrak{M}$, such that

$$\min_{1\le j\le N} \|(1-\chi_\delta)(u-u_j)\|_{W_2^{l_1}(\mathcal{G}_\delta)} \le \varepsilon'$$

for every $u \in \mathfrak{M}$. Hence

$$\min_{1\le j\le N} \|(1-\chi_\delta)(u-u_j)\|_{V_{2,\gamma}^{l_1}(\mathcal{G})} \le c_1(\delta)\, \varepsilon',$$

with a constant c_1 independent of ε'. Furthermore, the inequality

$$\|\chi_\delta(u-u_j)\|_{V_{2,\gamma}^{l_1}(\mathcal{G})} \le \delta^{(\gamma-l_1)-(\beta-l)} \, \|\chi_\delta(u-u_j)\|_{V_{2,\beta}^{l}(\mathcal{G})} \le c_2\, c_0\, \delta^{(\gamma-l_1)-(\beta-l)}$$

is valid for every $u \in \mathfrak{M}$, $j = 1, \dots, N$, where the constant c_2 is independent of δ. Choosing δ, ε' such that $c_2\, c_0\, \delta^{(\gamma-l_1)-(\beta-l)} + c_1(\delta)\, \varepsilon' < \varepsilon$, we obtain (6.2.1). Hence $\mathfrak{M}$ is precompact in $V^{l_1}_{2,\gamma}(\mathcal{G})$. ■

Similarly it can be proved that the imbeddings

$$V^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \subset V^{l_1-1/2}_{2,\gamma}(\partial\mathcal{G}), \qquad \tilde{V}^{l,k}_{2,\beta}(\mathcal{G}) \subset \tilde{V}^{l_1,k}_{2,\gamma}(\mathcal{G})$$

are continuous if $l \ge l_1$, $\beta_\tau - l \le \gamma_\tau - l_1$ for $\tau = 1, \dots, d$, and compact if $l > l_1$, $\beta_\tau - l < \gamma_\tau - l_1$. Moreover, the space $\tilde{V}^{l,k}_{2,\beta}(\mathcal{G})$ is dense in $\tilde{V}^{l_1,k}_{2,\gamma}(\mathcal{G})$ for $l \ge l_1$, $\beta_\tau - l \le \gamma_\tau - l_1$, $\tau = 1, \dots, d$.

6.2.2. Formulation of the problem. We consider the boundary value problem

$$L(x, \partial_x)u = f \quad \text{in } \mathcal{G}, \tag{6.2.2}$$

$$B(x, \partial_x)u + C(x, \partial_x)\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}, \tag{6.2.3}$$

where L is a differential operator of order $2m$, B is a vector of differential operators B_k, $\operatorname{ord} B_k \le \mu_k$, and C is a matrix of tangential differential operators $C_{k,j}$ on $\partial\mathcal{G}\backslash\mathcal{S}$, $\operatorname{ord} C_{k,j} \le \mu_k + \tau_j$. The coefficients of L, B_k, and $C_{k,j}$ are assumed to be infinitely differentiable in $\overline{\mathcal{G}}\backslash\mathcal{S}$. Throughout this chapter, we suppose that the orders (with respect to differentiation) of the operators B_k are less than $2m$. Then the vector B admits the representation

$$Bu\,|_{\partial\mathcal{G}\backslash\mathcal{S}} = Q(x, \partial_x) \cdot \mathcal{D}u|_{\partial\mathcal{G}\backslash\mathcal{S}}\,, \tag{6.2.4}$$

where Q is a $(m+J)\times 2m$-matrix of tangential differential operators $Q_{k,j}$, $\operatorname{ord} Q_{k,j} \le \mu_k - j + 1$, $Q_{k,j} \equiv 0$ if $\mu_k - j + 1 < 0$.

Furthermore, we suppose that the following condition analogous to the stabilization condition in Section 5.5 is satisfied for the coefficients in a neighbourhood of every conical point $x^{(\tau)}$.

DEFINITION 6.2.1. The operator

$$P(x, \partial_x) = \sum_{|\alpha| \le k} p_\alpha(x)\, \partial_x^\alpha \tag{6.2.5}$$

is said to be an *admissible operator* of order k in a neighbourhood of the conical point $x^{(\tau)}$ if the coefficients p_α have the form

$$p_\alpha(x) = r^{|\alpha|-k}\, p^{(0)}_\alpha(\omega, r) \tag{6.2.6}$$

in this neighbourhood, where $p^{(0)}_\alpha$ is infinitely differentiable in $\overline{\Omega}_\tau \times \mathbb{R}_+$, continuous in $\overline{\Omega}_\tau \times \overline{\mathbb{R}}_+$, and

$$(r\partial_r)^j\, \partial_\omega^\gamma \left(p^{(0)}_\alpha(\omega, r) - p^{(0)}_\alpha(\omega, 0)\right) \to 0 \quad \text{as } r \to 0 \tag{6.2.7}$$

uniformly with respect to $\omega \in \overline{\Omega}_\tau$. Here ω are coordinates on the unit sphere with center in $x^{(\tau)}$ and $r = |x - x^{(\tau)}|$ denotes the distance to $x^{(\tau)}$.

If $P(x, \partial_x)$ is an admissible operator of order k with the coefficients (6.2.6), then the operator

$$P^{(\tau)}(x, \partial_x) = \sum_{|\alpha| \le k} r^{|\alpha|-k}\, p^{(0)}_\alpha(\omega, 0)\, \partial_x^\alpha$$

is called the *leading part of P at the point $x^{(\tau)}$*.

Analogously, admissible tangential operators on $\partial\mathcal{G}\backslash\mathcal{S}$ and their leading parts at the conical point $x^{(\tau)}$ are defined.

Note that the order (with respect to differentiation) of an admissible operator of order k can be strictly less than k.

The leading part $P^{(\tau)}(x,\partial_x)$ of the admissible operator (6.2.5) is considered as a differential operator in the cone $\mathcal{K}_\tau$. Since $r^{|\alpha|}\partial_x^\alpha$ can be written in the form $\sum_{j\le|\alpha|} p_j(\omega,\partial_\omega)\,(r\partial_r)^j$, the leading part $P^{(\tau)}$ is a model operator in $\mathcal{K}_\tau$.

Clearly, every model operator is admissible. Moreover, e.g., every operator (6.2.5) with coefficients $p_\alpha \in C^\infty(\overline{\mathcal{G}})$ is admissible in a neighbourhood of each conical point. In this case the leading part at the point $x^{(\tau)}$ is equal to

$$P^\circ(x^{(\tau)},\partial_x) = \sum_{|\alpha|=k} a_\alpha(x^{(\tau)})\,\partial_x^\alpha\,.$$

It can be easily verified that every differential operator of order $k \le l$ with smooth coefficients in $\overline{\mathcal{G}}\backslash\mathcal{S}$ which is admissible in a neighbourhood of every conical point $x^{(\tau)}$ continuously maps the space $V^l_{2,\beta}(\mathcal{G})$ into $V^{l-k}_{2,\beta}(\mathcal{G})$. Furthermore, condition (6.2.7) ensures the validity of the following assertion (cf. Lemma 5.5.1).

LEMMA 6.2.2. *Let P be an admissible operator of order k in a neighbourhood of the conical point $x^{(\tau)}$ and let ε be a sufficiently small positive real number. Then there exists a constant c_ε such that*

$$\left\|\left(P(x,\partial_x) - P^{(\tau)}(x,\partial_x)\right)u\right\|_{V^{l-k}_{2,\beta_\tau}(\mathcal{K}_\tau)} \le c_\varepsilon\,\|u\|_{V^l_{2,\beta_\tau}(\mathcal{K}_\tau)}$$

for every $u \in V^l_{2,\beta}(\mathcal{K}_\tau)$ equal to zero outside the ball $|x-x^{(\tau)}|<\varepsilon$ (extending u by zero outside the ball $|x-x^{(\tau)}|<\varepsilon$, the function u can be simultaneously considered as a function in $\mathcal{G}$ and $\mathcal{K}_\tau$). The factor c_ε tends to zero as $\varepsilon \to 0$.

We suppose in the sequel that L, B_k, and $C_{k,j}$ are admissible operators of order $2m$, μ_k, and $\mu_k+\tau_j$, respectively, in a neighbourhood of each conical point $x^{(\tau)}$. Outside $\mathcal{S}$ the coefficients of L, B_k, $C_{k,j}$ are assumed to be smooth. Furthermore, we suppose that the boundary value problem (6.2.2), (6.2.3) is elliptic, i.e., condition (i) in Definition 3.1.2 is satisfied for each $x^{(0)} \in \overline{\mathcal{G}}\backslash\mathcal{S}$ and condition (ii) is satisfied for each $x^{(0)} \in \partial\mathcal{G}\backslash\mathcal{S}$, where the numbers μ_k and τ_j are the same as above. Obviously, the operator $\mathcal{A}$ of the boundary value problem (6.2.2), (6.2.3) continuously maps the space

$$V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{6.2.8}$$

into

$$V^{l-2m}_{2,\beta}(\mathcal{G}) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}). \tag{6.2.9}$$

for $l \ge 2m$. Here $V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$, $V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})$ denote the products of the spaces $V^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{G})$, $j=1,\dots,J$, and $V^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{G})$, $k=1,\dots,m+J$, respectively.

Let $(u,\underline{u})$ be a solution of the boundary value problem (6.2.2), (6.2.3). We suppose that the support of $(u,\underline{u})$ is contained in the neighbourhood $\mathcal{U}_\tau$ of $x^{(\tau)}$. Passing to the coordinates ω, t, where $t=\log r=\log|x-x^{(\tau)}|$ and ω are coordinates on the unit sphere $|x-x^{(\tau)}|=1$, the pair $(u,\underline{u})$ can be considered as a solution of a problem of the form (5.5.1), (5.5.2), i.e.,

$$(6.2.10)\quad \mathcal{L}(\omega,t,\partial_\omega,\partial_t)\,u = e^{2mt}f \quad \text{in } \mathcal{C}_\tau,$$

$$(6.2.11)\quad \mathcal{B}_k(\omega,t,\partial_\omega,\partial_t)\,u + \sum_{j=1}^{J}\mathcal{C}_{k,j}(\omega,t,\partial_\omega,\partial_t+\tau_j)\,e^{-\tau_j t}u_j = e^{\mu_k t}g_k \text{ on } \partial\mathcal{C}_\tau\,,$$

$k = 1,\dots,m+J$, in the cylinder $\mathcal{C}_\tau = \Omega_\tau\times\mathbb{R}$. From our conditions on the coefficients of L, B_k, and $C_{k,j}$ it follows that the coefficients of $\mathcal{L}$, $\mathcal{B}_k$, and $\mathcal{C}_{k,j}$ stabilize at infinity.

6.2.3. Extension of the operator of the boundary value problem. By Theorem 3.1.1, the following Green formula is satisfied for all $u, v \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})$, $\underline{u}\in C_0^\infty(\partial\mathcal{G}\backslash\mathcal{S})^J$, $\underline{v}\in C_0^\infty(\partial\mathcal{G}\backslash\mathcal{S})^{m+J}$:

$$(6.2.12)\quad \int\limits_{\mathcal{G}} Lu\cdot\bar{v}\,dx + \int\limits_{\partial\mathcal{G}\backslash\mathcal{S}}\big(Bu + C\underline{u},\underline{v}\big)_{\mathbb{C}^{m+J}}\,d\sigma$$

$$= \int\limits_{\mathcal{G}} u\cdot\overline{L^+v}\,dx + \int\limits_{\partial\mathcal{G}\backslash\mathcal{S}}\big(\mathcal{D}u, Pv + Q^+\underline{v}\big)_{\mathbb{C}^{2m}}\,d\sigma + \int\limits_{\partial\mathcal{G}\backslash\mathcal{S}}\big(\underline{u}, C^+\underline{v}\big)_{\mathbb{C}^J}\,d\sigma\,,$$

where P is a vector of differential operators of order $2m-j$.

LEMMA 6.2.3. *If L, B_k, and $C_{k,j}$ are admissible operators of order $2m$, μ_k, and $\mu_k+\tau_j$, respectively, then the operators L^+, P, Q^+, and C^+ in (6.2.12) are also admissible. More precisely: L^+ is an admissible operator of order $2m$, P is a vector of admissible operators P_j of order $2m-j$, Q^+ is a matrix of admissible operators $Q^+_{k,j}$ of order μ_k-j+1 ($Q^+_{k,j}\equiv 0$ if $\mu_k-j+1<0$), and C^+ is a matrix of admissible operators $C^+_{k,j}$ of order $\mu_k+\tau_j$.*

Proof: Suppose that the support of $(u,\underline{u})$ is contained in the neighbourhood $\mathcal{U}_\tau$ of $x^{(\tau)}$. Then in (6.2.12) the domain $\mathcal{G}$ can be replaced by $\mathcal{K}_\tau$. Using the coordinates ω, t which were introduced above, we get

$$Lu = e^{-2mt}\,\mathcal{L}(\omega,t,\partial_\omega,\partial_t)u,\qquad B_k u = e^{-\mu_k t}\,\mathcal{B}_k(\omega,t,\partial_\omega,\partial_t)u \quad\text{and}$$
$$C_{k,j}u_j = e^{-(\mu_k+\tau_j)t}\,\mathcal{C}_{k,j}(\omega,t,\partial_\omega,\partial_t)u_j\,,$$

where the coefficients of $\mathcal{L}$, $\mathcal{B}_k$, and $\mathcal{C}_{k,j}$ stabilize for $t\to-\infty$. Hence there exist differential operators $\mathcal{L}^+$, $\mathcal{P}_j$, $\mathcal{C}^+_{k,j}$, and $\mathcal{Q}^+_{k,j}$ such that the Green formula

$$\int\limits_{\mathcal{C}_\tau}\mathcal{L}(t,\partial_t)u\cdot\overline{w}\,d\omega\,d\tau + \sum_{k=1}^{m+J}\int\limits_{\partial\mathcal{C}_\tau}\Big(\mathcal{B}_k(t,\partial_t)u + \sum_{j=1}^{J}\mathcal{C}_{k,j}(t,\partial_t+\tau_j)\psi_j\Big)\,\overline{w}_k\,d\varsigma\,dt$$

$$= \int\limits_{\mathcal{C}_\tau} u\cdot\overline{\mathcal{L}^+(t,\partial_t+2m-n)w}\,d\omega\,dt$$

$$+\sum_{j=1}^{2m}\int\limits_{\partial\mathcal{C}_\tau} D^{j-1}_{\nu(\omega)}u\,\Big(\overline{\mathcal{P}_j(t,\partial_t+2m-n)w} + \sum_{k=1}^{m+J}\overline{\mathcal{Q}^+_{k,j}(t,\partial_t+\mu_k+1-n)w_k}\Big)\,d\varsigma\,dt$$

$$+\sum_{j=1}^{J}\int\limits_{\partial\mathcal{C}_\tau}\psi_j\cdot\sum_{k=1}^{m+J}\overline{\mathcal{C}^+_{k,j}(t,\partial_t+\mu_k+1-n)w_k}\,d\varsigma\,dt$$

is satisfied (here, for the sake of brevity, we have omitted the arguments ω, ∂_ω of the differential operators in this formula). The coefficients of the operators $\mathcal{L}^+$, $\mathcal{P}_j$, $\mathcal{C}^+_{k,j}$, and $\mathcal{Q}^+_{k,j}$ stabilize at infinity (see Section 5.5).

Setting $w = e^{(n-2m)t}v$ and $w_k = e^{(n-1-\mu_k)t}v_k$ for $k = 1, \dots, m+J$, we obtain the Green formula (6.2.12), where

$$L^+(x, \partial_x) = e^{-2mt}\, \mathcal{L}^+(\omega, t, \partial_\omega, \partial_t)$$

and analogous formulas are valid for P_j, $C^+_{k,j}$, $Q^+_{k,j}$. This proves the lemma. ■

By Lemma 6.2.3, the operator $\mathcal{A}^+$ of the formally adjoint problem

$$(6.2.13) \qquad L^+v = f \quad \text{in } \mathcal{G},$$

$$(6.2.14) \qquad Pv + Q^+\underline{v} = \underline{g}\,, \quad C^+\underline{v} = \underline{h} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}$$

continuously maps

$$(6.2.15) \qquad V^{2m-l}_{2,-\beta}(\mathcal{G}) \times \prod_{k=1}^{m+J} V^{-l+\mu_k+1/2}_{2,-\beta}(\partial\mathcal{G})$$

for $l \le 0$ into the space

$$(6.2.16) \qquad V^{-l}_{2,-\beta}(\mathcal{G}) \times \prod_{j=1}^{2m} V^{-l+j-1/2}_{2,-\beta}(\partial\mathcal{G}) \times \prod_{j=1}^{J} V^{-l-\tau_j+1/2}_{2,-\beta}(\partial\mathcal{G})\,.$$

In the same way as it was carried out for a model operator in the cone $\mathcal{K}$ (see Lemma 6.1.9), the admissible operator L can be continuously extended to the space $\tilde{V}^{l,2m}_{2,\beta}(\mathcal{G})$ with $l < 2m$. If $0 < l < 2m$, then we write the differential operator L in the form

$$L(x, \partial_x) = \sum_{|\alpha| \le 2m-l} \partial_x^\alpha\, L_\alpha(x, \partial_x)\,,$$

where L_α are admissible operators of order $\le l$. Analogously to Lemma 6.1.8, the formula

$$(6.2.17) \qquad \int_{\mathcal{G}} Lu \cdot \overline{v}\, dx = \sum_{|\alpha| \le 2m-l} \int_{\mathcal{G}} L_\alpha(x, \partial_x)u \cdot \overline{(-\partial_x)^\alpha v}\, dx + \sum_{j=l+1}^{2m} \int_{\partial\mathcal{G}} D_\nu^{j-1}u \cdot \overline{P_j v}\, d\sigma + \sum_{j=1}^{l} \int_{\partial\mathcal{G}} D_\nu^{j-1}u \cdot \overline{P_{l,j} v}\, d\sigma$$

holds for $u, v \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})$. Here P_j are the same operators as in the Green formula (6.2.12) and $P_{l,j}$ are admissible operators of order $\le 2m-j$ such that the functional

$$v \to \sum_{j=1}^{l} \int_{\partial\mathcal{G}} D_\nu^{j-1}u \cdot \overline{P_{l,j} v}\, d\sigma$$

is continuous on $V^{2m-l}_{2,-\beta}(\mathcal{G})$ for arbitrary $u \in V^l_{2,\beta}(\mathcal{G})$ and arbitrary $\beta \in \mathbb{R}^d$.

Using formulas (6.2.17) and (6.2.12), we obtain the following assertions (cf. Lemma 6.1.9, Theorem 6.1.2).

LEMMA 6.2.4. *The operator*

$$\tilde{V}^{2m,2m}_{2,\beta-(l-2m)\vec{1}}(\mathcal{G}) \ni \left(u, \mathcal{D}u|_{\partial\mathcal{G}\setminus\mathcal{S}}\right) \to Lu \in V^{0}_{2,\beta-(l-2m)\vec{1}}(\mathcal{G})$$

(here $\vec{1}$ denotes the tuple $(1,\dots,1)\in\mathbb{R}^d$) can be uniquely extended to a continuous operator

$$\tilde{V}^{l,2m}_{2,\beta}(\mathcal{G}) \ni (u,\underline{\phi}) \to f \in V^{2m-l}_{2,-\beta}(\mathcal{G})^*, \quad l<2m. \tag{6.2.18}$$

Here the functional $f=L(u,\underline{\phi})$ is given by the equality

$$(f,v)_{\mathcal{G}} = (u, L^+v)_{\mathcal{G}} + \sum_{j=1}^{2m} \left(\phi_j, P_j v\right)_{\partial\mathcal{G}}, \quad v\in V^{2m-l}_{2,-\beta}(\mathcal{G}), \tag{6.2.19}$$

if $l\le 0$ and by the equality

$$(f,v)_{\mathcal{G}} = \sum_{|\alpha|\le 2m-l} \int_{\mathcal{G}} L_\alpha(x,\partial_x)u\cdot\overline{(-\partial_x)^\alpha v}\,dx + \sum_{j=l+1}^{2m}(\phi_j\,,P_jv)_{\partial\mathcal{G}} \tag{6.2.20}$$
$$+\sum_{j=1}^{l}\left(D_\nu^{j-1}u\,,P_{l,j}v\right)_{\partial\mathcal{G}}, \quad v\in V^{2m-l}_{2,-\beta}(\mathcal{G}),$$

if $0<l<2m$.

THEOREM 6.2.1. *The operator*

$$\tilde{V}^{2m,2m}_{2,\beta-(l-2m)\vec{1}}(\mathcal{G}) \times V^{2m+\underline{\tau}-1/2}_{2,\beta-(l-2m)\vec{1}}(\partial\mathcal{G}) \ni \left(u,\mathcal{D}u|_{\partial\mathcal{G}\setminus\mathcal{S}}\,,\underline{u}\right) \tag{6.2.21}$$
$$\to \left(Lu,\ Bu|_{\partial\mathcal{G}\setminus\mathcal{S}}+C\underline{u}\right) \in V^{0}_{2,\beta-(l-2m)\vec{1}}(\mathcal{G}) \times V^{2m-\underline{\mu}-1/2}_{2,\beta-(l-2m)\vec{1}}(\partial\mathcal{G})$$

can be uniquely extended to a continuous operator

$$\mathcal{A}\,:\tilde{V}^{l,2m}_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \to \tilde{V}^{l-2m,0}_{2,\beta}(\mathcal{G}) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{6.2.22}$$

with $l<2m$. This extension has the form

$$(u,\underline{\phi},\underline{u}) \to \left(L(u,\underline{\phi}),\ Q\underline{\phi}+C\underline{u}\right)$$

where L is the operator (6.2.18) and Q is given by (6.2.4).

In particular, in the case $l\le 0$ the functional $f=L(u,\underline{\phi})\in V^{2m-l}_{2,-\beta}(\mathcal{G})^*$ and the vector-function $\underline{g}=Q\underline{\phi}+C\underline{u}$ satisfy the equality

$$(f,v)_{\mathcal{G}} + (\underline{g},\underline{v})_{\partial\mathcal{G}} = (u,L^+v)_{\mathcal{G}} + (\underline{\phi},Pv+Q^+\underline{v})_{\partial\mathcal{G}} + (\underline{u},C^+\underline{v})_{\partial\mathcal{G}} \tag{6.2.23}$$

for all $v\in V^{2m-l}_{2,-\beta}(\mathcal{G})$ and $\underline{v}\in V^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{G})$.

Clearly, the restriction of the operator (6.2.21) to the space

$$\tilde{V}^{l,2m}_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{6.2.24}$$

with $l>2m$ is also a continuous mapping from (6.2.24) into

$$\tilde{V}^{l-2m,0}_{2,\beta}(\mathcal{G}) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}). \tag{6.2.25}$$

Therefore, the operator (6.2.22) is continuous for arbitrary integer l.

Formula (6.2.23) means that in the case $l\le 0$ the operator (6.2.22) is adjoint to the operator $\mathcal{A}^+$ of the formally adjoint problem (6.2.13), (6.2.14) which maps the space (6.2.15) into (6.2.16).

6.3. Solvability of elliptic boundary value problems in bounded domains with conical points

The goal of this section is to prove the Fredholm property for the operator $\mathcal{A}$ of the boundary value problem (6.2.2), (6.2.2) in weighted Sobolev spaces. To this end, we derive an a priori estimate for the solutions and describe the relations between the cokernel of the operator $\mathcal{A}$ and the kernel of the operator of the formaly adjoint boundary value problem. Furthermore, we study the dependence of the index on l and β.

6.3.1. A priori estimates for the solutions. We consider the boundary value problem (6.2.2), (6.2.3) in the domain $\mathcal{G}$. As in the previous section, we suppose that there exists a set $\mathcal{S}$ of d conical points $x^{(1)}, \dots, x^{(d)}$ such that $\partial\mathcal{G}\backslash\mathcal{S}$ is smooth. For the sake of simplicity, it is assumed again that $\mathcal{G}$ coincides with a cone $\mathcal{K}_\tau$ in a neighbourhood $\mathcal{U}_\tau$ of every conical point $x^{(\tau)}$.

A regularity assertion for the solution. In the beginning of this section we prove a regularity assertion for the solution of the boundary value problem (6.2.2), (6.2.3), where only the ellipticity of the problem in $\overline{\mathcal{G}}\backslash\mathcal{S}$ is required.

LEMMA 6.3.1. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible. Furthermore, let ζ, η be smooth functions with support in the neighbourhood $\mathcal{U}_\tau$ of the conical point $x^{(\tau)}$ satisfying the conditions*

$$\zeta\,\eta = \zeta\,, \quad D_\nu^j\zeta|_{\partial\mathcal{G}\backslash\mathcal{S}} = D_\nu^j\eta|_{\partial\mathcal{G}\backslash\mathcal{S}} = 0 \quad \textit{for } j = 1, 2, \dots, 2m-1.$$

If $(u, \underline{\phi}, \underline{u})$ is a solution of the boundary value problem (6.2.2), (6.2.3) such that

$$\eta(u, \underline{\phi}, \underline{u}) \in \tilde{V}_{2,\beta_\tau}^{l,2m}(\mathcal{K}_\tau) \times V_{2,\beta_\tau}^{l+\underline{\tau}-1/2}(\partial\mathcal{K}_\tau), \tag{6.3.1}$$

and

$$\eta(f, \underline{g}) \in \tilde{V}_{2,\beta_\tau+1}^{l-2m+1,0}(\mathcal{K}_\tau) \times V_{2,\beta_\tau+1}^{l-\underline{\mu}+1/2}(\partial\mathcal{K}_\tau), \tag{6.3.2}$$

then $\zeta(u, \underline{\phi}, \underline{u}) \in \tilde{V}_{2,\beta_\tau+1}^{l+1,2m}(\mathcal{K}_\tau) \times V_{2,\beta_\tau+1}^{l+\underline{\tau}+1/2}(\partial\mathcal{K}_\tau)$. Furthermore, the following estimate holds with a constant c independent of $(u, \underline{\phi}, \underline{u})$:

$$\|\zeta(u, \underline{\phi}, \underline{u})\|_{l+1,\beta_\tau+1;\mathcal{K}_\tau} \le c\Big(\|\zeta f\|_{\tilde{V}_{2,\beta_\tau+1}^{l-2m+1,0}(\mathcal{K}_\tau)} + \|\zeta \underline{g}\|_{V_{2,\beta_\tau+1}^{l-\underline{\mu}+1/2}(\partial\mathcal{K}_\tau)} + \|\eta(u, \underline{\phi}, \underline{u})\|_{l,\beta_\tau;\mathcal{K}_\tau}\Big) \tag{6.3.3}$$

Here $\|\cdot\|_{l,\beta_\tau;\mathcal{K}_\tau}$ denotes the norm in (6.3.1).

Proof: Obviously,

$$\mathcal{A}\,\zeta(u, \underline{\phi}, \underline{u}) = (f^{(1)}, \underline{g}^{(1)}) \stackrel{def}{=} \zeta(f, \underline{g}) + [\mathcal{A}, \zeta]\,\eta(u, \underline{\phi}, \underline{u})\,,$$

where $[\mathcal{A}, \zeta] = \mathcal{A}\zeta - \zeta\mathcal{A}$ denotes the commutator of $\mathcal{A}$ and ζ. By the given conditions on $(u, \underline{\phi}, \underline{u})$ and $(f, \underline{g})$, the pair $(f^{(1)}, \underline{g}^{(1)})$ is an element of the space (6.3.2). Using the coordinates ω, t, where ω are coordinates on the unit sphere $|x - x^{(\tau)}| = 1$ and $t = \log|x - x^{(\tau)}|$, we get the equation

$$\begin{aligned} \mathfrak{A}(t, \partial_t)\,\zeta(u, \underline{\phi}, e^{-\tau_1 t}u_1, \dots, e^{-\tau_J t}u_J) \\ = (e^{2mt}f^{(1)}, e^{\mu_1 t}g_1^{(1)}, \dots, e^{\mu_{m+J} t}g_{m+J}^{(1)}), \end{aligned} \tag{6.3.4}$$

where

$$\begin{aligned}\mathfrak{A}(t,\partial_t)\,:\,\tilde{\mathcal{W}}^{l,2m}_{2,\beta_\tau-l+n/2}(\mathcal{C}_\tau)\times\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta_\tau-l+n/2}(\partial\mathcal{C}_\tau)\\ \to\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta_\tau-l+n/2}(\mathcal{C}_\tau)\times\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta_\tau-l+n/2}(\partial\mathcal{C}_\tau)\end{aligned}$$

is the operator of the boundary value problem (6.2.10), (6.2.11). Since $(f^{(1)},\underline{g}^{(1)})$ is an element of the space (6.3.2), the right-hand side of (6.3.4) belongs to the space

$$\tilde{\mathcal{W}}^{l-2m+1,0}_{2,\beta_\tau-l+n/2}(\mathcal{C}_\tau)\times\mathcal{W}^{l-\underline{\mu}+1/2}_{2,\beta_\tau-l+n/2}(\partial\mathcal{C}_\tau).$$

By means of Lemma 5.5.3, we conclude from this that $\zeta(u,\underline{\phi})\in\tilde{\mathcal{W}}^{l+1,2m}_{2,\beta_\tau-l+n/2}(\mathcal{C}_\tau)$ and $\zeta e^{-\tau_j t}u_j\in\mathcal{W}^{l+\underline{\tau}+1/2}_{2,\beta_\tau-l+n/2}(\partial\mathcal{C}_\tau)$ for $j=1,\dots,J$. Furthermore a corresponding estimate for $(u,\underline{\phi})$ and $e^{-\tau_j t}u_j$ holds. This leads to the assertion of our lemma if we return to the Cartesian coordinates. ■

REMARK 6.3.1. Suppose that the function η in Lemma 6.3.1 is equal to one in a neighbourhood of $\operatorname{supp}\zeta$. Then analogously to Lemma 3.2.3, there exists a constant c independent of $(u,\underline{\phi},\underline{u})$ such that

$$(6.3.5)\qquad \sum_{j=1}^{2m}\|\zeta\phi_j\|_{V^{l-j+1/2}_{2,\beta_\tau}(\partial\mathcal{K}_\tau)}\le c\Big(\|\eta u\|_{\tilde{V}^{l,0}_{2,\beta_\tau}(\mathcal{K}_\tau)}+\|\eta f\|_{\tilde{V}^{l-2m,0}_{2,\beta_\tau}(\partial\mathcal{K}_\tau)}\Big).$$

Hence (6.3.3) implies the following estimate:

$$\begin{aligned}\|\zeta(u,\underline{\phi},\underline{u})\|_{l+1,\beta_\tau+1;\mathcal{K}_\tau}\ \le\ & c\Big(\|\eta f\|_{\tilde{V}^{l-2m+1,0}_{2,\beta_\tau+1}(\mathcal{K}_\tau)}+\|\eta g\|_{V^{l-\underline{\mu}+1/2}_{2,\beta_\tau+1}(\partial\mathcal{K}_\tau)}\\ &+\|\eta u\|_{\tilde{V}^{l,0}_{2,\beta_\tau}(\mathcal{K}_\tau)}+\|\eta\underline{u}\|_{V^{l+\underline{\tau}-1/2}_{2,\beta_\tau}(\mathcal{K}_\tau)}\Big)\end{aligned}$$

(cf. Remark 6.1.3, Remark 5.5.1).

The operator pencils connected with the boundary value problem. Let the operators L, B_k, and $C_{k,j}$ be admissible in a neighbourhood of the conical point $x^{(\tau)}$. By $\mathcal{A}_\tau$ we denote the operator of the model problem

$$(6.3.6)\quad L^{(\tau)}(x,\partial_x)u=f\qquad\text{in }\mathcal{K}_\tau\,,$$

$$(6.3.7)\quad B^{(\tau)}_k(x,\partial_x)u+\sum_{j=1}^{J}C^{(\tau)}_{k,j}(x,\partial_x)u_j=g_k\ \text{on }\partial\mathcal{K}_\tau\backslash\{x^{(\tau)}\}\,,\ k=1,\dots,m+J,$$

where $L^{(\tau)}$, $B^{(\tau)}_k$, $C^{(\tau)}_{k,j}$ are the leading parts of L, B_k, $C_{k,j}$ at the point $x^{(\tau)}$ (see Definition 6.2.1).

We write the differential operators of problem (6.3.6), (6.3.7) in the form

$$\begin{aligned}&L^{(\tau)}(x,\partial_x)=r^{-2m}\mathcal{L}^{(\tau)}(\omega,\partial_\omega,r\partial_r),\qquad B^{(\tau)}_k(x,\partial_x)=r^{-\mu_k}\,\mathcal{B}^{(\tau)}_k(\omega,\partial_\omega,r\partial_r)\\ &C^{(\tau)}_{k,j}(x,\partial_x)=r^{-\mu_k-\tau_j}\,\mathcal{C}^{(\tau)}_{k,j}(\omega,\partial_\omega,r\partial_r),\end{aligned}$$

where $r=|x-x^{(\tau)}|$ and ω are coordinates on the unit sphere $|x-x^{(\tau)}|=1$. By $\mathfrak{A}_\tau(\lambda)$ we denote the operator of the boundary value problem

$$(6.3.8)\qquad \mathcal{L}^{(\tau)}(\omega,\partial_\omega,\lambda)u=f\quad\text{in }\Omega_\tau,$$

$$(6.3.9)\qquad \mathcal{B}^{(\tau)}_k(\omega,\partial_\omega,\lambda)u+\sum_{j=1}^{J}\mathcal{C}^{(\tau)}_{k,j}(\omega,\partial_\omega,\lambda+\tau_j)u_j=g_k\ \text{on }\partial\Omega_\tau,$$

$$k=1,\dots,m+J,$$

where Ω_τ is the intersection of the cone $\mathcal{K}_\tau$ with the unit sphere $|x - x^{(\tau)}| = 1$.

A sharper estimate. Under the additional assumption that there are no eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ on the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ for $\tau = 1, \dots, d$, the estimate in Lemma 6.3.1 can be sharpened.

LEMMA 6.3.2. *Let the conditions of Lemma 6.3.1 be satisfied. Additionally we assume that the line* $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ *does not contain eigenvalues of* $\mathfrak{A}_\tau(\lambda)$. *Then every solution*

$$(u, \underline{\phi}, \underline{u}) \in \tilde{V}_{2,\beta}^{l,2m}(\mathcal{G}) \times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G}) \tag{6.3.10}$$

of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$ *satisfies the estimate*

$$\begin{aligned} \|\zeta(u, \underline{\phi}, \underline{u})\|_{l,\beta_\tau;\mathcal{K}_\tau} \leq\ & c\Big(\|\zeta f\|_{\tilde{V}_{2,\beta_\tau}^{l-2m,0}(\mathcal{K}_\tau)} + \|\zeta \underline{g}\|_{V_{2,\beta_\tau}^{l-\underline{\mu}-1/2}(\partial\mathcal{K}_\tau)} \\ & + \|\eta(u, \underline{\phi}, \underline{u})\|_{l-1,\beta_\tau;\mathcal{K}_\tau}\Big), \end{aligned} \tag{6.3.11}$$

where $\|\cdot\|_{l,\beta;\mathcal{K}_\tau}$ *denotes the norm in the space (6.3.1) and the constant c is independent of* $(u, \underline{\phi}, \underline{u})$.

Proof: First we suppose that the support of ζ is sufficiently small. Then the norm of $(\mathcal{A}_\tau - \mathcal{A})\,\zeta(u, \underline{\phi}, \underline{u})$ in the space

$$\tilde{V}_{2,\beta_\tau}^{l-2m,0}(\mathcal{K}_\tau) \times V_{2,\beta_\tau}^{l-\underline{\mu}-1/2}(\partial\mathcal{K}_\tau), \tag{6.3.12}$$

is less than $\varepsilon\|\zeta(u, \underline{\phi}, \underline{u})\|_{l,\beta_\tau;\mathcal{K}_\tau}$ with a small positive real number ε (see Lemma 6.2.2). Under the assumptions of the theorem, the operator $\mathcal{A}_\tau$ realizes an isomorphism from (6.3.1) onto (6.3.12). Since the commutator $[\mathcal{A}, \zeta] = \mathcal{A}\zeta - \zeta\mathcal{A}$ continuously maps

$$\tilde{V}_{2,\beta_\tau}^{l-1,2m}(\mathcal{K}_\tau) \times V_{2,\beta_\tau}^{l+\underline{\tau}-3/2}(\partial\mathcal{K}_\tau),$$

into the space (6.3.12), it follows from the equation

$$\mathcal{A}_\tau\,\zeta(u, \underline{\phi}, \underline{u}) = \zeta(f, \underline{g}) + [\mathcal{A}, \zeta]\,\eta(u, \underline{\phi}, \underline{u}) + (\mathcal{A}_\tau - \mathcal{A})\,\zeta(u, \underline{\phi}, \underline{u})$$

that

$$\begin{aligned} \|\zeta(u, \underline{\phi}, \underline{u})\|_{l,\beta_\tau;\mathcal{K}_\tau} \leq\ & c\Big(\|\zeta f\|_{\tilde{V}_{2,\beta_\tau}^{l-2m,0}(\mathcal{K}_\tau)} + \|\zeta \underline{g}\|_{V_{2,\beta_\tau}^{l-\underline{\mu}-1/2}(\partial\mathcal{K}_\tau)} \\ & + \|\eta(u, \underline{\phi}, \underline{u})\|_{l-1,\beta_\tau;\mathcal{K}_\tau} + \varepsilon\,\|\zeta(u, \underline{\phi}, \underline{u})\|_{l,\beta_\tau;\mathcal{K}_\tau}\Big). \end{aligned}$$

The term $\varepsilon\,\|\zeta(u, \underline{\phi}, \underline{u})\|_{l,\beta_\tau;\mathcal{K}_\tau}$ on the right-hand side can be omitted if ε is sufficiently small, i.e., the support of ζ is sufficiently small.

Now let ζ be an arbitrary smooth function with support in $\mathcal{U}_\tau$ and let χ be a smooth function equal to one in a neighbourhood of $x^{(\tau)}$ with sufficiently small support. Then the estimate (6.3.11) with $\chi\zeta$ instead of ζ is valid. Furthermore, by Lemma 3.2.4, the estimate (6.3.11) with $(1-\chi)\zeta$ instead of ζ is valid. This implies the desired inequality. ∎

From Lemma 6.3.2 and from the corresponding local a priori estimate for boundary value problems in smooth domains (see Lemma 3.2.4) we can conclude the following global estimate.

THEOREM 6.3.1. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible and there are no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ on the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ for $\tau = 1, \dots, d$. Then every solution $(u, \underline{\phi}, \underline{u}) \in \tilde{V}_{2,\beta}^{l,2m}(\mathcal{G}) \times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G})$ of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$ satisfies the estimate*

$$(6.3.13) \qquad \|(u, \underline{\phi}, \underline{u})\|_{l,\beta;\mathcal{G}} \le c \Big(\|f\|_{\tilde{V}_{2,\beta}^{l-2m,0}(\mathcal{G})} + \|\underline{g}\|_{V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})} + \|(u, \underline{\phi}, \underline{u})\|_{l-1,\beta;\mathcal{G}} \Big),$$

where $\|\cdot\|_{l,\beta;\mathcal{G}}$ denotes the norm in the space (6.3.10).

REMARK 6.3.2. By (6.3.5) and Lemma 3.2.3, the term $\|(u, \underline{\phi}, \underline{u})\|_{l-1,\beta;\mathcal{G}}$ on the right-hand side of (6.3.13) can be replaced by

$$\|u\|_{\tilde{V}_{2,\beta}^{l-1,0}(\mathcal{G})} + \|\underline{u}\|_{V_{2,\beta}^{l+\underline{\tau}-3/2}(\partial\mathcal{G})} .$$

Consequently, under the assumptions of Theorem 6.3.1, every solution

$$(6.3.14) \qquad (u, \underline{u}) \in V_{2,\beta}^{l}(\mathcal{G}) \times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G}), \quad l \ge 2m,$$

of the boundary value problem (6.2.2), (6.2.3) satisfies the inequality

$$(6.3.15) \qquad \|(u, \underline{u})\|_{l,\beta;\mathcal{G}} \le c \Big(\|f\|_{V_{2,\beta}^{l-2m}(\mathcal{G})} + \|\underline{g}\|_{V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})} + \|(u, \underline{u})\|_{l-1,\beta;\mathcal{G}} \Big),$$

where $\|\cdot\|_{l,\beta;\mathcal{G}}$ denotes the norm in (6.3.14).

Necessity of the condition on the eigenvalues. Analogously to Lemma 5.2.5, it can be shown that the condition on the eigenvalues of $\mathfrak{A}_\tau(\lambda)$ in Theorem 6.3.1 is necessary for the validity of the inequality (6.3.13).

We assume that λ_0 is an eigenvalue of $\mathfrak{A}_\tau(\lambda)$ on the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ and $(\varphi^{(0)}, \underline{\varphi}^{(0)}) = (\varphi^{(0)}, \varphi_1^{(0)}, \dots, \varphi_J^{(0)})$ is an eigenvector of $\mathfrak{A}_\tau(\lambda)$ corresponding to this eigenvalue. Let ε be a sufficiently small positive real number such that

$$\{x \in \mathcal{G} : |x - x^{(\tau)}| < 2\varepsilon\} = \{x \in \mathcal{K}_\tau : |x - x^{(\tau)}| < 2\varepsilon\}.$$

Furthermore, let T be a real number greater than $|\log\varepsilon|$ and χ_T an infinitely differentiable function on the positive real half-axis such that

$$\chi_T(r) = 1 \ \text{ for } e^{-T} < r < \varepsilon, \qquad \chi_T(r) = 0 \ \text{ for } r < e^{-T-1},\ r > 2\varepsilon,$$

and

$$|(r\partial_r)^j \chi_T(r)| < c_j \quad \text{for } j = 0, 1, 2, \dots .$$

Here the constants c_j do not depend on T. We consider the functions

$$(6.3.16) \qquad \begin{cases} u^{(T)} = \chi_T(r)\, r^{\lambda_0}\, \varphi^{(0)}(\omega), & \phi_j^{(T)} = D_\nu^{j-1} u^{(T)}|_{\partial\mathcal{G}\setminus\mathcal{S}}, \ j = 1, \dots, 2m \\ u_j^{(T)} = \chi_T(r)\, r^{\lambda_0+\tau_j}\, \varphi_j^{(0)}(\omega), & j = 1, \dots, J, \end{cases}$$

where $r = |x - x^{(\tau)}|$ and ω are coordinates on the unit sphere $|x - x^{(\tau)}| = 1$.

Since $(u^{(T)}, \underline{\phi}^{(T)}, \underline{u}^{(T)}) = 0$ for $|x - x^{(\tau)}| > 2\varepsilon$, there exists a constant c_1 independent of ε, T such that the inequality

$$\|(u^{(T)}, \underline{\phi}^{(T)}, \underline{u}^{(T)})\|_{l-1,\beta;\mathcal{G}} \le c_1\, \varepsilon\, \|(u^{(T)}, \underline{\phi}^{(T)}, \underline{u}^{(T)})\|_{l,\beta;\mathcal{G}}$$

is valid. Furthermore, from the conditions on the coefficients L, B_k, $C_{k,j}$ it follows that

$$\|(\mathcal{A}-\mathcal{A}_\tau)\,(u^{(T)},\underline{\phi}^{(T)},\underline{u}^{(T)})\|_{\tilde V^{l-2m,0}_{2,\beta}(\mathcal{G})\times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})} \le c_2(\varepsilon)\,\|(u^{(T)},\underline{\phi}^{(T)},\underline{u}^{(T)})\|_{l,\beta;\mathcal{G}}\,,$$

where $c_2(\varepsilon)$ tends to zero as $\varepsilon\to 0$ (cf. Lemma 6.2.2).

Suppose there exists a constant c such that the inequality (6.3.13) is satisfied for every $(u,\underline{\phi},\underline{u})\in \tilde V^{l,2m}_{2,\beta}(\mathcal{G})\times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$. Then for sufficiently small ε we have

$$\|(u^{(T)},\underline{\phi}^{(T)},\underline{u}^{(T)})\|_{l,\beta;\mathcal{G}} \le 2c\,\|\mathcal{A}_\tau(u^{(T)},\underline{\phi}^{(\tau)},\underline{u}^{(T)})\|_{\tilde V^{l-2m,0}_{2,\beta}(\mathcal{G})\times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})} \tag{6.3.17}$$

for each triple $(u^{(T)},\underline{\phi}^{(T)},\underline{u}^{(T)})$ defined by (6.3.16), $T>-\log\varepsilon$. It can be easily seen that the left side of (6.3.17) tends to infinity for fixed ε as $T\to+\infty$. On the other hand, from the equation

$$\begin{aligned}\mathfrak{A}_\tau(r\partial_r)\,\chi_T(r)\,r^{\lambda_0}\,(\varphi,\underline{\varphi}) &= r^{\lambda_0}\,\mathfrak{A}_\tau(r\partial_r+\lambda_0)\,\chi_T(r)\,(\varphi,\underline{\varphi})\\ &= r^{\lambda_0}\sum_{j\ge 1}\frac{1}{j!}\,(r\partial_r)^j\chi_T(r)\,\mathfrak{A}^{(j)}(\lambda_0)(\varphi,\underline{\varphi})\end{aligned}$$

and from our assumptions on the function χ_T it follows that the right side of (6.3.17) is bounded by a constant c independent of T. Thus, we have proved the following assertion.

LEMMA 6.3.3. *If the line* $\operatorname{Re}\lambda=-\beta_\tau+l-n/2$ *contains eigenvalues of* $\mathfrak{A}_\tau(\lambda)$ *for at least one* τ, *then there does not exist a finite constant* c *such that the inequality (6.3.13) is satisfied for each solution* $(u,\underline{\phi},\underline{u})\in\tilde V^{l,2m}_{2,\beta}(\mathcal{G})\times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ *of the equation* $\mathcal{A}\,(u,\underline{\phi},\underline{u})=(f,\underline{g})$.

6.3.2. Relations between the adjoint operator and the operator of the formally adjoint problem. We suppose that the operators L, B_k, $C_{k,j}$ are admissible. Then the operator

$$\mathcal{A}\,:\;(u,\underline{u})\to\big(Lu,\;Bu|_{\partial\mathcal{G}\backslash\mathcal{S}}+C\underline{u}\big)$$

of the boundary value problem (6.2.2), (6.2.3) continuously maps the space (6.2.8) into (6.2.9) for arbitrary integer $l\ge 2m$. Therefore, the adjoint operator $\mathcal{A}^*$ realizes a continuous mapping from

$$V^{l-2m}_{2,\beta}(\mathcal{G})^*\times V^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{G}) \tag{6.3.18}$$

into the space

$$V^{l}_{2,\beta}(\mathcal{G})^*\times V^{-l-\underline{\tau}+1/2}_{2,-\beta}(\partial\mathcal{G}) \tag{6.3.19}$$

for $l\ge 2m$. Furthermore, by Theorem 6.2.1, the operator

$$\begin{aligned}\tilde V^{2m,2m}_{2,-\beta+l\vec 1}(\mathcal{G})\times V^{\underline{\mu}+1/2}_{2,-\beta+l\vec 1}(\partial\mathcal{G})\ni(v,\underline{\psi},\underline{v})&\to\big(L^+v,\;Pv+Q^+\underline{v},\;C^+\underline{v}\big)\\ &\in V^{0}_{2,-\beta+l\vec 1}(\mathcal{G})\times\prod_{j=1}^{2m}V^{j-1/2}_{2,-\beta+l\vec 1}(\partial\mathcal{G})\times\prod_{j=1}^{J}V^{-\tau_j+1/2}_{2,-\beta+l\vec 1}(\partial\mathcal{G})\end{aligned}$$

of the formally adjoint problem (6.2.13), (6.2.14) can be uniquely extended to a continuous operator $\mathcal{A}^+$ which maps

$$\tilde{V}_{2,-\beta}^{-l+2m,2m}(\mathcal{G}) \times V_{2,-\beta}^{-l+\underline{\mu}+1/2}(\partial\mathcal{G}) \tag{6.3.20}$$

into the space

$$\tilde{V}_{2,-\beta}^{-l,0}(\mathcal{G}) \times \prod_{j=1}^{2m} V_{2,-\beta}^{-l+j-1/2}(\partial\mathcal{G}) \times \prod_{j=1}^{J} V_{2,-\beta}^{-l-\tau_j+1/2}(\partial\mathcal{G}) \tag{6.3.21}$$

with $l > 0$. Let T be the matrix of tangential differential operators defined by the equality

$$Pv|_{\partial\mathcal{G}\backslash\mathcal{S}} = T \cdot \mathcal{D}v|_{\partial\mathcal{G}\backslash\mathcal{S}}\,, \quad v \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S}).$$

From the admissibility of the operators P_j it follows that the components $T_{j,k}$ of the matrix T are admissible operators. If the differential operator L is elliptic and admissible, then the operator T realizes an isomorphism

$$\prod_{j=1}^{2m} V_{2,-\beta}^{-l+2m+j-1/2}(\partial\mathcal{G}) \to \prod_{j=1}^{2m} V_{2,-\beta}^{-l+j-1/2}(\partial\mathcal{G})$$

(cf. Remark 3.1.2). Repeating the proof of Lemma 3.3.1, we get the following assertion.

LEMMA 6.3.4. *Let $(v, \underline{\psi}, \underline{v})$ and $(f, \underline{g}, \underline{h})$ be elements of the spaces (6.3.20) and (6.3.21), respectively, where $l \geq 2m$. Furthermore, let the functional $F \in V_{2,\beta}^l(\mathcal{G})^*$ be defined by the equality*

$$(u, F)_\mathcal{G} = (u, f)_\mathcal{G} + (\mathcal{D}u, \underline{g})_{\partial\mathcal{G}}\,, \qquad u \in V_{2,\beta}^l(\mathcal{G}), \tag{6.3.22}$$

Then $(v, \underline{\psi}, \underline{v})$ is a solution of the equation

$$\mathcal{A}^+ (v, \underline{\psi}, \underline{v}) = (f, \underline{g}, \underline{h})$$

if and only if

$$\mathcal{A}^* (v, \underline{v}) = (F, \underline{h})$$

and $T\underline{\psi} + Q^+\underline{v} = \underline{g}$.

Motivated by the representation (6.3.22) for the functional F we introduce the space $D_{2,\beta}^{l,k}(\mathcal{G})$ for integer l and nonnegative integer k as follows (cf. the definition of the spaces $D_2^{l,k}$ in Section 3.3).

For $l \leq -k$ we set $D_{2,\beta}^{l,k}(\mathcal{G}) = V_{2,-\beta}^{-l}(\mathcal{G})^*$, while for $l > -k$ the space $D_{2,\beta}^{l,k}(\mathcal{G})$ is defined as the set of all functionals $F \in V_{2,-\beta}^k(\mathcal{G})^*$ which have the form

$$(u, F)_\mathcal{G} = (u, f)_\mathcal{G} + \sum_{j=1}^{k} (D_\nu^{j-1}u, g_j)_{\partial\mathcal{G}}\,, \qquad u \in V_{2,-\beta}^k(\mathcal{G}), \tag{6.3.23}$$

where $f \in \tilde{V}_{2,\beta}^{l,0}(\mathcal{G})$, $g_j \in V_{2,\beta}^{l+j-1/2}(\partial\mathcal{G})$.

REMARK 6.3.3. Since the functional

$$u \to \sum_{j=1}^{-l} (D_\nu^{j-1}u, g_j)_{\partial\mathcal{G}}$$

belongs to the space $V_{2,-\beta}^{-l}(\mathcal{G})^*$ for given $g_j \in V_{2,\beta}^{l+j-1/2}(\partial\mathcal{G})$, $l < 0$, the space $D_{2,\beta}^{l,k}(\mathcal{G})$ can be defined in the case $-k < l < 0$ as the set of all functionals $F \in V_{2,-\beta}^{k}(\mathcal{G})^*$ which have the form

$$(u, F)_{\mathcal{G}} = (u, f)_{\mathcal{G}} + \sum_{j=-l+1}^{k} (D_\nu^{j-1} u, g_j)_{\partial\mathcal{G}}\,, \qquad u \in V_{2,-\beta}^{k}(\mathcal{G}),$$

where $f \in \tilde{V}_{2,\beta}^{l,0}(\mathcal{G})$, $g_j \in V_{2,\beta}^{l+j-1/2}(\partial\mathcal{G})$. In contrast to (6.3.23), this representation is unique.

The norm of the functional F in the space $D_{2,\beta}^{l,k}(\mathcal{G})$ is defined as the infimum of the sum

$$\|f\|_{\tilde{V}_{2,\beta}^{l,0}(\mathcal{G})} + \sum_{j=1}^{k} \|g_j\|_{V_{2,\beta}^{l+j-1/2}(\partial\mathcal{G})}\,,$$

where f, g_j satisfy (6.3.23).

Note that, by our notation, the space $D_{2,\beta}^{l,0}(\mathcal{G})$ coincides with the space $\tilde{V}_{2,\beta}^{l,0}(\mathcal{G})$ for arbitrary integer l, i.e, with $V_{2,\beta}^{l}(\mathcal{G})$ for $l \geq 0$ and with $V_{2,-\beta}^{-l}(\mathcal{G})^*$ for $l < 0$.

From Lemma 6.3.4 it follows that the adjoint operator $\mathcal{A}^*$ realizes a continuous mapping from

$$D_{2,-\beta}^{-l+2m,0}(\mathcal{G}) \times V_{2,-\beta}^{-l+\underline{\mu}+1/2}(\partial\mathcal{G}) \tag{6.3.24}$$

into the space

$$D_{2,-\beta}^{-l,2m}(\mathcal{G}) \times V_{2,-\beta}^{-l-\underline{\tau}+1/2}(\partial\mathcal{G}) \tag{6.3.25}$$

for arbitrary integer l.

6.3.3. A regularity assertion for the adjoint operator. Applying the regularity assertion in Lemma 6.3.1 to the operator $\mathcal{A}^+$ and using the relation between the operators $\mathcal{A}^*$ and $\mathcal{A}^+$ given in Lemma 6.3.4, we obtain the following theorem (cf. Theorem 3.3.1, Corollary 3.3.1).

THEOREM 6.3.2. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible in a neighbourhood of each conical point $x^{(\tau)}$. If $(v, \underline{v}) \in D_{2,-\beta}^{-l+2m,0}(\mathcal{G}) \times V_{2,-\beta}^{-l+\underline{\mu}+1/2}(\partial\mathcal{G})$ is a solution of the equation $\mathcal{A}^*(v, \underline{v}) = (F, \underline{h})$ and $(F, \underline{h})$ belongs to the space*

$$D_{2,-\beta+q\vec{1}}^{-l+q,2m}(\mathcal{G}) \times V_{2,-\beta+q\vec{1}}^{-l+q-\underline{\tau}+1/2}(\partial\mathcal{G})\,, \tag{6.3.26}$$

where q is an arbitrary integer, $q \geq 0$, and $\vec{1}$ denotes the vector $(1, \dots, 1) \in \mathbb{R}^d$, then $(v, \underline{v}) \in D_{2,-\beta+q\vec{1}}^{-l+2m+q,0}(\mathcal{G}) \times V_{2,-\beta+q\vec{1}}^{-l+q+\underline{\mu}+1/2}(\partial\mathcal{G})$ and

$$\|(v, \underline{v})\|_{-l+2m+q,-\beta+q\vec{1}} \leq c\Big(\|(F, \underline{h})\| + \|(v, \underline{v})\|_{-l+2m,-\beta}\Big).$$

Here $\|\cdot\|_{-l+2m,-\beta}$ denotes the norm in (6.3.24), whereas the norm of $(F, \underline{h})$ is taken in the space (6.3.26).

Moreover, for $q \geq l$ the pair $(v, \underline{v})$ is a solution of the formally adjoint boundary value problem (6.2.13), (6.2.14), where

$$f \in V^{-l+q}_{2,-\beta+q\vec{1}}(\mathcal{G}) \quad \text{and} \quad \underline{g} \in \prod_{j=1}^{2m} V^{-l+q+j-1/2}_{2,-\beta+q\vec{1}}(\partial\mathcal{G}) \tag{6.3.27}$$

are determined by the equality

$$(u, F)_{\mathcal{G}} = (u, f)_{\mathcal{G}} + (\mathcal{D}u, \underline{g})_{\partial\mathcal{G}}\,, \quad u \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S}). \tag{6.3.28}$$

Proof: According to the assumptions of the theorem, the functional F has the representation (6.3.28) with f, $\underline{g}$ as in (6.3.27). By Lemma 6.3.4, there exists a vector-function

$$\underline{\psi} \in \prod_{j=1}^{2m} V^{-l+2m-j+1/2}_{2,-\beta}(\partial\mathcal{G})$$

such that $(v, \underline{\psi}, \underline{v})$ is a solution of the equation $\mathcal{A}^+(v, \underline{\psi}, \underline{v}) = (f, \underline{g}, \underline{h})$. Hence from Lemma 6.3.1 it follows that $v \in D^{-l+2m+q,0}_{2,-\beta+q\vec{1}}(\mathcal{G})$, $\underline{v} \in V^{-l+q+\underline{\mu}+1/2}_{2,-\beta+q\vec{1}}(\partial\mathcal{G})$ and the desired inequality for the pair $(v, \underline{v})$ is satisfied. ∎

Analogously, Theorem 6.3.1 implies the following a priori estimate for the solutions of the adjoint equation.

LEMMA 6.3.5. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem are admissible in a neighbourhood of each conical point $x^{(\tau)}$. Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ on the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ for $\tau = 1, \dots, d$. Then for every element $(v, \underline{v})$ of the space (6.3.24) the estimate*

$$\|(v, \underline{v})\|_{-l+2m,-\beta} \leq c\left(\|\mathcal{A}^*(v, \underline{v})\| + \|(v, \underline{v})\|_{-l+2m-1,-\beta}\right).$$

is satisfied. Here $\|\cdot\|_{-l+2m,-\beta}$ denotes the norm in the space (6.3.24), whereas the norm of $\mathcal{A}^*(v, \underline{v})$ is taken in the space (6.3.25).

6.3.4. The Fredholm property for the operator of the boundary value problem. Now we prove the Fredholm property for the operator (6.2.22) of the boundary value problem (6.2.2), (6.2.3). In the sequel, this operator will be denoted by $\mathcal{A}_{l,\beta}$. Furthermore, let

$$\begin{aligned} \mathcal{A}^+_{l,\beta}\ :\ & \tilde{V}^{l,2m}_{2,\beta}(\mathcal{G}) \times V^{l-2m+\underline{\mu}+1/2}_{2,\beta}(\partial\mathcal{G}) \\ & \to \tilde{V}^{l-2m,0}_{2,\beta}(\mathcal{G}) \times \prod_{j=1}^{2m} V^{l-2m+j-1/2}_{2,\beta}(\partial\mathcal{G}) \times \prod_{j=1}^{J} V^{l-2m-\tau_j+1/2}_{2,\beta}(\partial\mathcal{G}) \end{aligned}$$

be the operator of the formally adjoint boundary value problem (6.2.13), (6.2.14).

If $l \geq 2m$, then $\mathcal{A}_{l,\beta}$ can be identified with the operator

$$\begin{aligned} & V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \ni (u, \underline{u}) \\ & \qquad \to \left(Lu,\ Bu|_{\partial\mathcal{G}\backslash\mathcal{S}} + C\underline{u}\right) \in V^{l-2m}_{2,\beta}(\mathcal{G}) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}). \end{aligned} \tag{6.3.29}$$

We denote the adjoint operator to (6.3.29) by $\mathcal{A}^*_{l,\beta}$. In the case $l < 2m$ the operator $\mathcal{A}^*_{l,\beta}$ is defined as the restriction of $\mathcal{A}^*_{2m,\beta+(2m-l)\vec{1}}$ to the space (6.3.24).

Moreover, we introduce the following three sets. The set $\mathcal{N}_\beta$ consits of all

$$(u, \underline{\phi}, \underline{u}) \in \bigcap_{l \ge 2m} \left(\tilde{V}^{l,2m}_{2,\beta+l\vec{1}}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta+l\vec{1}}(\partial\mathcal{G}) \right)$$

such that $(u, \underline{u})$ is a solution of the homogeneous boundary value problem (6.2.2), (6.2.3). Analogously, $\mathcal{N}^+_\beta$ is the set of all

$$(v, \underline{\psi}, \underline{v}) \in \bigcap_{l \ge 2m} \left(\tilde{V}^{l,2m}_{2,\beta+l\vec{1}}(\mathcal{G}) \times V^{l-2m+\underline{\mu}+1/2}_{2,\beta+l\vec{1}}(\partial\mathcal{G}) \right)$$

such that $(v, \underline{v})$ is a solution of the homogeneous formally adjoint problem (6.2.13), (6.2.14). Finally, we set

$$\mathcal{N}^*_\beta = \left\{ (v, \underline{v}) : \left(v, \mathcal{D}v|_{\partial\mathcal{G}\setminus\mathcal{S}}, \underline{v} \right) \in \mathcal{N}^+_\beta \right\}.$$

LEMMA 6.3.6. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible. Then the kernels of $\mathcal{A}_{l,\beta}$, $\mathcal{A}^+_{l,\beta}$, and $\mathcal{A}^*_{l,\beta}$ are finite-dimensional and depend only on $\beta_1 - l, \dots, \beta_d - l$. More precisely, we have*

$$\ker \mathcal{A}_{l,\beta} = \mathcal{N}_{\beta - l\vec{1}}, \quad \ker \mathcal{A}^+_{l,\beta} = \mathcal{N}^+_{\beta-l\vec{1}}, \quad \ker \mathcal{A}^*_{l,\beta} = \mathcal{N}^*_{-\beta+(l-2m)\vec{1}}.$$

If additionally the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ does not contain eigenvalues of $\mathfrak{A}_\tau(\lambda)$ for $\tau = 1, \dots, d$, then the range of the operators $\mathcal{A}_{l,\beta}$, $\mathcal{A}^+_{l,\beta}$, and $\mathcal{A}^*_{l,\beta}$ is closed.

Proof: Since the domain of the operator $\mathcal{A}_{l+1,\beta+\vec{1}}$ is contained in the domain of the operator $\mathcal{A}_{l,\beta}$, the kernel of $\mathcal{A}_{l+1,\beta+\vec{1}}$ is a subset of $\ker \mathcal{A}_{l,\beta}$. On the other hand, from Lemma 6.3.1 it follows that every element of $\ker \mathcal{A}_{l,\beta}$ belongs to the kernel of $\mathcal{A}_{l+1,\beta+\vec{1}}$. Hence $\ker \mathcal{A}_{l,\beta} = \mathcal{A}_{l+1,\beta+\vec{1}}$ for every l and every β, i.e., $\ker \mathcal{A}_{l,\beta} = \mathcal{N}_{\beta-l\vec{1}}$.

If no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ lie on the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ for $\tau = 1, \dots, d$, then the a priori estimate (6.3.13) is satisfied. Therefore, by Lemma 3.4.1, the kernel of $\mathcal{A}_{l,\beta}$ is finite-dimensional and the range of $\mathcal{A}_{l,\beta}$ is closed.

Obviously, for every tuple $\beta = (\beta_1, \dots, \beta_d)$ there exists a tuple $\gamma = (\gamma_1, \dots, \gamma_d)$, $\gamma \ge \beta$ (i.e., $\gamma_\tau \ge \beta_\tau$ for $\tau = 1, \dots, d$), such that the line $\operatorname{Re}\lambda = \gamma_\tau - l + n/2$ does not contain eigenvalues of $\mathfrak{A}_\tau(\lambda)$ and therefore the kernel of $\mathcal{A}_{l,\gamma}$ is finite-dimensional. Since $\ker \mathcal{A}_{l,\beta} \subset \mathcal{A}_{l,\gamma}$, it follows that $\dim\ker \mathcal{A}_{l,\beta} < \infty$. Thus, we have shown that the kernel of $\mathcal{A}_{l,\beta}$ is finite-dimensional for arbitrary $l \ge 2m$, $\beta \in \mathbb{R}^d$.

Analogously the assertions of the lemma for the operators $\mathcal{A}^+_{l,\beta}$ and $\mathcal{A}^*_{l,\beta}$ hold. ∎

As a consequence of Lemma 6.3.6, we obtain the following theorem.

THEOREM 6.3.3. *Suppose that the boundary value problem (6.2.2), (6.2.3) is elliptic in $\overline{\mathcal{G}}\setminus\mathcal{S}$ and the operators L, B_k, $C_{k,j}$ are admissible. Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ on the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ for $\tau = 1, \dots, d$. Then the operator $\mathcal{A}_{l,\beta}$ is Fredholm. The kernel of $\mathcal{A}_{l,\beta}$ is the finite-dimensional spaces $\mathcal{N}_{\beta-l\vec{1}}$ defined before Lemma 6.3.6, while the range of $\mathcal{A}_{l,\beta}$ consists of all*

$$(f, \underline{g}) \in \tilde{V}^{l-2m,0}_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$$

satisfying the condition

$$(f, v)_{\mathcal{G}} + (\underline{g}, \underline{v})_{\partial\mathcal{G}} = 0 \qquad \text{for all } (v, \underline{v}) \in \mathcal{N}^*_{-\beta+(l-2m)\vec{1}}. \tag{6.3.30}$$

Proof: Condition (i) in the definition of the Fredholm property (see Definition 3.4.1) follows Lemma 6.3.6. It remains to show that the cokernel of $\mathcal{A}_{l,\beta}$ has finite dimension.

First let l be not greater than zero. Then $\mathcal{A}_{l,\beta}$ is adjoint to the operator

$$V_{2,-\beta}^{2m-l}(\mathcal{G}) \times V_{2,-\beta}^{-l+\underline{\mu}+1/2}(\partial\mathcal{G}) \ni (v, \underline{v}) \to \big(L^+ v,\ Pv + Q^+\underline{v},\ C^+\underline{v}\big)$$
$$\in V_{2,-\beta}^{-l}(\mathcal{G}) \times \Big(\prod_{j=1}^{2m} V_{2,-\beta}^{-l+j-1/2}(\partial\mathcal{G})\Big) \times V_{2,-\beta}^{-l-\underline{\tau}+1/2}(\partial\mathcal{G}).$$

(see Theorem 6.2.1). The kernel of the last operator coincides with the finite-dimensional set $\mathcal{N}^*_{-\beta+(l-2m)\vec{1}}$. Since $\mathcal{R}(\mathcal{A}_{l,\beta})$ is closed, the equation $\mathcal{A}_{l,\beta}(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$ is solvable in the space (6.3.10) if and only if condition (6.3.30) is satisfied for all $(v, \underline{v}) \in \mathcal{N}^*_{-\beta+(l-2m)\vec{1}}$. This proves the Fredholm property of the operator $\mathcal{A}_{l,\beta}$ for $l \le 0$.

We assume now that $l \ge 0$. Then we conclude from the regularity assertion in Lemma 6.3.1 that

$$\mathcal{R}(\mathcal{A}_{l,\beta}) = \mathcal{R}(\mathcal{A}_{0,\beta-l\vec{1}}) \cap \big(\tilde{V}_{2,\beta}^{l-2m,0}(\mathcal{G}) \times V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})\big).$$

Hence by the first part of the proof, the pair $(f, \underline{g})$ belongs to the range of $\mathcal{A}_{l,\beta}$ if and only if $f \in \tilde{V}_{2,\beta}^{l-2m,0}(\mathcal{G})$, $\underline{g} \in V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})$, and the condition (6.3.30) is satisfied for every element $(v, \underline{v})$ of the finite-dimensional space $\mathcal{N}^*_{-\beta+(l-2m)\vec{1}}$. This proves the Fredholm property of the operator $\mathcal{A}_{l,\beta}$ for $l \ge 0$. ■

Analogously the Fredholm property of the operator $\mathcal{A}^*_{l,\beta}$ and $\mathcal{A}^+_{l,\beta}$ holds.

THEOREM 6.3.4. *Let the assumptions of Theorem 6.3.3 be satisfied. Then the operator $\mathcal{A}^*_{l,\beta}$ is a Fredholm operator. The kernel of $\mathcal{A}^*_{l,\beta}$ is the finite-dimensional space $\mathcal{N}^*_{-\beta+(l-2m)\vec{1}}$, while the range of $\mathcal{A}^*_{l,\beta}$ consists of all*

$$(F, \underline{h}) \in D_{2,-\beta}^{-l,2m}(\mathcal{G}) \times V_{2,-\beta}^{-l-\underline{\tau}+1/2}(\partial\mathcal{G})$$

satisfying the condition

$$(u, F)_{\mathcal{G}} + (\underline{u}, \underline{h})_{\partial\mathcal{G}} = 0$$

for all $(u, \underline{u})$ such that $\big(u, \mathcal{D}u|_{\partial\mathcal{G}\setminus\mathcal{S}}, \underline{u}\big) \in \mathcal{N}_{\beta-l\vec{1}}$.

THEOREM 6.3.5. *Suppose that the boundary value problem (6.2.2), (6.2.3) is elliptic in $\overline{\mathcal{G}}\backslash\mathcal{S}$ and the operators L, B_k, $C_{k,j}$ are admissible. Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ on the line $\operatorname{Re}\lambda = \beta_\tau + 2m - l - n/2$ for $\tau = 1, \dots, d$. Then the operator $\mathcal{A}^+_{l,\beta}$ is Fredholm. The kernel of this operator is the space $\mathcal{N}^+_{\beta-l\vec{1}}$, while the range of $\mathcal{A}^+_{l,\beta}$ consists of all*

$$(f, \underline{g}, \underline{h}) \in V_{2,\beta}^{l-2m}(\mathcal{G}) \times \prod_{j=1}^{2m} V_{2,\beta}^{l-2m+j-1/2}(\partial\mathcal{G}) \times \prod_{j=1}^{J} V_{2,\beta}^{l-2m-\tau_j+1/2}(\partial\mathcal{G}).$$

such that

$$(f, u)_{\mathcal{G}} + (\underline{g}, \underline{\phi})_{\partial\mathcal{G}} + (\underline{h}, \underline{u})_{\partial\mathcal{G}} = 0$$

for all $(u, \underline{\phi}, \underline{u}) \in \mathcal{N}_{-\beta+(l-2m)\vec{1}}$.

REMARK 6.3.4. By Lemma 6.3.3, the condition on the eigenvalues of $\mathfrak{A}_\tau(\lambda)$ in Theorem 6.3.3 is necessary for the Fredholm property of the operator $\mathcal{A}_{l,\beta}$. If this condition is not satisfied, then the range of the operator $\mathcal{A}_{l,\beta}$ is not closed (cf. Lemma 3.4.1).

6.3.5. A property of the index. Now we study the dependence of the index of the operator $\mathcal{A}_{l,\beta}$ on l and β. By Lemma 6.3.6, the index depends only on the difference $\beta - l\vec{1}$. For this reason, we fix an integer number $l \geq 2m$ in what follows.

On the structure of the solution.

THEOREM 6.3.6. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible in a neighbourhood of the conical point $x^{(\tau)}$. Furthermore, we assume that $\beta = (\beta_1, \dots, \beta_d)$, $\gamma = (\gamma_1, \dots, \gamma_d)$ are tuples of real numbers such that $\gamma_\tau < \beta_\tau$ and the lines $\mathrm{Re}\,\lambda = -\beta_\tau + l - n/2$ and $\mathrm{Re}\,\lambda = -\gamma_\tau + l - n/2$ do not contain eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$, $\tau = 1, \dots, d$. Let $(u, \underline{u}) \in V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ be a solution of the boundary value problem (6.2.2), (6.2.3), where $(f, \underline{g})$ belongs to the space*

$$V^{l-2m}_{2,\gamma}(\mathcal{G}) \times V^{l-\underline{\mu}-1/2}_{2,\gamma}(\partial\mathcal{G}). \tag{6.3.31}$$

Then there is the decomposition

$$(u, \underline{u}) = \sum_{j=1}^{\kappa} c_j U_j + (w, \underline{w}) \tag{6.3.32}$$

in a neighbourhood of the conical point $x^{(\tau)}$, where c_j are constants, κ is the sum of the algebraic multiplicities of all eigenvectors of $\mathfrak{A}_\tau(\lambda)$ lying in the strip $-\beta_\tau + l - n/2 < \mathrm{Re}\lambda < -\gamma_\tau + l - n/2$, $(w, \underline{w})$ is an element of the space

$$V^l_{2,\gamma}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G}), \tag{6.3.33}$$

and U_j are elements of the space $V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ with support near $x^{(\tau)}$ which satisfy the homogeneous equations (6.2.2), (6.2.3) in a neighbourhood of $x^{(\tau)}$ and are linearly independent modulo (6.3.33).

Proof: Let ζ be an infinitely differentiable function with support in a sufficiently small neighbourhood of $x^{(\tau)}$ equal to unity near $x^{(\tau)}$. Then $\zeta(u, \underline{u})$ satisfies the equation

$$\mathcal{A}\,\zeta(u, \underline{u}) = (f^{(1)}, \underline{g}^{(1)}),$$

where $(f^{(1)}, \underline{g}^{(1)}) = \zeta(f, \underline{g}) + [\mathcal{A}, \zeta]\,(u, \underline{u})$ and $[\mathcal{A}, \zeta] = \mathcal{A}\zeta - \zeta\mathcal{A}$ denotes the commutator of $\mathcal{A}$ and ζ. By our assumption on ζ, the pair $(f^{(1)}, \underline{g}^{(1)})$ is an element of the space (6.3.31) with support in a neighbourhood of $x^{(\tau)}$. Passing to the coordinates (ω, t), where ω are coordinates on the unit sphere $|x - x^{(\tau)}| = 1$ and $t = \log|x - x^{(\tau)}|$, we get

$$\mathfrak{A}(t, \partial_t)\,\zeta(u, e^{-\tau_1 t}u_1, \dots, e^{-\tau_J t}u_J) = (e^{2mt}f^{(1)}, e^{\mu_1 t}g_1^{(1)}, \dots, e^{\mu_{m+J} t}g_{m+J}^{(1)}).$$

Here $\mathfrak{A}(t, \partial_t)$ denotes the operator of problem (6.2.10), (6.2.11). Since the coefficients of $\mathfrak{A}(t, \partial_t)$ stabilize at infinity, we can apply Theorem 5.5.3 (see also Remark

5.5.2). Thus, we obtain

$$\zeta\,(u,\underline{u}) = \sum_{j=1}^{\kappa} c_j U_j + (w,\underline{w}), \tag{6.3.34}$$

where $(w,\underline{w})$ is an element of

$$V^l_{2,\gamma_\tau}(\mathcal{K}_\tau) \times V^{l+\underline{\tau}-1/2}_{2,\gamma_\tau}(\partial\mathcal{K}_\tau) \tag{6.3.35}$$

and $U_1,\dots,U_\kappa$ are linearly independent modulo the space (6.3.35). Without loss of generality, we may assume that the supports of U_j and $(w,\underline{w})$ are contained in a neighbourhood of $x^{(\tau)}$. Otherwise, we multiply (6.3.34) by a function $\eta \in C_0^\infty(\overline{\mathcal{K}}_\tau)$ equal to unity on the support of ζ and obtain (6.3.34) with ηU_j, $\eta(w,\underline{w})$ instead of U_j, $(w,\underline{w})$. Since $(1-\eta)U_j$ belongs to the space (6.3.35) for $j=1,\dots,\kappa$, the vector-functions $\eta U_1,\dots,\eta U_\kappa$ are also linearly independent modulo the space (6.3.35). This proves the theorem. ∎

COROLLARY 6.3.1. *Let $\beta = (\beta_1,\dots,\beta_d)$, $\gamma = (\gamma_1,\dots,\gamma_d)$ be tuples of real numbers such that $\gamma_\tau \le \beta_\tau$ for $\tau = 1,\dots,d$. We suppose that the operators of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible and the lines $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$, $\operatorname{Re}\lambda = -\gamma_\tau + l - n/2$ do not contain eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ for $\tau = 1,\dots,d$. Then the homogeneous problem (6.2.2), (6.2.3) has not more than*

$$\kappa = \sum_{\tau=1}^{d} \kappa^{(\tau)}$$

solutions in $V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ which are linearly independent modulo the space (6.3.33). Here $\kappa^{(\tau)}$ denotes the sum of the algebraic multiplicities of all eigenvalues of $\mathfrak{A}_\tau(\lambda)$ lying in the strip $-\beta_\tau + l - n/2 < \operatorname{Re}\lambda < -\gamma_\tau + l - n/2$.

Proof: Let ζ_τ, $\tau = 1,\dots,d$, be infinitely differentiable functions equal to one in a neighbourhood of $x^{(\tau)}$ with sufficiently small supports. Then by (6.3.34), every solution $(u,\underline{u}) \in V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ of the homogeneous boundary value problem (6.2.2), (6.2.3) admits the decomposition

$$\zeta_\tau(u,\underline{u}) = \sum_{j=1}^{\kappa^{(\tau)}} c_j^{(\tau)}\, U_j^{(\tau)} + (w^{(\tau)}, \underline{w}^{(\tau)}),$$

where $U_j^{(\tau)}$, $j = 1,\dots,\kappa^{(\tau)}$, are linearly independent modulo the space (6.3.33) and $(w^{(\tau)}, \underline{w}^{(\tau)})$ are elements of the space (6.3.33). Consequently, we obtain

$$(u,\underline{u}) = \sum_{\tau=1}^{d}\sum_{j=1}^{\kappa^{(\tau)}} c_j^{(\tau)}\, U_j^{(\tau)} + (w,\underline{w}), \tag{6.3.36}$$

where $(w,\underline{w}) = \sum_{\tau=1}^{d} (w^{(\tau)}, \underline{w}^{(\tau)}) + (1-\zeta_1-\cdots-\zeta_d)\,(u,\underline{u})$ is an element of the space (6.3.33). This proves our assertion. ∎

Dependence of the index on β. Let $\mathcal{A}_{l,\beta}$ be the operator of the boundary value (6.2.2), (6.2.3) mapping the space (6.2.8) into (6.2.9). Furthermore, let

$\{Z_1, \dots, Z_q\} \subset V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$, $q \le \kappa$, be a maximal set of solutions of the homogeneous boundary value problem (6.2.2), (6.2.3) linearly independent modulo the space (6.3.33). Then every solution $(u, \underline{u}) \in V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ of the homogeneous problem (6.2.2), (6.2.3) satisfies a congruence

$$(u, \underline{u}) \equiv \sum_{j=1}^{q} c_j\, Z_j \, \Big(\bmod V^l_{2,\gamma}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})\Big).$$

The set $\{Z_1, \dots, Z_q\}$ is called a *basis in* $\ker \mathcal{A}_{l,\beta}$ *linearly independent modulo the space (6.3.33).*

As it was shown in the proof of Corollary 6.3.1, every element Z_j of this basis satisfies a congruence

$$Z_j \equiv \sum_{s=1}^{\kappa} c_{j,s}\, U_s \, \Big(\bmod V^l_{2,\gamma}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})\Big), \tag{6.3.37}$$

where the set $\{U_1, \dots, U_\kappa\}$ consists of the elements $U^{(\tau)}_j$ ($\tau = 1, \dots, d$, $j = 1, \dots, \kappa^{(\tau)}$) in (6.3.36) and $(c_{j,s})$ is a $q \times \kappa-$matrix with rank equal to q. It may be assumed that the matrix $(c_{j,s})$ has the form (D_1, D_2), where D_1 is a nondegenerate $q \times q-$matrix. Therefore,

$$D_1^{-1} \underline{Z} \equiv (I_q\,,\, D_1^{-1} D_2)\, \underline{U} \, \Big(\bmod V^l_{2,\gamma}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})\Big).$$

Here I_q is the identity matrix and $\underline{Z}$, $\underline{U}$ denote the vectors $(Z_1, \dots, Z_q)$ and $(U_1, \dots, U_\kappa)$, respectively. Consequently, with no loss of generality, we may assume that

$$Z_j \equiv U_j + \sum_{s=q+1}^{\kappa} c_{j,s} U_s \, \Big(\bmod V^l_{2,\gamma}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})\Big). \tag{6.3.38}$$

Any basis $\{Z_1, \dots, Z_q\}$ in $\ker \mathcal{A}_{l,\beta}$ linearly independent modulo the space (6.3.33) which has the representation (6.3.38) is called *canonical.* Henceforth, we assume that some canonical basis is given.

LEMMA 6.3.7. *Let the conditions of Corollary 6.3.1 be satisfied and let q be the maximal number of solutions of the homogeneous boundary value problem (6.2.2), (6.2.3) in $V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ which are linearly independent modulo the space (6.3.33). Then the equation $\mathcal{A}^*(v, \underline{v}) = 0$ has exactly $\kappa - q$ solutions in*

$$V^{l-2m}_{2,\gamma}(\mathcal{G})^* \times V^{-l+\underline{\mu}+1/2}_{2,-\gamma}(\partial\mathcal{G}) \tag{6.3.39}$$

which are linearly independent modulo

$$V^{l-2m}_{2,\beta}(\mathcal{G})^* \times V^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{G}). \tag{6.3.40}$$

Proof: Let $\{Z^+_j\}_{j=1,\dots,p}$ be a basis in $\ker \mathcal{A}^*_{l,\gamma}$ linearly independent modulo the space (6.3.40).

We show first that $p + q \le \kappa$. Obviously, there exists a system $\{\Phi_s\}_{s=1,\dots,p}$ of elements of the space (6.3.31) such that

$$(Z^+_j, \Phi_s) = \delta_{j,s} \text{ for } j, s = 1, \dots, p \ \text{ and } \ ((v, \underline{v}), \Phi_s) = 0 \ \text{ for all } (v, \underline{v}) \in \ker \mathcal{A}^*_{l,\beta},$$

where $(\cdot, \cdot)$ denotes the extension of the scalar product in $L_2(\mathcal{G}) \times L_2(\partial\mathcal{G})^{m+J}$ to the product of the spaces (6.3.40) and (6.2.9). The last condition implies the existence of solutions $W_s \in V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$, $s = 1, \dots, p$, of the equation $\mathcal{A}W_s = \Phi_s$ (see Theorem 6.3.3, Theorem 6.3.4). Suppose there exists a linear combination

$$W = \sum_{s=1}^{p} c_s\, W_s + \sum_{s=1}^{q} d_s\, Z_s$$

of $W_1, \dots, W_p, Z_1, \dots, Z_q$ which belongs to the space (6.3.33). Then we obtain

$$0 = (\mathcal{A}^* Z_j^+\,,\, W) = (Z_j^+\,, \mathcal{A}W) = \sum_{s=1}^{p} c_s\,(Z_j^+\,,\,\Phi_s) + \sum_{s=1}^{q} d_s\,(Z_j^+\,,\,\mathcal{A}Z_s) = c_j$$

for $j = 1, \dots, p$. Furthermore, from the linear independence of $Z_1, \dots, Z_q$ modulo the space (6.3.33) it follows that $d_1 = \dots = d_q = 0$. Thus, $W_1, \dots, W_p, Z_1, \dots, Z_q$ are linear independent modulo the space (6.3.33). However, by Theorem 6.3.6, every of the elements $W_1, \dots, W_p, Z_1, \dots, Z_q$ satisfies a congruence of the form (6.3.37). This implies $p + q \le \kappa$.

We suppose now that $p < \kappa - q$. Since $\mathcal{A}U_s = 0$ in a neighbourhood of the conical points (see Theorem 6.3.6), we can set

$$d_{j,s} = (\mathcal{A}U_s\,,\, Z_j^+) \qquad \text{for } j = 1, \dots, p,\ \ s = q+1, \dots, \kappa.$$

Let the vector $(c_{q+1}, \dots, c_\kappa) \in \mathbb{C}^{\kappa-q}$ be a nontrivial solution of the system of the equations

$$\sum_{s=q+1}^{\kappa} d_{j,s}\, c_s = 0 \qquad (j = 1, \dots, p).$$

Then $(u, \underline{u}) = c_{q+1}U_{q+1} + \dots + c_\kappa U_\kappa$ satisfies the condition

$$(\mathcal{A}(u, \underline{u})\,,\, Z_j^+) = 0 \qquad \text{for } j = 1, \dots, p.$$

Consequently, $(\mathcal{A}(u, \underline{u})\,,\,(v, \underline{v})) = 0$ for all $(v, \underline{v}) \in \ker \mathcal{A}^*_{l,\gamma}$. Therefore, by Theorems 6.3.3, 6.3.4, $\mathcal{A}(u, \underline{u})$ belongs to the range of $\mathcal{A}_{l,\gamma}$. This means that there exists an element $(w, \underline{w})$ of the space (6.3.33) such that $Z_{q+1} = (u, \underline{u}) - (w, \underline{w})$ is a solution of the equation $\mathcal{A}Z_{q+1} = 0$. However, then $Z_1, \dots, Z_{q+1}$ form a system of $q + 1$ solutions of the homogeneous problem (6.2.2), (6.2.3) which are linearly independent modulo the space (6.3.33). This contradicts our assumption. The proof is complete. ■

REMARK 6.3.5. By Theorem 6.3.2, the pair $(v, \underline{v})$ is a solution of the equation $\mathcal{A}^*(v, \underline{v}) = 0$ in the space (6.3.39) if and only if $(v, \underline{v})$ belongs to the space

$$V^{2m}_{2,-\gamma+l\vec{1}}(\mathcal{G}) \times V^{\underline{\mu}+1/2}_{2,-\gamma+l\vec{1}}(\partial\mathcal{G}) \tag{6.3.41}$$

and $(v, \underline{v})$ is a solution of the homogeneous formally adjoint problem (6.2.13), (6.2.14). Therefore, under the conditions of Lemma 6.3.7, the homogeneous problem (6.2.13), (6.2.14) has exactly $\kappa - q$ solutions in the space (6.3.41) which are linearly independent modulo $V^{2m}_{2,-\beta+l\vec{1}}(\mathcal{G}) \times V^{\underline{\mu}+1/2}_{2,-\beta+l\vec{1}}(\partial\mathcal{G})$.

Furthermore, the assertion of Lemma 6.3.7 leads to the following interesting consequence for the index of the operator $\mathcal{A}$.

THEOREM 6.3.7. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible in a neighbourhood of each conical* *point* $x^{(\tau)}$. *If the lines* $\operatorname{Re}\lambda = -\beta + l - n/2$, $\operatorname{Re}\lambda = -\gamma_\tau + l - n/2$ *do not contain eigenvalues of the pencil* $\mathfrak{A}_\tau(\lambda)$ *for* $\tau = 1, \dots, d$, *then*

$$\operatorname{ind}\mathcal{A}_{l,\beta} = \operatorname{ind}\mathcal{A}_{l,\gamma} + \kappa,$$

where $\kappa = \kappa^{(1)} + \cdots + \kappa^{(d)}$ *and* $\kappa^{(\tau)}$ *denotes the sum of the multiplicities of all eigenvalues of* $\mathfrak{A}_\tau(\lambda)$ *lying in the strip* $-\beta_\tau + l - n/2 < \operatorname{Re}\lambda < -\gamma_\tau + l - n/2$.

Proof: By Theorems 6.3.3 and 6.3.4, the index of $\mathcal{A}_{l,\beta}$ is given by the equality

$$\operatorname{ind}\mathcal{A}_{l,\beta} = \dim\ker\mathcal{A}_{l,\beta} - \dim\ker\mathcal{A}^*_{l,\beta}\,.$$

The analogous equality is valid for the index of the operator $\mathcal{A}_{l,\gamma}$. We set $q = \dim\ker\mathcal{A}_{l,\beta} - \dim\ker\mathcal{A}_{l,\gamma}$. Then from Lemma 6.3.7 it follows that

$$\dim\ker\mathcal{A}^*_{l,\gamma} = \dim\ker\mathcal{A}^*_{l,\beta} + (\kappa - q).$$

Consequently, we obtain

$$\operatorname{ind}\mathcal{A}_{l,\beta} = (q + \dim\ker\mathcal{A}_{l,\gamma}) - (\dim\ker\mathcal{A}^*_{l,\gamma} + q - \kappa) = \operatorname{ind}\mathcal{A}_{l,\gamma} + \kappa.$$

The theorem is proved. ∎

A regularity assertion for the solution. From Lemma 6.3.1, Theorem 6.3.3, and Theorem 6.3.6 we obtain the following two corollaries.

COROLLARY 6.3.2. *Suppose that the operators* L, B_k, $C_{k,j}$ *of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible in a neighbourhood of the conical points* $x^{(\tau)}$. *Furthermore, we assume that* $\beta = (\beta_1, \dots, \beta_d)$, $\gamma = (\gamma_1, \dots, \gamma_d)$ *are tuples of real numbers such that the closed strip between the lines* $\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$ *and* $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$ $\quad(l_1, l_2 \geq 2m)$ *does not contain eigenvalues of the pencil* $\mathfrak{A}_\tau(\lambda)$ *for* $\tau = 1, \dots, d$. *Then every solution* $(u, \underline{u}) \in V^{l_1}_{2,\beta}(\mathcal{G}) \times V^{l_1 + \underline{\tau} - 1/2}_{2,\beta}(\partial\mathcal{G})$ *of the boundary value problem (6.2.2), (6.2.3), where* $(f, \underline{g}) \in V^{l_2 - 2m}_{2,\gamma}(\mathcal{G}) \times V^{l_2 - \underline{\mu} - 1/2}_{2,\gamma}(\partial\mathcal{G})$, *belongs to the space* $V^{l_2}_{2,\gamma}(\mathcal{G}) \times V^{l_2 + \underline{\tau} - 1/2}_{2,\gamma}(\partial\mathcal{G})$.

COROLLARY 6.3.3. *Suppose that the operators* L, B_k, $C_{k,j}$ *of the elliptic boundary value problem (6.2.2), (6.2.3) are admissible in a neighbourhood of the conical points* $x^{(\tau)}$ *and the representation (6.2.4) is valid for the vector* B. *Furthermore, we assume that* $\beta = (\beta_1, \dots, \beta_d)$, $\gamma = (\gamma_1, \dots, \gamma_d)$ *are tuples of real numbers such that the closed strip between the lines* $\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$ *and* $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$, *where* $l_1, l_2 \geq 2m$, *does not contain eigenvalues of the pencil* $\mathfrak{A}_\tau(\lambda)$ *for* $\tau = 1, \dots, d$. *If the operator of the boundary value (6.2.2), (6.2.3) realizes an isomorphism*

$$\mathcal{A}_{l_1,\beta} \,:\, V^{l_1}_{2,\beta}(\mathcal{G}) \times V^{l_1 + \underline{\tau} - 1/2}_{2,\beta}(\partial\mathcal{G}) \to V^{l_1 - 2m}_{2,\beta}(\mathcal{G}) \times V^{l_1 - \underline{\mu} - 1/2}_{2,\beta}(\partial\mathcal{G}),$$

then it realizes an isomorphism

$$\mathcal{A}_{l_2,\gamma} \,:\, V^{l_2}_{2,\gamma}(\mathcal{G}) \times V^{l_2 + \underline{\tau} - 1/2}_{2,\gamma}(\partial\mathcal{G}) \to V^{l_2 - 2m}_{2,\gamma}(\mathcal{G}) \times V^{l_2 - \underline{\mu} - 1/2}_{2,\gamma}(\partial\mathcal{G}).$$

An analogous assertion is true if $l_1 < 2m$ *or* $l_2 < 2m$.

Proof: Under the assumptions on the eigenvalues of the pencil $\mathfrak{A}$, the kernels of the operators $\mathcal{A}_{l_1,\beta}$ and $\mathcal{A}_{l_2,\gamma}$ coincide. Furthermore, the sets $\mathcal{N}^*_{-\beta + (l_1 - 2m)\vec{1}}$ and $\mathcal{N}^*_{-\gamma + (l_2 - 2m)\vec{1}}$ coincide. Hence the cokernels of the operators $\mathcal{A}_{l_1,\beta}$ and $\mathcal{A}_{l_2,\gamma}$ have the same dimension. ∎

6.4. Asymptotics of the solution

In the last section we have proved a theorem on the structure of the solution for the case of admissible operators. In this section we give a more precise description of the functions U_j which appear in the decomposition (6.3.32). For this we need more restrictive assumptions on the coefficients of the operators in the boundary value problem (6.2.2), (6.2.3). The condition of admissibility is now replaced by the sharper condition of δ-admissibility.

Furthermore, we derive formulas for the coefficients in the asymptotics. Again $\mathcal{G}$ is a bounded domain with piecewise smooth boundary $\partial\mathcal{G}$ containing d conical points $x^{(1)}, \dots, x^{(d)}$. For simplicity, it will be assumed that for every conical point $x^{(\tau)}$ there exist a neighbourhood $\mathcal{U}_\tau$ and a cone $\mathcal{K}_\tau$ with vertex in $x^{(\tau)}$ such that $\mathcal{G} \cap \mathcal{U}_\tau = \mathcal{K}_\tau \cap \mathcal{U}_\tau$.

6.4.1. Decomposition of the solution of elliptic boundary value problems. The admissibility of the operators L, B_k and $C_{k,j}$ does not suffice to obtain a decomposition like (6.1.54) for the solution $(u, \underline{u})$ of the boundary value problem (6.2.2), (6.2.3). In the following, we consider the boundary value problem under the stronger condition of δ-admissibility defined below.

DEFINITION 6.4.1. Let δ be a positive real constant. The operator

$$P(x, \partial_x) = \sum_{|\alpha| \le k} p_\alpha(x)\, \partial_x^\alpha \tag{6.4.1}$$

is said to be a *δ-admissible operator of order k near the conical point* $x^{(\tau)}$ if the coefficients are infinitely differentiable in $\overline{\mathcal{G}}\backslash\mathcal{S}$ and in a neighbourhood of $x^{(\tau)}$ there is the representation

$$p_\alpha(x) = r^{|\alpha|-k} \Big(p_\alpha^{(0)}(\omega) + r^\delta\, p_\alpha^{(1)}(\omega, r) \Big),$$

where $r = |x - x^{(\tau)}|$, ω are coordinates on the unit sphere $|x - x^{(\tau)}| = 1$, $p_\alpha^{(0)} \in C^\infty(\overline{\Omega}_\tau)$, and $p_\alpha^{(1)}$ is an infinitely differentiable function in $\overline{\Omega}_\tau \times \mathbb{R}_+$ such that

$$|\,(r\partial_r)^j\, \partial_\omega^\gamma\, p_\alpha^{(1)}(\omega, r)| < c_{j,\gamma}\,, \qquad \omega \in \overline{\Omega}_\tau,\ r > 0, \tag{6.4.2}$$

for every multi-index γ and every integer $j \ge 0$. Here the constants $c_{j,\gamma}$ do not depend on ω and r.

Analogously, the δ-admissibility of tangential operators on $\partial\mathcal{G}\backslash\mathcal{S}$ is defined.

Example. Suppose the coefficients p_α of the operator (6.4.1) belong to $C^\infty(\overline{\mathcal{G}})$. Then the operator (6.4.1) is δ-admissible with $\delta = 1$. For $|\alpha| = k$ there is the representation

$$p_\alpha(x) = p_\alpha(x^{(\tau)}) + r\, p_\alpha^{(1)}(\omega, r), \qquad \text{where } p_\alpha^{(1)}(\omega, r) = r^{-1}\big(p_\alpha(x) - p_\alpha(x^{(\tau)})\big),$$

while for $|\alpha| < k$ we have

$$p_\alpha(x) = r^{|\alpha|-k}\big(0 + r\, p_\alpha^{(1)}(\omega, r)\big), \qquad \text{where } p_\alpha^{(1)}(\omega, r) = r^{k-1-|\alpha|}\, p_\alpha(x),$$

in a neighbourhood of $x^{(\tau)}$. It can be easily shown that the functions $p_\alpha^{(1)}$ satisfy condition (6.4.2).

If $P(x, \partial_x)$ is a δ-admissible operator of order k near $x^{(\tau)}$ and $P^{(\tau)}$ is the leading part of P at the point $x^{(\tau)}$ (see Definition 6.2.1), then there exists a constant c such that

(6.4.3) $$\|(P - P^{(\tau)})u\|_{V^{l-k}_{2,\beta_\tau-\delta}(\mathcal{K}_\tau)} \le c\,\|u\|_{V^l_{2,\beta_\tau}(\mathcal{K}_\tau)}$$

for all $u \in V^l_{2,\beta_\tau}(\mathcal{K}_\tau)$ with support in the neighbourhood $\mathcal{U}_\tau$ of $x^{(\tau)}$, $l \ge k$. This inequality sharpens the estimate given in Lemma 6.2.2. Clearly, an analogous assertion is valid for δ-admissible tangential differential operators on $\partial\mathcal{G}\backslash\mathcal{S}$.

Let $x^{(\tau)}$ be a fixed conical point of the boundary $\partial\mathcal{G}$ and let $\mathfrak{A}_\tau(\lambda)$ be the operator pencil of the parameter-depending boundary value problem (6.3.8), (6.3.9).The eigenvalues of this pencil are denoted by λ_μ. Furthermore, let

$$\{(\varphi^{(\mu)}_{j,s}, \underline{\varphi}^{(\mu)}_{j,s})\}_{j=1,\dots,I_\mu,\, s=0,\dots,\kappa_{\mu,j}-1}$$

be canonical systems of Jordan chains of $\mathfrak{A}_\tau(\lambda)$ corresponding to the eigenvalues λ_μ. We recall that the functions $\varphi^{(\mu)}_{j,s}$ and vector-functions

$$\underline{\varphi}^{(\mu)}_{j,s} = \big(\varphi^{(\mu)}_{1;j,s}, \dots, \varphi^{(\mu)}_{J;j,s}\big)$$

are infinitely differentiable in $\overline{\Omega}$ and on $\partial\Omega$, respectively.

As in Section 6.1, we introduce the functions

(6.4.4) $$u_{\mu,j,s} = r^{\lambda_\mu} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma\, \varphi^{(\mu)}_{j,s-\sigma}(\omega)$$

and vector-functions $\underline{u}_{\mu,j,s}$ with the components

(6.4.5) $$\underline{u}_{q;\mu,j,s} = r^{\lambda_\mu+\tau_q} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma\, \varphi^{(\mu)}_{q;j,s-\sigma}(\omega), \quad q = 1, \dots, J.$$

Then, as a consequence of Theorem 6.1.4, we obtain the following representation for the solution of the boundary value problem (6.2.2), (6.2.3) near the point $x^{(\tau)}$.

THEOREM 6.4.1. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (6.2.2), (6.2.3) are δ-admissible near the conical point $x^{(\tau)}$. Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ on the lines* $\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$ *and* $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$ *and the strip* $-\beta_\tau + l_1 - n/2 < \operatorname{Re}\lambda < -\gamma_\tau + l_2 - n/2$ *contains the eigenvalues $\lambda_1, \dots, \lambda_N$. Here l_1, l_2 are integer numbers, $l_1 \ge 2m$, $l_2 \ge 2m$, while β_τ, γ_τ are real numbers satisfying the inequalities*

$$0 < (l_2 - \gamma_\tau) - (l_1 - \beta_\tau) < \delta$$

Let η be an infinitely differentiable function with support in $\mathcal{U}_\tau$ equal to one in a neighbourhood of $x^{(\tau)}$. If $(u, \underline{u})$ is a solution of the boundary value problem (6.2.2), (6.2.3) such that $\eta\,(u, \underline{u}) \in V^{l_1}_{2,\beta_\tau}(\mathcal{K}_\tau) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta_\tau}(\partial\mathcal{K}_\tau)$ and $\eta(f, \underline{g})$ belongs to the space

(6.4.6) $$V^{l_2-2m}_{2,\gamma_\tau}(\mathcal{K}_\tau) \times V^{l_2-\underline{\mu}-1/2}_{2,\gamma_\tau}(\partial\mathcal{K}_\tau),$$

then there is the representation

(6.4.7) $$(u, \underline{u}) = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\,(u_{\mu,j,s}\,, \underline{u}_{\mu,j,s}) + (w, \underline{w})$$

in a neighbourhood of $x^{(\tau)}$. *Here* $c_{\mu,j,s}$ *are constants and*

$$(w,\underline{w}) \in V^{l_2}_{2,\gamma_\tau}(\mathcal{K}_\tau) \times V^{l_2+\underline{\tau}-1/2}_{2,\gamma_\tau}(\partial\mathcal{K}_\tau). \tag{6.4.8}$$

Proof: Without loss of generality, we may assume that $l_1 = l_2 = l$. Let ζ be an arbitrary smooth cut-off function equal to one in a neighbourhood of $x^{(\tau)}$ satisfying the equality $\zeta\eta = \zeta$. Then $\zeta(u,\underline{u})$ is a solution of the equation

$$\mathcal{A}_\tau\, \zeta(u,\underline{u}) = (f^{(1)}, \underline{g}^{(1)}),$$

where $\mathcal{A}_\tau$ denotes the operator of the model problem (6.3.6), (6.3.7) and

$$(f^{(1)}, \underline{g}^{(1)}) = \zeta\,(f,\underline{g}) + \zeta\,(\mathcal{A}_\tau - \mathcal{A})\,(u,\underline{u}) + \mathcal{A}_\tau\zeta(u,\underline{u}) - \zeta\mathcal{A}_\tau(u,\underline{u}).$$

Since the operators L, B_k and $C_{k,j}$ are δ-admissible, the term

$$\zeta(\mathcal{A}_\tau - \mathcal{A})(u,\underline{u}) = \zeta(\mathcal{A}_\tau - \mathcal{A})\eta(u,\underline{u})$$

belongs to the space $V^{l-2m}_{2,\beta_\tau-\delta}(\mathcal{K}_\tau) \times V^{l-\underline{\mu}-1/2}_{2,\beta_\tau-\delta}(\partial\mathcal{K})$ and, therefore, also to the space (6.4.6). The same assertion is true for the term $\mathcal{A}_\tau\zeta(u,\underline{u}) - \zeta\mathcal{A}_\tau(u,\underline{u})$, since this term is equal to zero in a neighbourhood of $x^{(\tau)}$. Hence we can apply Theorem 6.1.4 and obtain the representation (6.4.7). ∎

Using Theorem 6.1.5, we get an analogous assertion for the cases $l_1 \le 2m$, $l_2 \le 2m$.

6.4.2. Asymptotics for special right-hand sides. The asymptotic expansion in Theorem 6.4.1 is valid if the difference of $\beta_\tau - l_1$ and $\gamma_\tau - l_2$ does not exceed the number δ. This is a strong restriction even for differential operators with infinitely differentiable coefficients. To obtain an asymptotic decomposition of the form (6.4.7) without this restriction, we need more information on the structure of the differential operators L, B_k, and $C_{k,j}$.

We suppose now that the coefficients $a_\alpha(x)$ of L have the representation

$$a_\alpha(x) = r^{|\alpha|-2m}\Big(a^{(0)}_\alpha(\omega) + \sum_{\iota=1}^{s} r^{\delta_\iota}\, a^{(\iota)}_\alpha(\omega,\log r) + r^{\delta_{s+1}} a^{(s+1)}_\alpha(\omega,r)\Big) \tag{6.4.9}$$

in the neighbourhood $\mathcal{U}_\tau$ of the conical point $x^{(\tau)}$, where $\delta_1, \delta_2, \dots, \delta_{s+1}$ is a given sequence of complex numbers such that

$$0 < \operatorname{Re}\delta_1 \le \operatorname{Re}\delta_2 \le \ldots \le \operatorname{Re}\delta_s \le (\beta_\tau - l_1) - (\gamma_\tau - l_2) \le \operatorname{Re}\delta_{s+1}\,,$$

$a^{(0)}_\alpha \in C^\infty(\overline{\Omega}_\tau)$, $a^{(1)}_\alpha, \dots, a^{(s)}_\alpha$ are polynomials of $\log r$ with coefficients in $C^\infty(\overline{\Omega_\tau})$, and $a^{(s+1)}_\alpha$ is an infinitely differentiable function in $\overline{\Omega}_\tau \times \mathbb{R}_+$ which satisfies the condition

$$|\,(r\partial_r)^\mu\, \partial^\gamma_\omega\, a^{(s+1)}_\alpha(\omega,r)| < c_{\mu,\gamma} \tag{6.4.10}$$

for every integer $\mu \ge 0$ and every multi-index γ, where the constants $c_{\mu,\gamma}$ are independent of ω and r.

Analogous assumptions are imposed on the coefficients $b_{k;\alpha}$, $c_{k,j;\alpha}$ of B_k and $C_{k,j}$, i.e.,

$$b_{k;\alpha}(x) = r^{|\alpha|-\mu_k}\Big(b^{(0)}_{k;\alpha}(\omega) + \sum_{\iota=1}^{s} r^{\delta_\iota}\, b^{(\iota)}_{k;\alpha}(\omega,\log r) + r^{\delta_{s+1}} b^{(s+1)}_{k;\alpha}(\omega,r)\Big) \tag{6.4.11}$$

(6.4.12)

$$c_{k,j;\alpha}(x) = r^{|\alpha|-\mu_k-\tau_j}\Big(c^{(0)}_{k,j;\alpha}(\omega) + \sum_{\iota=1}^{s} r^{\delta_\iota}\, c^{(\iota)}_{k,j;\alpha}(\omega,\log r) + r^{\delta_{s+1}} c^{(s+1)}_{k,j;\alpha}(\omega,r)\Big),$$

where $b^{(\iota)}_{k;\alpha}$, $c^{(\iota)}_{k,j;\alpha}$ satisfy the same conditions as $a^{(\iota)}_\alpha$.

Note that the operators L, B_k, $C_{k,j}$ with the properties (6.4.9)–(6.4.12) are $(\operatorname{Re}\delta_1 - \varepsilon)$-admissible for arbitrary $\varepsilon > 0$.

Example. Let the coefficients a_α of L be infinitely differentiable in $\overline{\mathcal{G}}$. Then there is the representation

$$a_\alpha(x) = r^{|\alpha|-2m}\Big(\sum_{\iota=2m-|\alpha|}^{s} r^\iota\, a^{(\iota)}_\alpha(\omega) + r^{s+1}\, a^{(s+1)}_\alpha(\omega,r)\Big),$$

where $a^{(\iota)}_\alpha \in C^\infty(\overline{\Omega}_\tau)$ for $\iota = 2m-|\alpha|,\dots,s$ and $a^{(s+1)}_\alpha$ satisfies the condition (6.4.10). (In the case $|\alpha| \le 2m-s-1$ the terms $r^\iota\, a^{(\iota)}_\alpha(\omega)$ do not appear.) Consequently, the operator L has the above property with the numbers $\delta_\iota = \iota$ $(\iota = 1,\dots,s+1)$.

Furthermore, we suppose that the functions f and g_k on the right-hand side of the boundary value problem (6.2.2), (6.2.3) have the representations

$$f(x) = r^{-2m}\sum_{\iota=1}^{q} r^{\sigma_\iota} f^{(\iota)}(\omega,\log r) + f^{(q+1)}(x)\,, \tag{6.4.13}$$

$$g_k(x) = r^{-\mu_k}\sum_{\iota=1}^{q} r^{\sigma_\iota} g^{(\iota)}_k(\omega,\log r) + g^{(q+1)}_k(x), \quad k = 1,\dots,m+J, \tag{6.4.14}$$

in the neighbourhood $\mathcal{U}_\tau$ of $x^{(\tau)}$, where $\sigma_1,\dots,\sigma_q$ are complex numbers satisfying the condition

$$-\beta_\tau + l_1 - n/2 \le \operatorname{Re}\sigma_1 \le \ldots \le \operatorname{Re}\sigma_q \le -\gamma_\tau + l_2 - n/2\,,$$

$f^{(q+1)} \in V^{l_2-2m}_{2,\gamma_\tau}(\mathcal{K}_\tau)$, $g^{(q+1)}_k \in V^{l_2-\mu_k-1/2}_{2,\gamma_\tau}(\partial\mathcal{K}_\tau)$, and $f^{(\iota)}$, $g^{(\iota)}_k$ $(\iota = 1,\dots,q)$ are polynomials of $\log r$ with coefficients from $W^{l_2-2m}_2(\Omega_\tau)$ and $W^{l_2-\mu_k-1/2}_2(\partial\Omega_\tau)$, respectively.

For an arbitrary given complex number λ_0 we denote the set of all sums

$$\delta = \delta_{\iota_1} + \cdots + \delta_{\iota_k}$$

formed by the numbers δ_ι in (6.4.9) such that

$$\operatorname{Re}(\delta+\lambda_0) \le -\gamma_\tau + l_2 - n/2$$

by $\Lambda(\lambda_0)$. Then the following theorem holds.

THEOREM 6.4.2. *Suppose the boundary value problem (6.2.2), (6.2.3) is elliptic in* $\overline{\mathcal{G}}\backslash\mathcal{S}$ *and the coefficients of* L, B_k, $C_{k,j}$ *have the representations (6.4.9)–(6.4.12) in the neighbourhood* $\mathcal{U}_\tau$ *of the conical point* $x^{(\tau)}$. *Furthermore, we assume that the function* f *and the vector-function* $\underline{g} = (g_1,\dots,g_{m+J})$ *admit the decompositions (6.4.13), (6.4.14) in* $\mathcal{U}_\tau$ *and the lines* $\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$, $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$ *do not contain eigenvalues of the operator pencil* $\mathfrak{A}_\tau(\lambda)$.

Then every solution $(u,u_1,\dots,u_J)\in V^{l_1}_{2,\beta}(\mathcal{G})\times V^{l_1+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ *of the boundary value problem (6.2.2), (6.2.3) has the representation*

$$u=\sum_{\mu=1}^{N}\sum_{\delta\in\Lambda(\lambda_\mu)} r^{\lambda_\mu+\delta}\,u^{(1)}_{\mu,\delta}(\omega,\log r)+\sum_{\iota=1}^{q}\sum_{\delta\in\Lambda(\sigma_\iota)} r^{\sigma_\iota+\delta}u^{(2)}_{\mu,\delta}(\omega,\log r)+w\,,$$

$$u_j=\sum_{\mu=1}^{N}\sum_{\delta\in\Lambda(\lambda_\mu)} r^{\lambda_\mu+\tau_j+\delta}\,u^{(1)}_{j;\mu,\delta}(\omega,\log r)+\sum_{\iota=1}^{q}\sum_{\delta\in\Lambda(\sigma_\iota)} r^{\tau_j+\sigma_\iota+\delta}u^{(2)}_{j;\mu,\delta}(\omega,\log r)+w_j$$

in a neighbourhood of $x^{(\tau)}$. *Here* $\lambda_1,\dots,\lambda_N$ *are the eigenvalues of* $\mathfrak{A}_\tau(\lambda)$ *in the strip* $-\beta_\tau+l_1-n/2<\mathrm{Re}\,\lambda<-\gamma_\tau+l_2-n/2$, $w\in V^{l_2}_{2,\gamma_\tau}(\mathcal{K}_\tau)$, $w_j\in V^{l_2+\tau_j-1/2}_{2,\gamma_\tau}(\partial\mathcal{K}_\tau)$, *and* $u^{(k)}_{\mu,\delta}$, $u^{(k)}_{j;\mu,\delta}$ $(k=1,\,2)$ *are polynomials of* $\log r$ *with coefficients from* $W^{l_2}_2(\Omega_\tau)$ *and* $W^{l_2+\tau_j-1/2}_2(\partial\Omega_\tau)$, *respectively.*

Proof: Due to Lemma 6.3.1, it suffices to prove the theorem for $l_1=l_2=l$. Let $\mathcal{A}_\tau$ be the operator of the model problem (6.3.6), (6.3.7) in the cone $\mathcal{K}_\tau$ and let ζ be an arbitrary infinitely differentiable function with support in $\mathcal{U}_\tau$ which is equal to one in a neighbourhood of $x^{(\tau)}$. By Lemma 6.1.13, there exists a solution $(v,\underline{v})=(v,v_1,\dots,v_J)$ of the equation

$$\mathcal{A}_\tau\,(v,\underline{v})=(f,\underline{g})-(f^{(q+1)},\underline{g}^{(q+1)})$$

which has the form

$$v=\sum_{\iota=1}^{q} r^{\sigma_\iota}\,v^{(\iota)}(\omega,\log r)\,,\qquad v_j=\sum_{\iota=1}^{q} r^{\tau_j+\sigma_\iota}\,v^{(\iota)}_j(\omega,\log r),$$

where $v^{(\iota)}$, $v^{(\iota)}_j$ are polynomials of $\log r$ with coefficients from $W^l_2(\Omega_\tau)$ and $W^l_2(\partial\Omega_\tau)$, respectively. Then $\zeta((u,\underline{u})-(v,\underline{v}))$ satisfies the equation

$$\text{(6.4.15)}\quad \mathcal{A}_\tau\,\zeta\big((u,\underline{u})-(v,\underline{v})\big)\;=\;\zeta(f^{(q+1)},\underline{g}^{(q+1)})+\zeta(\mathcal{A}-\mathcal{A}_\tau)(u,\underline{u})+[\mathcal{A}_\tau,\zeta]\,\big((u,\underline{u})-(v,\underline{v})\big).$$

Here $[\mathcal{A}_\tau,\zeta]=\mathcal{A}_\tau\zeta-\zeta\mathcal{A}_\tau$ denotes the commutator of $\mathcal{A}_\tau$ and ζ. Under the given assumptions on the coefficients of L, B_k, $C_{k,j}$, the right side of (6.4.15) belongs to the space

$$V^{l-2m}_{2,\beta_\tau-Re\,\delta_1+\varepsilon}(\mathcal{K}_\tau)\times V^{l-\underline{\mu}-1/2}_{2,\beta_\tau-Re\,\delta_1+\varepsilon}(\partial\mathcal{K}_\tau)$$

with an arbitrary small positive ε which can be chosen such that the line $\mathrm{Re}\,\lambda=-(\beta_\tau-\mathrm{Re}\,\delta_1+\varepsilon)+l-n/2$ does not contain eigenvalues of $\mathfrak{A}_\tau(\lambda)$. Applying Theorem 6.1.4, we get

$$\zeta(u-v)\;=\;\sum_\mu r^{\lambda_\mu}\,w^{(\mu)}(\omega,\log r)+w\,,$$

$$\zeta(u_j-v_j)\;=\;\sum_\mu r^{\lambda_\mu+\tau_j}\,w^{(\mu)}_j(\omega,\log r)+w_j\,,\quad j=1,\dots,J,$$

where λ_μ are the eigenvalues of $\mathfrak{A}_\tau(\lambda)$ which are situated in the strip

$$-\beta_\tau+l-n/2<\mathrm{Re}\,\lambda<-(\beta_\tau-\mathrm{Re}\,\delta_1+\varepsilon)+l-n/2$$

and $w^{(\mu)}$, $w_j^{(\mu)}$ are polynomials of $\log r$ with coefficients in the spaces $W_2^l(\Omega_\tau)$ and $W_2^{l+\tau_j-1/2}(\partial\Omega_\tau)$, respectively. Furthermore, $(w, \underline{w})$ is a solution of the equation

$$\mathcal{A}_\tau\,(w, \underline{w}) = \mathcal{A}_\tau\,\zeta\,((u, \underline{u}) - (v, \underline{v})).$$

Thus, in a neighbourhood of $x^{(\tau)}$ the equation

$$\begin{aligned}\mathcal{A}(w, \underline{w}) &= \mathcal{A}(u, \underline{u}) - \mathcal{A}_\tau(v, \underline{v}) - (\mathcal{A} - \mathcal{A}_\tau)\,((u, \underline{u}) - (w, \underline{w})) \\ &= (f^{(q+1)}, \underline{g}^{(q+1)}) - (\mathcal{A} - \mathcal{A}_\tau)\,((u, \underline{u}) - (w, \underline{w}))\end{aligned}$$

holds. By our assumption on the coefficients of the operator L,

$$(L - L^{(\tau)})\,(u - w) = (L - L^{(\tau)})\,\Big(v + \sum_\mu r^{\lambda_\mu} w^{(\mu)}\Big)$$

is a finite sum of terms of the form

$$r^{\sigma_j + \delta_\iota - 2m} z(\omega, \log r) \quad \text{and} \quad r^{\lambda_\mu + \delta_\iota - 2m} z(\omega, \log r),$$

where z is a polynomial of $\log r$ with coefficients in $W_2^{l-2m}(\Omega_\tau)$. Analogous representations hold for $(B_k - B_k^{(\tau)})(u - w)$ and $(C_{k,j} - C_{k,j}^{(\tau)})(\underline{u} - \underline{w})$.

Repeating this procedure with $(w, \underline{w})$ instead of $(u, \underline{u})$, we get the assertion of the theorem in a finite number of steps. ■

As a special case we consider the boundary value problem (6.2.2), (6.2.3) for operators with coefficients in $C^\infty(\overline{\mathcal{G}})$. In this case $\delta_\iota = \iota$ for $\iota = 1, \dots, s$ and, therefore, the following assertion holds.

COROLLARY 6.4.1. *Suppose that the coefficients of the operators L, B_k, $C_{k,j}$ are infinitely differentiable in $\overline{\mathcal{G}}$ and the boundary value problem (6.2.2), (6.2.3) is elliptic. Furthermore, we assume that the lines* $\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$ *and* $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$ *do not contain eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$. If*

$$\eta\,(f, \underline{g}) \in V_{2,\gamma_\tau}^{l_2-2m}(\mathcal{K}_\tau) \times V_{2,\gamma_\tau}^{l_2-\underline{\mu}-1/2}(\partial\mathcal{K}_\tau),$$

where η is a smooth cut-off function with support in $\mathcal{U}_\tau$ which is equal to one near $x^{(\tau)}$, then the solution $(u, u_1, \dots, u_J) \in V_{2,\beta}^{l_1}(\mathcal{G}) \times V_{2,\beta}^{l_1+\underline{\tau}-1/2}(\partial\mathcal{G})$ of the boundary value (6.2.2), (6.2.3) admits the decomposition

$$\begin{aligned}u &= \sum_{\mu=1}^N \sum_k r^{\lambda_\mu + k}\, u_{\mu,k}(\omega, \log r) + w, \\ u_j &= \sum_{\mu=1}^N \sum_k r^{\lambda_\mu + \tau_j + k}\, u_{j;\mu,k}(\omega, \log r) + w_j\,, \qquad j = 1, \dots, J,\end{aligned}$$

in a neighbourhood of $x^{(\tau)}$. Here λ_μ are the eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ in the strip $-\beta_\tau + l_1 - n/2 < \operatorname{Re}\lambda < -\gamma_\tau + l_2 - n/2$, *the summation in both formulas is extended over the set of all nonnegative integer* $k \le -\gamma_\tau + l - n/2 - \operatorname{Re}\lambda_\mu$, $w \in V_{2,\gamma_\tau}^{l_2}(\mathcal{K}_\tau)$, $w_j \in V_{2,\gamma_\tau}^{l_2+\tau_j-1/2}(\partial\mathcal{K}_\tau)$, *and $u_{\mu,k}$, $u_{j;\mu,k}$ are polynomials of* $\log r$ *with coefficients from $W_2^{l_2}(\Omega_\tau)$ and $W_2^{l_2+\tau_j-1/2}(\partial\Omega_\tau)$, respectively.*

6.4.3. Formulas for the coefficients in the asymptotics. Now we derive formulas for the coefficients $c_{\mu,j,s}$ in Theorem 6.4.1. While the singular functions $u_{\mu,j,s}$, $\underline{u}_{\mu,j,s}$ in (6.4.7) depend only on the operators L, B_k, $C_{k,j}$ and on the geometry of the domain near the conical point $x^{(\tau)}$, the coefficients $c_{\mu,j,s}$ depend also on the functions f and g_k on the right-hand side of the boundary value problem (6.2.2), (6.2.3). We show that they are linear functionals of f and $\underline{g}$ and give a representation of these functionals.

A formula for the coefficients which contains the solution of the boundary value problem. Let λ_μ be the eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ and let

$$\{(\varphi_{j,s}^{(\mu)}, \underline{\varphi}_{j,s}^{(\mu)})\}_{j=1,\dots,I_\mu,\, s=0,1,\dots,\kappa_{\mu,j}-1}$$

be the canonical systems of Jordan chains of $\mathfrak{A}_\tau(\lambda)$ corresponding to the eigenvalue λ_μ which were introduced in the beginning of this section. By $\mathfrak{A}_\tau^+(\lambda)$ we denote the formally adjoint operator to $\mathfrak{A}_\tau(\overline{\lambda})$, i.e., the operator of the formally adjoint boundary value problem to (6.3.8), (6.3.9) (cf. (6.1.36)–(6.1.38)). Furthermore, let

$$\{(\psi_{j,s}^{(\mu)}, \underline{\psi}_{j,s}^{(\mu)})\} = \{(\psi_{j,s}^{(\mu)}, \psi_{1;j,s}^{(\mu)}, \dots, \psi_{m+J;j,s}^{(\mu)})\}$$

be canonical systems of Jordan chains of $\mathfrak{A}_\tau^+(\lambda)$ corresponding to the eigenvalues $\overline{\lambda}_\mu$ such that the biorthonormality condition (cf. (5.1.6))

$$\begin{aligned}(6.4.16)\quad &\sum_{p=0}^{\sigma}\sum_{q=p+1}^{p+s+1}\frac{1}{q!}\langle\mathfrak{A}_\tau^{(q)}(\lambda_\mu)\,(\varphi_{j,p+s+1-q}^{(\mu)}, \underline{\varphi}_{j,p+s+1-q}^{(\mu)}),\, (\psi_{k,\sigma-p}^{(\mu)}, \underline{\psi}_{k,\sigma-p}^{(\mu)})\rangle_2\\ &= \delta_{j,k}\,\delta_{s,\kappa_{\mu,k}-1-\sigma}\end{aligned}$$

is satisfied for $j = 1,\dots,I_\mu$, $s = 0,\dots,\kappa_{\mu,j}-1$, $\sigma = 0,\dots,\kappa_{\mu,k}-1$. Here $\langle\cdot,\cdot\rangle_2$ denotes the scalar product in $L_2(\Omega_\tau)\times L_2(\partial\Omega_\tau)^{m+J}$. By Lemma 6.1.11, the functions

$$(6.4.17)\qquad v_{\mu,j,s} = -r^{-\overline{\lambda}_\mu+2m-n}\sum_{\sigma=0}^{s}\frac{1}{\sigma!}(-\log r)^\sigma\,\psi_{j,s-\sigma}^{(\mu)}(\omega)$$

and the vector-functions $\underline{v}_{\mu,j,s}$ with the components

(6.4.18)

$$v_{k;\mu,j,s} = -r^{-\overline{\lambda}_\mu+\mu_k+1-n}\sum_{\sigma=0}^{s}\frac{1}{\sigma!}(-\log r)^\sigma\,\psi_{k;j,s-\sigma}^{(\mu)}(\omega),\quad k=1,\dots,m+J,$$

are solutions of the equation

$$\mathcal{A}_\tau^+\,(v_{\mu,j,s}, \underline{v}_{\mu,j,s}) = 0\,,$$

where $\mathcal{A}_\tau^+$ is the operator of the formally adjoint model problem to (6.3.6), (6.3.7).

THEOREM 6.4.3. *Let $\mathcal{U}_\tau$ be a neighbourhood of the conical point $x^{(\tau)}$, and let ζ, η be smooth cut-off functions with support in $\mathcal{U}_\tau$ which are equal to one in a neighbourhood of $x^{(\tau)}$ and satisfy the equality $\zeta\eta = \zeta$.*

We suppose that the conditions of Theorem 6.4.1 are satisfied and $(v, \underline{v})$ is a solution of the equations

$$\begin{aligned}&L^+ v = 0 \quad \textit{in } \mathcal{G}\cap\mathcal{U}_\tau,\\ &Pv + Q^+\underline{v} = 0,\quad C^+\underline{v} = 0 \quad \textit{on } \partial\mathcal{G}\cap\mathcal{U}_\tau\backslash\{x^{(\tau)}\}\end{aligned}$$

which has the form

$$(v, \underline{v}) = (v_{\mu,j,\kappa_{\mu,j}-1-s}, \underline{v}_{\mu,j,\kappa_{\mu,j}-1-s}) + (v^{(1)}, \underline{v}^{(1)}), \tag{6.4.19}$$

where $\eta(v^{(1)}, \underline{v}^{(1)}) \in V^{2m}_{2,-\beta_\tau+l_1}(\mathcal{K}_\tau) \times V^{\underline{\mu}+1/2}_{2,-\beta_\tau+l_1}(\partial\mathcal{K}_\tau)$. *Then the constants* $c_{\mu,j,s}$ *in Theorem 6.4.1 are determined by the equality*

$$c_{\mu,j,s} = \big(L(\zeta u), v\big)_{\mathcal{G}} + \big(B(\zeta u) + C(\zeta \underline{u})\,,\, \underline{v}\big)_{\partial\mathcal{G}}\,. \tag{6.4.20}$$

Proof: Let ε be a sufficiently small positive real number and $\zeta_\varepsilon(x) = \zeta(\varepsilon^{-1}x)$, $\eta_\varepsilon(x) = \eta(\varepsilon^{-1}x)$. By the Green formula (6.2.12), we have

$$\big(L(\zeta_\varepsilon - \zeta)u\,,\, v\big)_{\mathcal{G}} + \big(B(\zeta_\varepsilon - \zeta)u + C(\zeta_\varepsilon - \zeta)\underline{u}\,,\, \underline{v}\big)_{\partial\mathcal{G}} = 0.$$

Hence the right-hand side of (6.4.20) is equal to

$$(L(\zeta_\varepsilon u), v)_{\mathcal{G}} + (B(\zeta_\varepsilon u) + C(\zeta_\varepsilon \underline{u})\,,\, \underline{v})_{\partial\mathcal{G}}\,.$$

We write this expression in the form

$$\begin{aligned}
&\Big(L^{(\tau)}(\zeta_\varepsilon u), v_{\mu,j,\kappa_{\mu,j}-1-s}\Big)_{\mathcal{K}_\tau} + \Big(B^{(\tau)}(\zeta_\varepsilon u) + C^{(\tau)}(\zeta_\varepsilon \underline{u}), \underline{v}_{\mu,j,\kappa_{\mu,j}-1-s}\Big)_{\partial\mathcal{K}_\tau}\\
&+ \Big((L - L^{(\tau)})(\zeta_\varepsilon u), v\Big)_{\mathcal{K}_\tau} + \Big((B - B^{(\tau)})(\zeta_\varepsilon u) + (C - C^{(\tau)})(\zeta_\varepsilon \underline{u}), \underline{v}\Big)_{\partial\mathcal{K}_\tau}\\
&+ \Big(L^{(\tau)}(\zeta_\varepsilon u), v^{(1)}\Big)_{\mathcal{K}_\tau} + \Big(B^{(\tau)}(\zeta_\varepsilon u) + C^{(\tau)}(\zeta_\varepsilon \underline{u}), \underline{v}^{(1)}\Big)_{\partial\mathcal{K}_\tau}
\end{aligned} \tag{6.4.21}$$

By Theorem 6.1.6, the sum of the first two terms is equal to $c_{\mu,j,s}$. Furthermore, it follows from the δ-admissibility of L, that the third term does not exceed the quantity

$$c\,\|\zeta_\varepsilon u\|_{V^{l_1}_{2,\beta_\tau}(\mathcal{K}_\tau)} \cdot \|\eta v\|_{V^{2m}_{2,-\gamma_\tau+l_2}(\mathcal{K}_\tau)}$$

(cf. inequality (6.4.3)). Consequently, the third and analogously the fourth term in (6.4.21) tend to zero as $\varepsilon \to 0$. Similarly, the fifth term is majorized by

$$c\,\|\zeta_\varepsilon u\|_{V^{l_1}_{2,\beta_\tau}(\mathcal{K}_\tau)} \cdot \|\eta v^{(1)}\|_{V^{2m}_{2,-\beta_\tau+l_1}(\mathcal{K}_\tau)}\,.$$

Therefore, the fifth and analogously the last term in (6.4.21) also tend to zero. This proves the theorem. ■

Representation of the coefficients by the right-hand sides of the boundary value problem. Suppose that

$$(f, \underline{g}) \in V^{l_2-2m}_{2,\gamma}(\mathcal{G}) \times V^{l_2-\underline{\mu}-1/2}_{2,\gamma}(\partial\mathcal{G}),$$

where $\gamma = (\gamma_1, \dots, \gamma_d)$ satisfies the condition $\beta_\tau - l_1 - \delta \le \gamma_\tau - l_2 \le \beta_\tau - l_1$. If the assumptions of Theorem 6.4.1 are satisfied for each index $\tau = 1, \dots, d$, then every solution $(u, \underline{u}) \in V^{l_1}_{2,\beta}(\mathcal{G}) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta}(\mathcal{G})$ of the boundary value problem (6.2.2), (6.2.3) admits the representation

$$(u, \underline{u}) = \sum_{\tau=1}^{d} \sum_{\mu=1}^{N_\tau} \sum_{j=1}^{I_{\tau,\mu}} \sum_{s=0}^{\kappa^{(\tau)}_{\mu,j}-1} c^{(\tau)}_{\mu,j,s}\, \zeta_\tau\, (u^{(\tau)}_{\mu,j,s}, \underline{u}^{(\tau)}_{\mu,j,s}) + (w, \underline{w})$$

with a function $w \in V^{l_2}_{2,\gamma}(\mathcal{G})$ and a vector-function $\underline{w} \in V^{l_2+\underline{\tau}-1/2}_{2,\gamma}(\mathcal{G})$. Here ζ_τ are smooth cut-off functions equal to one in a neighbourhood of $x^{(\tau)}$ with support in $\mathcal{U}_\tau$, and $u^{(\tau)}_{\mu,j,s}$, $\underline{u}^{(\tau)}_{\mu,j,s}$ are functions of the form (6.4.4) and (6.4.5), respectively.

Let $U_1, \dots, U_\kappa$ be an ordered set of the elements

$$\zeta_\tau\,(u^{(\tau)}_{\mu,j,s}, \underline{u}^{(\tau)}_{\mu,j,s})$$

in this asymptotic decomposition. Furthermore, let $V_1, \dots, V_\kappa$ be an ordered set of the corresponding elements

$$\zeta_\tau\,(v^{(\tau)}_{\mu,j,s}, \underline{v}^{(\tau)}_{\mu,j,s})$$

defined by (6.4.17), (6.4.18). We say that the sets $\{U_j\}$, $\{V_j\}$ are *ordered compatibly* if

$$U_j = \zeta_\tau\,(u^{(\tau)}_{\mu,j,s}, \underline{u}^{(\tau)}_{\mu,j,s}) \quad \text{simultaneously with} \quad V_j = (v^{(\tau)}_{\mu,j,\kappa_{\mu,j}-1-s}, \underline{v}^{(\tau)}_{\mu,j,\kappa_{\mu,j}-1-s}).$$

LEMMA 6.4.1. *Suppose the conditions of Theorem 6.4.1 are satisfied for each $\tau = 1, \dots, d$ and the sets $\{U_j\}$, $\{V_j\}$ are ordered compatibly. Furthermore, we suppose that $\{Z_j\}_{j=1,\dots,q}$ $(0 \le q \le \kappa)$ is a canonical basis in* $\ker \mathcal{A}_{l_1,\beta}$ *linearly independent modulo the space $V^{l_2}_{2,\gamma}(\mathcal{G}) \times V^{l_2+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})$ (see Section 6.3) satisfying the congruence*

$$Z_j \equiv U_j + \sum_{s=q+1}^{\kappa} c_{j,s}\, U_s \,\Big(\text{mod } V^{l_2}_{2,\gamma}(\mathcal{G}) \times V^{l_2+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})\Big). \tag{6.4.22}$$

Then there exist solutions

$$Z_s^+ \in V^{2m}_{2,-\gamma+l_2\vec{1}}(\mathcal{G}) \times V^{\underline{\mu}+1/2}_{2,-\gamma+l_2\vec{1}}(\partial\mathcal{G}), \quad s = q+1, \dots, \kappa, \tag{6.4.23}$$

of the homogeneous formally adjoint boundary value problem (6.2.13), (6.2.14) satisfying the congruences

$$Z_s^+ \equiv V_s - \sum_{j=1}^{q} \bar{c}_{j,s}\, V_j \,\Big(\text{mod } V^{2m}_{2,-\beta+l_1\vec{1}}(\mathcal{G}) \times V^{\underline{\mu}+1/2}_{2,-\beta+l_1\vec{1}}(\partial\mathcal{G})\Big) \tag{6.4.24}$$

for $s = q+1, \dots, \kappa$.

Proof: By Lemma 6.3.7 (see also Remark 6.3.5), there exist exactly $\kappa - q$ solutions $Z^+_{q+1}, \dots, Z^+_\kappa$ of the homogeneous formally adjoint problem (6.2.13), (6.2.14) in the space (6.4.23) which are linearly independent modulo

$$V^{2m}_{2,-\beta+l_1\vec{1}}(\mathcal{G}) \times V^{\underline{\mu}+1/2}_{2,-\beta+l_1\vec{1}}(\partial\mathcal{G}). \tag{6.4.25}$$

From Theorem 6.4.1 it follows that these solutions satisfy the congruence

$$Z_s^+ \equiv \sum_{j=1}^{\kappa} d_{s,j}\, V_j \,\Big(\text{mod } V^{2m}_{2,-\beta+l_1\vec{1}}(\mathcal{G}) \times V^{\underline{\mu}+1/2}_{2,-\beta+l_1\vec{1}}(\partial\mathcal{G})\Big).$$

for $s = q+1, \dots, \kappa$ with certain constants $d_{j,s}$. Furthermore, we have

$$0 = \big(\mathcal{A}Z_j, Z_s^+\big) = \sum_{\tau=1}^{d} \big(\mathcal{A}(\zeta_\tau Z_j), Z_s^+\big) + \big(\mathcal{A}(\zeta_0 Z_j), Z_s^+\big)\,,$$

where $\zeta_0 = 1 - \zeta_1 - \cdots - \zeta_d$ and $(\cdot,\cdot)$ denotes the scalar product in $L_2(\mathcal{G}) \times L_2(\partial\mathcal{G})^{m+J}$. Obviously, the last term on the right-hand side is equal to zero. Theorem 6.4.3 yields

$$\sum_{\tau=1}^{d} \big(\mathcal{A}(\zeta_\tau Z_j), Z_s^+\big) = \overline{d}_{s,j} + \sum_{k=q+1}^{\kappa} c_{j,k}\, \overline{d}_{s,k}$$

Therefore, the coefficients $d_{s,j}$ satisfy the equalities

$$d_{s,j} = -\sum_{k=q+1}^{\kappa} d_{s,k}\, \overline{c}_{j,k} \qquad \text{for } j = 1, \dots, q,\ s = q+1, \dots, \kappa,$$

i.e., the matrix $D = (d_{s,j})_{s=q+1,\dots,\kappa,\, j=1,\dots,\kappa}$ has the form

$$D = \tilde{D} \cdot (-\tilde{C}^*\,,\ I_{\kappa-q}),$$

where $\tilde{D}$, $\tilde{C}$ are the matrices $\tilde{D} = (d_{s,k})_{s,k=q+1,\dots,\kappa}$, $\tilde{C} = (c_{j,k})_{j=1,\dots,q,\, k=q+1,\dots,\kappa}$ and $I_{\kappa-q}$ denotes the $(\kappa - q) \times (\kappa - q)$ identity matrix. Since $Z_{q+1}^+, \dots, Z_\kappa^+$ are linearly independent modulo the space (6.4.25), the rank of the matrix D is equal to $\kappa - q$. Hence $\tilde{D}$ is a nondegenerate matrix and in the space spanned by $Z_{q+1}^+, \dots, Z_\kappa^+$ it is possible to choose a basis such that the congruence (6.4.24) is valid. This proves the lemma. ■

Now we proceed to the main theorem.

THEOREM 6.4.4. *Suppose that the conditions of Theorem 6.4.1 are satisfied for every $\tau = 1, \dots, d$ and $Z_1, \dots, Z_q$ is a canonical basis in $\ker \mathcal{A}_{l_1,\beta}$ linearly independent modulo*

$$V_{2,\gamma}^{l_2}(\mathcal{G}) \times V_{2,\gamma}^{l_2+\underline{\tau}-1/2}(\partial\mathcal{G}). \tag{6.4.26}$$

Furthermore, we assume that $(f, \underline{g})$ is an element of the space

$$V_{2,\gamma}^{l_2-2m}(\mathcal{G}) \times V_{2,\gamma}^{l_2-\underline{\mu}-1/2}(\partial\mathcal{G})$$

such that the problem (6.2.2), (6.2.3) is solvable in (6.4.26). Then for any constants $c_1, \dots, c_q$ there exists a solution $(u, \underline{u}) \in V_{2,\beta}^{l_1}(\mathcal{G}) \times V_{2,\beta}^{l_1+\underline{\tau}-1/2}(\partial\mathcal{G})$ of the problem (6.2.2), (6.2.3) satisfying the congruence

$$(u, \underline{u}) \equiv \sum_{j=1}^{q} c_j\, U_j + \sum_{s=q+1}^{\kappa} d_s\, U_s \ \Big(\text{mod } V_{2,\gamma}^{l_2}(\mathcal{G}) \times V_{2,\gamma}^{l_2+\underline{\tau}-1/2}(\partial\mathcal{G})\Big). \tag{6.4.27}$$

The constants d_s are determined by the equalities

$$d_s = (f, v^{(s)})_{\mathcal{G}} + (\underline{g}, \underline{v}^{(s)})_{\partial\mathcal{G}} + \sum_{j=1}^{q} c_j\, c_{j,s}\,, \quad s = q+1, \dots, \kappa, \tag{6.4.28}$$

where the constants $c_{j,s}$ are the coefficients in the congruence (6.4.22) and $Z_s^+ = (v^{(s)}, \underline{v}^{(s)})$ are solutions of the homogeneous formally adjoint problem (6.2.13), (6.2.14) in the space (6.4.23) such that the congruence (6.4.24) is satisfied.

Proof: The existence of $(u,\underline{u})$ is obvious. If $(w,\underline{w})$ is a solution of problem (6.2.2), (6.2.3) in the space (6.4.26), then

$$(u,\underline{u}) = (w,\underline{w}) + \sum_{j=1}^{q} c_j\, Z_j$$

is a solution of (6.2.2), (6.2.3) satisfying the congruence (6.4.27).

Now let $(u,\underline{u})$ be a solution of the problem (6.2.2), (6.2.3) such that the congruence (6.4.27) with the given constants $c_1,\dots,c_q$ is satisfied. Then the solution

$$(w,\underline{w}) \stackrel{def}{=} (u,\underline{u}) - \sum_{j=1}^{q} c_j\, Z_j$$

satisfies the congruence

$$(w,\underline{w}) \equiv \sum_{s=q+1}^{\kappa} c'_s\, U_s \ \Big(\bmod\ V^{l_2}_{2,\gamma}(\mathcal{G}) \times V^{l_2+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})\Big),$$

where $c'_s = d_s - \sum_{j=1}^{q} c_j\, c_{j,s}$. Furthermore, let ζ_τ be smooth functions with support in $\mathcal{U}_\tau$ equal to one near $x^{(\tau)}$, $\tau = 1,\dots,d$. We set $\zeta_0 = 1 - \zeta_1 - \cdots - \zeta_d$. Then we have

$$(f, v^{(s)})_{\mathcal{G}} + (\underline{g}, \underline{v}^{(s)})_{\partial\mathcal{G}} = \sum_{\tau=0}^{d} \Big(\big(L(\zeta_\tau u), v^{(s)}\big)_{\mathcal{G}} + \big(B(\zeta_\tau u) + C(\zeta_\tau \underline{u}), \underline{v}^{(s)}\big)_{\partial\mathcal{G}} \Big).$$

By the Green formula, the term with $\tau = 0$ on the right-hand side vanishes. Therefore, from Theorem 6.4.3 it follows that the right-hand side of the above equality is equal to c'_s. Consequently, we get

$$c'_s = d_s - \sum_{j=1}^{q} c_j\, c_{j,s} = (f, v^{(s)})_{\mathcal{G}} + (\underline{g}, \underline{v}^{(s)})_{\partial\mathcal{G}}\,.$$

This proves the theorem. ∎

6.4.4. Coefficients formulas in terms of the classical Green formula. Now let the boundary value problem

$$Lu = f \quad \text{in } \mathcal{G}, \tag{6.4.29}$$

$$Bu = \underline{g} \quad \text{on } \partial\mathcal{G} \tag{6.4.30}$$

be given, where L is an elliptic differential operator of order $2m$ and B is a vector of differential operators B_k $(k = 1,\dots,m)$ of order $\mu_k < 2m$ which form a normal system on $\partial\mathcal{G}\backslash\mathcal{S}$. This system can be completed by differential operators B_k $(k = m+1,\dots,2m)$ of order $\mu_k < 2m$ to a Dirichlet system of order $2m$ on $\partial\mathcal{G}\backslash\mathcal{S}$. Then the classical Green formula

$$\int_{\mathcal{G}} Lu\cdot\overline{v}\,dx + \sum_{k=1}^{m}\int_{\partial\mathcal{G}} B_k u\cdot\overline{B'_{k+m}v}\,d\sigma = \int_{\mathcal{G}} u\cdot\overline{L^+ v}\,dx + \sum_{k=1}^{m}\int_{\partial\mathcal{G}} B_{k+m}u\cdot\overline{B'_k v}\,d\sigma$$

is satisfied for all $u,\ v \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})$. Here B'_k are differential operators of order $\mu'_k = 2m - 1 - \mu_{k+m}$ for $k = 1,\dots,m$ and of order $\mu'_k = 2m-1-\mu_{k-m}$ for $k = m+1,\dots,2m$. As before it is assumed that the operators L and B_k are δ-admissible near $x^{(\tau)}$. Then the operators L^+, B'_k are also δ-admissible.

As in the previous subsection let

$$\{\varphi_{j,s}^{(\mu)}\}_{j=1,\dots,I_\mu,\, s=0,\dots,\kappa_{\mu,j}-1} \quad \text{and} \quad \{(\psi_{j,s}^{(\mu)}, \underline{\psi}_{j,s}^{(\mu)})\}_{j=1,\dots,I_\mu,\, s=0,\dots,\kappa_{\mu,j}-1}$$

be canonical systems of Jordan chains of $\mathfrak{A}_\tau(\lambda)$ and $\mathfrak{A}_\tau^+(\lambda)$ corresponding to the eigenvalues λ_μ and $\overline{\lambda}_\mu$, respectively, which satisfy the biorthonormality condition (6.4.16).

Using Theorems 6.4.1 and 6.4.3, we obtain the following result (cf. Theorem 6.1.7).

THEOREM 6.4.5. *Let $u \in V_{2,\beta}^{l_1}(\mathcal{G})$, $l_1 \geq 2m$, be a solution of the elliptic boundary value problem (6.4.29), (6.4.30), where*

$$\eta\,(f, \underline{g}) \in V_{2,\gamma_\tau}^{l_2-2m}(\mathcal{K}_\tau) \times V_{2,\gamma_\tau}^{l-\underline{\mu}-1/2}(\partial\mathcal{K}\tau), \quad l_2 \geq 2m, \ 0 < (l_2 - \gamma_\tau) - (l_1 - \beta_\tau) < \delta.$$

Here η is a smooth cut-off function equal to one near $x^{(\tau)}$ with support in $\mathcal{U}_\tau$. We suppose that there are no eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ on the lines $\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$, $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$ while the strip $-\beta_\tau + l_1 - n/2 < \operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$ contains the eigenvalues $\lambda_1, \dots, \lambda_N$. Then u has the representation

$$u = \sum_{\mu=1}^{N} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\, u_{\mu,j,s} + w \tag{6.4.31}$$

in a neighbourhood of the conical point $x^{(\tau)}$, where $c_{\mu,j,s}$ are constants, the functions $u_{\mu,j,s}$ are given by (6.4.4), and $w \in V_{2,\gamma_\tau}^{l_2}(\mathcal{K}_\tau)$. The coefficients $c_{\mu,j,s}$ in (6.4.31) are determined by the formula

$$c_{\mu,j,s} = \big(L(\zeta u), v\big)_{\mathcal{G}} + \sum_{k=1}^{m} \big(B_k(\zeta u)\,,\, B'_{k+m} v\big)_{\partial\mathcal{G}}\,. \tag{6.4.32}$$

Here ζ is an arbitrary smooth cut-off function equal to one in a neighbourhood of $x^{(\tau)}$ such that $\zeta\eta = \zeta$ and v is a solution of the homogeneous formally adjoint boundary value problem

$$L^+ v = 0 \quad \text{in } \mathcal{G}, \qquad B'_k v = 0 \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S} \quad (k = 1, \dots, m) \tag{6.4.33}$$

which has the form

$$v = v_{\mu,j,\kappa_{\mu,j}-1-s} + v^{(1)}\,, \qquad \eta v^{(1)} \in V_{2,-\beta_\tau+2m}^{2m}(\mathcal{K}_\tau)\,,$$

where $v_{\mu,j,s}$ is given by (6.4.17).

Proof: The first assertion follows immediately from Theorem 6.4.1. We show the validity of formula (6.4.32).

Let v be a solution of the homogeneous formally adjoint problem (6.4.33) of the given form and let $v_k = B'_{k+m} v|_{\partial\mathcal{G}}$, $k = 1, \dots, m$. Then $(v, \underline{v}) = (v, v_1, \dots, v_m)$ is a solution of the homogeneous formally adjoint (with respect to the Green formula (6.2.12)) boundary value problem

$$L^+ v = 0 \quad \text{in } \mathcal{G}, \qquad Pv + Q^+ \underline{v} = 0 \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}$$

(cf. Lemma 3.1.1). This solution has the form (6.4.19). Applying Theorem 6.4.3, we get (6.4.32). ∎

Analogously, the assertion of Theorem 6.4.4 can be modified. Suppose that

$$(f, \underline{g}) \in V_{2,\gamma}^{l_2-2m}(\mathcal{G}) \times V_{2,\gamma}^{l_2-\underline{\mu}-1/2}(\partial\mathcal{G})$$

and the assumptions of Theorem 6.4.5 are satisfied for every $\tau = 1, \dots, d$. Then every solution $u \in V_{2,\beta}^{l_1}(\mathcal{G})$ of the boundary value problem (6.4.29), (6.4.30) has the representation

$$u = \sum_{\tau=1}^{d} \sum_{\mu=1}^{N_\tau} \sum_{j=1}^{I_{\tau,\mu}} \sum_{s=0}^{\kappa_{\mu,j}^{(\tau)}-1} c_{\mu,j,s}^{(\tau)} \, \zeta_\tau \, u_{\mu,j,s}^{(\tau)} + w,$$

where ζ_τ are infinitely differentiable functions equal to one near $x^{(\tau)}$, $u_{\mu,j,s}^{(\tau)}$ are functions of the form (6.4.4), and $w \in V_{2,\gamma}^{l_2}(\mathcal{G})$.

Let $U_1, \dots, U_\kappa$ be an ordered set of the elements $\zeta_\tau u_{\mu,j,s}^{(\tau)}$ and let $V_1, \dots, V_\kappa$ be the set of the corresponding elements $\zeta_\tau v_{\mu,j,s}^{(\tau)}$ defined by (6.4.17) such that $\{U_j\}$ and $\{V_j\}$ are ordered compatibly.

THEOREM 6.4.6. *We assume that the conditions of Theorem 6.4.5 are satisfied for every $\tau = 1, \dots, d$ and $Z_1, \dots, Z_q$ is a canonical basis in $\ker \mathcal{A}_{l,\beta}$ linearly independent modulo $V_{2,\gamma}^{l_2}(\mathcal{G})$ satisfying the congruence*

$$Z_j \equiv U_j + \sum_{s=q+1}^{\kappa} c_{j,s} \, U_s \, \Big(\bmod \ V_{2,\gamma}^{l_2}(\mathcal{G})\Big).$$

If the problem (6.4.29), (6.4.30) is solvable in $V_{2,\gamma}^{l_2}(\mathcal{G})$ for the given elements $f \in V_{2,\gamma}^{l_2-2m}(\mathcal{G})$, $\underline{g} \in V_{2,\gamma}^{l_2-\underline{\mu}-1/2}(\partial\mathcal{G})$, then for any constants $c_1, \dots, c_q$ there exists a solution $u \in V_{2,\beta}^{l_1}(\mathcal{G})$ satisfying the congruence

$$u = \sum_{j=1}^{q} c_j \, U_j + \sum_{s=q+1}^{\kappa} d_s \, U_s \, \Big(\bmod \ V_{2,\gamma}^{l_2}(\mathcal{G})\Big).$$

The constants d_s are determined by the formula

$$d_s = (f, v^{(s)})_{\mathcal{G}} + \sum_{k=1}^{m} (g_k \, , \, B'_{k+m} v^{(s)})_{\partial\mathcal{G}} + \sum_{j=1}^{q} c_j \, c_{j,s} \, ,$$

where $v^{(s)}$ are solutions of the homogeneous formally adjoint problem (6.4.33) which satisfy the congruence.

$$v^{(s)} \equiv V_s - \sum_{j=1}^{q} \bar{c}_{j,s} \, V_j \, \Big(\bmod \ V_{2,-\beta+l_1\vec{1}}^{2m}(\mathcal{G})\Big).$$

(Such solutions $v^{(s)}$ always exist, cf. Lemma 6.4.1.)

6.5. Boundary value problems with parameter in domains with conical points

This section is devoted to elliptic boundary value problems with parameter in a bounded domain with conical point. Such problems play a crucial role for the study of elliptic boundary value problems in domains with more complicated singularities on the boundary (edges, polyhedral vertices, cuspidal points).

Parameter-dependent problems in smooth bounded domains have been already studied in Section 3.6. The main result of that section was the unique solvability for purely imaginary parameters with sufficiently large modulus and an a priori estimate for the solution in parameter-dependent norms. The goal of this section is to obtain an analogous result for domains with conical points. In the beginning we consider the parameter-dependent model problem in a cone. We state the conditions under which the model problem with purely imaginary parameters is uniquely solvable in a certain class of weighted Sobolev spaces and derive a priori estimates for the solutions.

6.5.1. The model problem with parameter in a cone.

Ellipticity of the model problem with parameter. Let $\mathcal{K}$ be the same cone as in Section 6.1. A differential operator $P(x,\partial_x,\mu)$ is said to be a *parameter-depending model operator of order* k in $\mathcal{K}$ if P has the form

$$P(x,\partial_x,\mu) = r^{-k}\,\mathcal{P}(\omega,\partial_\omega,r\partial_r,r\mu) = r^{-k}\sum_{i+j\le k} p_{i,j}(\omega,\partial_\omega)\,(r\mu)^i\,(r\partial_r)^j\,,$$

where $p_{i,j}(\omega,\partial_\omega)$ are differential operators of order $\le k-i-j$ with smooth coefficients in $\overline{\Omega}$. Analogously, parameter-depending model operators on $\partial\mathcal{K}\backslash\{0\}$ are defined.

We consider the boundary value problem

$$L(x,\partial_x,\mu)\,u = f \quad \text{in } \mathcal{K}, \tag{6.5.1}$$

$$B(x,\partial_x,\mu)\,u + C(x,\partial_x,\mu)\,\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{K}\backslash\{0\}, \tag{6.5.2}$$

where L is a parameter-depending model operator of order $2m$, B is a vector of parameter-depending model operators B_k of order μ_k, and C is a matrix of parameter-depending model operators $C_{k,j}$ of order $\mu_k+\tau_j$, $j=1,\dots,J$, $k=1,\dots,m+J$, which are tangential on $\partial\mathcal{K}\backslash\{0\}$. We will always assume that every of the operators B_k contains only derivatives of order less than $2m$. Then there exists a $(m+J)\times 2m$-matrix $Q(x,\partial_x,\mu)$ of tangential parameter-depending operators on $\partial\mathcal{K}\backslash\{0\}$ such that

$$B(x,\partial_x,\mu)\,u\,\big|_{\partial\mathcal{K}\backslash\{0\}} = Q(x,\partial_x,\mu)\cdot\mathcal{D}u\big|_{\partial\mathcal{K}\backslash\{0\}}\,. \tag{6.5.3}$$

Let $\mathcal{A}(\mu)$ denote the operator of the boundary value problem (6.5.1), (6.5.2). As in Section 3.6 (cf. Definition 3.6.1), we define the ellipticity with parameter.

DEFINITION 6.5.1. Problem (6.5.1), (6.5.2) is said to be *elliptic with parameter* if the problem

$$L(x,\partial_x,\partial_t)\,u = f \quad \text{in } \mathcal{K}\times\mathbb{R}, \tag{6.5.4}$$

$$B(x,\partial_x,\partial_t)\,u + C(x,\partial_x,\mu)\,\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{K}\times\mathbb{R}, \tag{6.5.5}$$

is elliptic, i.e., the differential operator $L(x,\partial_x,\partial_t)$ is elliptic in $(\overline{\mathcal{K}}\backslash\{0\})\times\mathbb{R}$, and condition (ii) in Definition 3.1.2 is satisfied for all $(x^{(0)},t^{(0)})\in(\partial\mathcal{K}\backslash\{0\})\times\mathbb{R}$.

Obviously, the boundary value problem

$$L(x,\partial_x,0)\,u = f \quad \text{in } \mathcal{K}, \tag{6.5.6}$$

$$B(x,\partial_x,0)\,u + C(x,\partial_x,0)\,\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{K} \tag{6.5.7}$$

is a model problem (without parameter) on $\mathcal{K}$. By Theorem 6.1.1, Remark 6.1.1, the operator $\mathcal{A}(0)$ of this problem is an isomorphism

$$V^l_{2,\beta}(\mathcal{K})\times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K}) \to V^{l-2m}_{2,\beta}(\mathcal{K})\times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$$

for $l \ge 2m$ if and only if problem (6.5.6), (6.5.7) is elliptic and no eigenvalues of the corresponding operator pencil $\mathfrak{A}(\lambda)$ lie on the line $\operatorname{Re}\lambda = -\beta + l - n/2$.

The formally adjoint model problem. Analogously to formula (3.6.9), we obtain the following Green formula which is valid for all $u,\, v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$, $\underline{u} \in C_0^\infty(\partial\mathcal{K}\backslash\{0\})^J$, $\underline{v} \in C_0^\infty(\partial\mathcal{K}\backslash\{0\})^{m+J}$:

$$\begin{aligned}(6.5.8)\qquad & \int\limits_{\mathcal{K}} L(\mu)u\cdot\overline{v}\,dx + \int\limits_{\partial\mathcal{K}\backslash\{0\}} \big(B(\mu)u + C(\mu)\underline{u},\, \underline{v}\big)_{\mathbb{C}^{m+J}}\,d\sigma \\ &= \int\limits_{\mathcal{K}} u\cdot\overline{L^+(\overline{\mu})v}\,dx + \int\limits_{\partial\mathcal{K}\backslash\{0\}} \big(\mathcal{D}u,\, P(\overline{\mu})v + Q^+(\overline{\mu})\underline{v}\big)_{\mathbb{C}^{2m}}\,d\sigma \\ &\quad + \int\limits_{\partial\mathcal{K}\backslash\{0\}} \big(\underline{u}, C^+(\overline{\mu})\underline{v}\big)_{\mathbb{C}^J}\,d\sigma.\end{aligned}$$

Here $L^+(\overline{\mu})$ is the formally adjoint operator to $L(\mu) = L(x,\partial_x,\mu)$, $C^+(\overline{\mu})$, $Q^+(\overline{\mu})$ are the adjoint operators to $C(\mu) = C(x,\partial_x,\mu)$ and $Q(\mu) = Q(x,\partial_x,\mu)$, respectively, and $P(\mu)$ is a vector of parameter-depending model operators $P_j(x,\partial_x,\mu)$ of order $2m-j$, $j = 1,\dots,2m$.

By Lemma 3.6.1, the boundary value problem

$$(6.5.9)\qquad L^+(-\mu)\,v = f \quad \text{in } \mathcal{K},$$

$$(6.5.10)\qquad P(-\mu)\,v + Q^+(-\mu)\,\underline{v} = \underline{g}, \quad C^+(-\mu)\,\underline{v} = \underline{h} \quad \text{on } \partial\mathcal{K}\backslash\{0\}$$

is elliptic with parameter if and only if problem (6.5.1), (6.5.2) is elliptic with parameter. For purely imaginary μ problem (6.5.9), (6.5.10) is formally adjoint to problem (6.5.1), (6.5.2).

The spaces $E^l_{2,\beta}$. We introduce the space $E^l_{2,\beta}(\mathcal{K})$ with the norm

$$\|u\|_{E^l_{2,\beta}(\mathcal{K})} = \Big(\int\limits_{\mathcal{K}} \sum_{|\alpha|\le l} \big(r^{2\beta} + r^{2(\beta-l+\alpha)}\big)|D_x^\alpha u|^2\,dx\Big)^{1/2}.$$

By $E^{l-1/2}_{2,\beta}(\partial\mathcal{K})$ we denote the space of traces of functions from $E^l_{2,\beta}(\mathcal{K})$, $l \ge 1$, on $\partial\mathcal{K}$. Note that $E^l_{2,\beta}(\mathcal{K}) \subset E^{l_1}_{2,\beta_1}(\mathcal{K})$ and $E^{l-1/2}_{2,\beta}(\partial\mathcal{K}) \subset E^{l_1-1/2}_{2,\beta_1}(\partial\mathcal{K})$ if $l \ge l_1$, $\beta_1 \le \beta \le \beta_1 + l - l_1$.

It can be easily verified that the operator

$$(6.5.11)\qquad \mathcal{A}(\mu) : E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K}) \to E^{l-2m}_{2,\beta}(\mathcal{K}) \times E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$$

is continuous. Here and in what follows, it is assumed that l is an integer, $l \ge 2m$, $l > \max\mu_k$. Note that then $l + \tau_j \ge 1$ for $j = 1,\dots,J$. (Otherwise, the j-th column of C is equal to zero, i.e., the unknown u_j does not appear in the boundary conditions.)

We are interested in the properties of the operator $\mathcal{A}(\mu)$ for purely imaginary μ. First we consider this operator for the parameter $\mu = \pm i$.

6.5.2. A priori estimates for solutions of the parameter-depending model problem. We derive a priori estimates for the solutions of the problem

$$L(x, \partial_x, \pm i)\, u = f \quad \text{in } \mathcal{K}, \tag{6.5.12}$$

$$B(x, \partial_x, \pm i)\, u + C(x, \partial_x, \pm i)\, \underline{u} = \underline{g} \quad \text{on } \partial\mathcal{K}\backslash\{0\}, \tag{6.5.13}$$

in the class of the spaces $E^l_{2,\beta}$.

LEMMA 6.5.1. *Let ζ, η be infinitely differentiable functions on $\overline{\mathcal{K}}$ such that $\operatorname{supp}\zeta \subset \operatorname{supp}\eta \subset \{x \in \overline{\mathcal{K}} : c_1 T < |x| < c_2 T\}$, $\eta \equiv 1$ on $\operatorname{supp}\zeta$, and $|D^\alpha_x \zeta| + |D^\alpha_x \eta| \le c_\alpha T^{-|\alpha|}$ for all multi-indices α with constants c_α independent of T. We suppose that problem (6.5.1), (6.5.2) is elliptic with parameter. Then for every solution $(u, \underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ of problem (6.5.12), (6.5.13) the inequality*

$$\begin{aligned}\|\zeta u\|_{E^l_{2,\beta}(\mathcal{K})} + \|\zeta \underline{u}\|_{E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \;\le\; & c\Big(\|\eta f\|_{E^{l-2m}_{2,\beta}(\mathcal{K})} + \|\eta \underline{g}\|_{E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})} \\ & + T^{\beta-l}\,\|\eta u\|_{L_2(\mathcal{K})} + \sum_{j=1}^{J} T^{\beta-l-\tau_j+1/2}\,\|\eta u_j\|_{L_2(\partial\mathcal{K})}\Big)\end{aligned}$$

holds, where the constant c is independent of $(u, \underline{u})$ and T.

Proof: We set $\zeta_T(x) = \zeta(Tx)$, $\eta_T(x) = \eta(Tx)$, $v(x,t) = e^{\pm Tti} u(Tx)$, and $\underline{v}(x,t) = e^{\pm Tti} \underline{u}(Tx)$. By our assumptions on ζ, η, the supports of ζ_T and η_T are contained in $\{x \in \mathcal{K} : c_1 < |x| < c_2\}$. Moreover, the derivatives of ζ_T and η_T have upper bounds independent of T. Hence by Corollary 3.2.1, there is the estimate

$$\begin{aligned}&\|\zeta_T v\|_{W^l_2(\mathcal{K}\times(-1,1))} + \|\zeta_T \underline{v}\|_{W^{l+\underline{\tau}-1/2}_2(\partial\mathcal{K}\times(-1,1))} \\ &\le c\Big(\|\eta_T \mathcal{A}(\partial_t)(v, \underline{v})\|_{W^{l-2m}_2(\mathcal{K}\times(-2,2))\times W^{l-\underline{\mu}-1/2}_2(\partial\mathcal{K}\times(-2,2))} \\ &\quad + \|\eta_T v\|_{L_2(\mathcal{K}\times(-2,2))} + \|\eta \underline{v}\|_{L_2(\partial\mathcal{K}\times(-2,2))^J}\Big).\end{aligned} \tag{6.5.14}$$

Substituting $Tx = y$, we obtain

$$\begin{aligned}\|\zeta_T v\|^2_{W^l_2(\mathcal{K}\times(-1,1))} \;\ge\; & c\,T^{-n} \sum_{j+|\alpha|\le l} T^{2(j+|\alpha|)} \int_{\mathcal{K}} |D^\alpha_y(\zeta(y)\, u(y))|^2\, dy \\ \ge\; & c\,T^{-n+2l-2\beta} \|\zeta u\|^2_{E^l_{2,\beta}(\mathcal{K})}\end{aligned}$$

Furthermore,

$$L(x, \partial_x, \partial_t)\, v(x,t) = e^{\pm Tti}\, L(x, \partial_x, \pm Ti)\, u(Tx) = e^{\pm Tti}\, T^{2m}\, L(y, \partial_y, \pm i)\, u(y)$$

and, consequently,

$$\|\eta_T L(x, \partial_x, \partial_t)\, v\|^2_{W^{l-2m}_2(\mathcal{K}\times(-2,2))} \le c\,T^{-n+2l-2\beta} \|\eta f\|^2_{E^{l-2m}_{2,\beta}(\mathcal{K})}\,.$$

Analogous estimates are valid for the other terms in (6.5.14). This leads to the desired inequality. ∎

Analogously to Lemma 6.1.1, it can be shown that the norm in the space $E^l_{2,\beta}(\mathcal{K})$ is equivalent to

$$\|u\| = \Big(\sum_{k=-\infty}^{+\infty} \|\zeta_k u\|^2_{E^l_{2,\beta}(\mathcal{K})}\Big)^{1/2}, \tag{6.5.15}$$

where ζ_k are infinitely differentiable functions on $\overline{\mathcal{K}}$ satisfying the conditions (6.1.3). An analogous assertion is valid for the norm in $E^{l-1/2}_{2,\beta}(\partial\mathcal{K})$.

Using this and the previous lemma, we obtain the following result.

LEMMA 6.5.2. *Suppose that problem (6.5.1), (6.5.2) is elliptic with parameter. If* $(u,\underline{u}) \in E^{l_1}_{2,\beta_1}(\mathcal{K}) \times E^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K})$ *is a solution of problem (6.5.12), (6.5.13), where* $f \in E^l_{2,\beta}(\mathcal{K})$, $\underline{g} \in E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$, $\beta - l = \beta_1 - l_1$, *then* $u \in E^l_{2,\beta}(\partial\mathcal{K})$, $\underline{u} \in E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$, *and*

$$(6.5.16)\quad \|u\|_{E^l_{2,\beta}(\mathcal{K})} + \|\underline{u}\|_{E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \le c\left(\|f\|_{E^{l-2m}_{2,\beta}(\mathcal{K})} + \|\underline{g}\|_{E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})} + \|u\|_{E^{l_1}_{2,\beta_1}(\mathcal{K})} + \|\underline{u}\|_{E^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K})}\right).$$

Proof: Let ζ_k be infinitely differentiable functions in $\overline{\mathcal{K}}$ satisfying the conditions (6.1.3). Furthermore, let $\eta_k = \zeta_{k-1}+\zeta_k+\zeta_{k+1}$. Then $\eta_k = 1$ on supp ζ_k and Lemma 6.5.1 implies

$$\|\zeta_k u\|^2_{E^l_{2,\beta}(\mathcal{K})} + \|\zeta_k \underline{u}\|^2_{E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \le c\left(\|\eta_k f\|^2_{E^{l-2m}_{2,\beta}(\mathcal{K})} + \|\eta_k \underline{g}\|^2_{E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})} + \|\eta_k u\|^2_{E^{l_1}_{2,\beta_1}(\mathcal{K})} + \|\eta_k \underline{u}\|^2_{E^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K})}\right)$$

Summing up over all integer k and using the above mentioned equivalence of the norm in $E^l_{2,\beta}(\mathcal{K})$ to the expression (6.5.15), we obtain (6.5.16). ■

Estimate (6.5.16) can be sharpened if the line $\operatorname{Re}\lambda = -\beta + l - n/2$ does not contain eigenvalues of the pencil $\mathfrak{A}$.

THEOREM 6.5.1. *Suppose that problem (6.5.1), (6.5.2) is elliptic with parameter.*

1) *If no eigenvalues of the pencil* $\mathfrak{A}(\lambda)$ *lie on the line* $\operatorname{Re}\lambda = -\beta + l - n/2$, *then every solution* $(u,\underline{u}) \in E^l_{2,\beta}(\partial\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ *of problem (6.5.12), (6.5.13) satisfies the estimate*

$$(6.5.17)\quad \|u\|_{E^l_{2,\beta}(\mathcal{K})} + \|\underline{u}\|_{E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \le c\left(\|f\|_{E^{l-2m}_{2,\beta}(\mathcal{K})} + \|\underline{g}\|_{E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})} + \|u\|_{L_2(S_0)} + \|\underline{u}\|_{L_2(S_1)^J}\right),$$

where S_0, S_1 *are compact subsets of* $\overline{\mathcal{K}}\backslash\{0\}$ *and* $\partial\mathcal{K}\backslash\{0\}$, *respectively.*

2) *Conversely, if every solution* $(u,\underline{u}) \in E^l_{2,\beta}(\partial\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ *of problem (6.5.12), (6.5.13) satisfies the estimate (6.5.17), then there are no eigenvalues of the pencil* $\mathfrak{A}$ *on the line* $\operatorname{Re}\lambda = -\beta + l - n/2$.

Proof: 1) Let ζ_k, η_k be the same cut-off functions as in the proof of Lemma 6.5.2. We set $\zeta = \sum_{k=-\infty}^{-N-1} \zeta_k$, where N is a sufficiently large integer. Then by Theorem 6.1.1, the estimate

$$\|\zeta u\|_{V^l_{2,\beta}(\mathcal{K})} + \|\zeta \underline{u}\|_{V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \le c\,\|\mathcal{A}(0)\,\zeta(u,\underline{u})\|$$
$$\le c\left(\|\zeta(f,\underline{g})\| + \|(\mathcal{A}(0) - \mathcal{A}(\pm i))\,\zeta(u,\underline{u})\| + \|[\mathcal{A}(\pm i),\zeta]\,(u,\underline{u})\|\right)$$

holds, where $\|\cdot\|$ denotes the norm in $V_{2,\beta}^{l-2m}(\mathcal{K}) \times V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{K})$, and $[\mathcal{A}(\pm i), \zeta] = \mathcal{A}(\pm i)\,\zeta - \zeta\,\mathcal{A}(\pm i)$ is the commutator of $\mathcal{A}(\pm i)$ and ζ. Here

$$\|(\mathcal{A}(0) - \mathcal{A}(\pm i))\,\zeta(u,\underline{u})\| \le c\Big(\|\zeta u\|_{E_{2,\beta}^{l-1}(\mathcal{K})} + \sum_{j:\,l+\tau_j\ge 2} \|\zeta u_j\|_{E_{2,\beta}^{l+\tau_j-3/2}(\partial\mathcal{K})}\Big)$$

and

$$\|[\mathcal{A}(\pm i), \zeta]\,u\| \le c\Big(\|\eta u\|_{W_2^{l-1}(\mathcal{K})} + \sum_{j:\,l+\tau_j\ge 2} \|\eta u_j\|_{W_2^{l+\tau_j-3/2}(\partial\mathcal{K})}\Big),$$

where η is an arbitrary smooth function equal to one on the set $\{x \in \mathcal{K} :\ 2^{-N-1} < |x| < 2^{-N}\}$. (Note that in the case $l+\tau_j = 1$ we have $\mu_k + \tau_j \le 0$ and, consequently, $\operatorname{ord} C_{k,j} = 0$ for all k. Therefore, the summation in the above inequalities is extended only over all j such that $l + \tau_j \ge 2$.) Consequently, for sufficiently large N we get

$$\|\zeta u\|_{V_{2,\beta}^{l}(\mathcal{K})} + \|\zeta\underline{u}\|_{V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{K})} \le c\Big(\|\zeta f\|_{E_{2,\beta}^{l-2m}(\mathcal{K})} + \|\zeta\underline{g}\|_{E_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{K})}$$
$$+ \|\eta u\|_{W_2^{l-1}(\mathcal{K})} + \sum_{j:\,l+\tau_j\ge 2} \|\eta u_j\|_{W_2^{l+\tau_j-3/2}(\partial\mathcal{K})}\Big).$$

Furthermore, Lemma 6.5.1 yields

$$\|\zeta_k u\|_{E_{2,\beta}^{l}(\mathcal{K})} + \|\zeta_k\underline{u}\|_{E_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{K})} \le c\Big(\|\eta_k f\|_{E_{2,\beta}^{l-2m}(\mathcal{K})} + \|\eta_k\underline{g}\|_{E_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{K})}$$
$$+ 2^{k(\beta-l)}\|\eta_k u\|_{L_2(\mathcal{K})} + \sum_{j=1}^{J} 2^{k(\beta-l-\tau_j+1/2)}\,\|\eta_k u_j\|_{L_2(\partial\mathcal{K})}\Big)$$

with a constant c independent of k, where $\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1}$. For $k \ge N$ the last two terms on the right are majorized by

$$2^{-Nl}\,\|\eta_k u\|_{E_{2,\beta}^{l}(\mathcal{K})} + \sum_{j=1}^{J} 2^{-N(l+\tau_j-1/2)}\,\|\eta_k u_j\|_{E_{2,\beta}^{l+\tau_j-1/2}(\partial\mathcal{K})}.$$

Using the equivalence of the norm in $E_{2,\beta}^{l}(\mathcal{K})$ to the expression (6.5.15), we obtain

$$\begin{aligned}\|u\|_{E_{2,\beta}^{l}(\mathcal{K})} + \|\underline{u}\|_{E_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{K})} \ \le\ & c\Big(\|f\|_{E_{2,\beta}^{l-2m}(\mathcal{K})} + \|\underline{g}\|_{E_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{K})} \\ & + \|u\|_{W_2^{l-1}(S_0)} + \sum_{j:\,l+\tau_j\ge 2} \|u_j\|_{W_2^{l+\tau_j-3/2}(S_1)} + 2^{-Nl}\|u\|_{E_{2,\beta}^{l}(\mathcal{K})} \\ & + \sum_{j=1}^{J} 2^{-N(l+\tau_j-1/2)}\,\|u_j\|_{E_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{K})}\Big),\end{aligned}$$

where

$$S_0 = \{x \in \overline{\mathcal{K}} :\ 2^{-N-2} \le |x| \le 2^{N+2}\} \quad \text{and} \quad S_1 = \partial\mathcal{K} \cap S_0.$$

Applying the inequality

$$\|u\|_{W_2^{l-1}(S_0)} \le \varepsilon\,\|u\|_{E_{2,\beta}^{l}(\mathcal{K})} + C(\varepsilon)\,\|u\|_{L_2(S_0)}$$

and an analogous inequality for the norm of u_j in the space $W_2^{l+\tau_j-3/2}(S_1)$, we obtain (6.5.17) if ε is sufficiently small and N is sufficiently large.

2) We prove now that the nonexistence of eigenvalues of $\mathfrak{A}$ on the line $\operatorname{Re}\lambda = -\beta + l - n/2$ is necessary for the validity of (6.5.17).

Let $(\varphi^{(0)}, \underline{\varphi}^{(0)})$ be an eigenfunction of the pencil $\mathfrak{A}(\lambda)$ corresponding to the eigenvalue λ_0, $\operatorname{Re}\lambda_0 = -\beta + l - n/2$. We consider the functions

$$u^{(T)} = \chi_T(r)\, r^{\lambda_0}\, \varphi^{(0)}(\omega)\,, \qquad u_j^{(T)} = \chi_T(r)\, r^{\lambda_0+\tau_j}\, \varphi_j^{(0)}(\omega)\,, \quad j = 1, \dots, J,$$

$T > 0$, where χ_T are infinitely differentiable functions on the positive real half-axis such that

$$\chi_T(t) = 1 \ \text{ for } e^{-T} < r < 1, \qquad \chi_T(t) = 0 \ \text{ for } r < e^{-T-1},\ r > 2, \ \text{and}$$
$$|(r\partial_r)^j \chi_T(r)| < c_j \quad \text{for } j = 0, 1, 2, \dots$$

Then the norm of $(u^{(T)}, \underline{u}^{(T)})$ in $E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ tends to infinity as $T \to \infty$. Analogously to the proof of Lemma 6.3.3, it can be shown that the norm of $\mathcal{A}(\pm i)\,(u^{(T)}, \underline{u}^{(T)})$ in $E^{l-2m}_{2,\beta}(\mathcal{K}) \times E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$ is bounded. Hence there does not exist a constant c such that the inequality (6.5.17) is satisfied for all $(u, \underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$. The proof of the lemma is complete. ■

6.5.3. Bijectivity of the operator of the model problem.

Bijectivity of the operator $\mathcal{A}(\pm i)$. From Lemma 3.4.1 and Theorem 6.5.1 we conclude that the range of the operator (6.5.11) with $\mu = \pm i$ is closed and its kernel has finite dimension if problem (6.5.1), (6.5.2) is elliptic with parameter and the line $\operatorname{Re}\lambda = -\beta + l - n/2$ does not contain eigenvalues of the pencil $\mathfrak{A}$. Furthermore, under these conditions, the same assertions are true for the operator

$$(6.5.18)\quad \mathcal{A}^+(\mp i) \,:\, E^q_{2,-\beta+l-2m+q}(\mathcal{K}) \times E^{q-2m+\underline{\mu}+1/2}_{2,-\beta+l-2m+q}(\partial\mathcal{K})$$
$$\to E^{q-2m}_{2,-\beta+l-2m+q}(\mathcal{K}) \times \Big(\prod_{j=1}^{2m} E^{q-2m+j-1/2}_{2,-\beta+l-2m+q}(\partial\mathcal{K})\Big) \times E^{q-2m-\underline{\tau}+1/2}_{2,-\beta+l-2m+q}(\partial\mathcal{K}),$$

$q \geq 2m$, of the formally adjoint problem

$$L^+(\mp i)\, v = f \qquad \text{in } \mathcal{K},$$
$$P(\mp i)\, v + Q^+(\mp i)\, \underline{v} = \underline{g}, \quad C^+(\mp i)\, \underline{v} = \underline{h} \quad \text{on } \partial\mathcal{K}\backslash\{0\}.$$

The following theorem can be proved analogously to Theorems 6.3.3 and 6.3.5.

THEOREM 6.5.2. *Suppose that problem (6.5.1), (6.5.2) is elliptic with parameter and there are no eigenvalues of the pencil* $\mathfrak{A}$ *on the line* $\operatorname{Re}\lambda = -\beta + l - n/2$. *Then the operators (6.5.11) with* $\mu = \pm i$ *and (6.5.18) are Fredholm. The kernels of these operators depend only on* $\beta - l$. *The range of the operator (6.5.11) with* $\mu = \pm i$ *consists of all* $(f, \underline{g}) \in E^{l-2m}_{2,\beta}(\mathcal{K}) \times E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$ *such that*

$$(f, v)_{\mathcal{K}} + (\underline{g}, \underline{v})_{\partial\mathcal{K}} = 0$$

for all $(v, \underline{v})$ *from the kernel of the operator (6.5.18), while the range of the operator (6.5.18) is the set of all*

$$(f, \underline{g}, \underline{h}) \in E^{q-2m}_{2,-\beta+l-2m+q}(\mathcal{K}) \times \Big(\prod_{j=1}^{2m} E^{q-2m+j-1/2}_{2,-\beta+l-2m+q}(\partial\mathcal{K})\Big) \times E^{q-2m-\underline{\tau}+1/2}_{2,-\beta+l-2m+q}(\partial\mathcal{K})$$

such that

$$(u, f)_{\mathcal{K}} + (\mathcal{D}u, \underline{g})_{\partial\mathcal{K}} + (\underline{u}, \underline{h})_{\partial\mathcal{K}} = 0$$

for all solutions $(u,\underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ *of the homogeneous problem (6.5.12), (6.5.13).*

In the following we suppose that the operator (6.5.11) is an isomorphism for $\mu = \pm i$, $\beta = \beta_0$, and $l = l_0$. Then by Theorem 6.5.2, the operator (6.5.18) is also an isomorphism for $\beta = \beta_0$, $l = l_0$, and $q \geq 2m$.

Let $\lambda_- < \operatorname{Re}\lambda < \lambda_+$ be the widest strip containing the line $\operatorname{Re}\lambda = -\beta_0 + l_0 - n/2$ which is free of eigenvalues of the pencil $\mathfrak{A}$.

LEMMA 6.5.3. *Suppose that the operator (6.5.11) is an isomorphism for* $\mu = \pm i$, $\beta = \beta_0$, $l = l_0$. *If* $(u,\underline{u}) \in E^{l_1}_{2,\beta_1}(\mathcal{K}) \times E^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K})$ *is a solution of problem (6.5.12), (6.5.13), where*

$$f \in \bigcap_{j=1}^{2} E^{l_j-2m}_{2,\beta_j}(\mathcal{K}), \qquad \underline{g} \in \bigcap_{j=1}^{2} E^{l_j-\underline{\mu}-1/2}_{2,\beta_j}(\partial\mathcal{K}),$$

$\lambda_- < -\beta_j + l_j - n/2 < \lambda_+$ *for* $j = 1,\ 2$, *then* $(u,\underline{u}) \in E^{l_2}_{2,\beta_2}(\mathcal{K}) \times E^{l_2+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{K})$.

Proof: Due to Lemma 6.5.2, it suffices to prove the lemma for $l_1 = l_2 = l$.

First let $\beta_1 - 1 \leq \beta_2 < \beta_1$. We have

$$\mathcal{A}(0)\,(u,\underline{u}) = (f,\underline{g}) + \big(\mathcal{A}(0) - \mathcal{A}(\pm i)\big)\,(u,\underline{u}),$$

where $(u,\underline{u}) \in V^l_{2,\beta_1}(\mathcal{K}) \times V^{l+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K})$, $(f,\underline{g}) \in V^{l-2m}_{2,\beta_2}(\mathcal{K}) \times V^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{K})$, and

$$\begin{aligned}\big(\mathcal{A}(0) - \mathcal{A}(\pm i)\big)\,(u,\underline{u}) \;&\in\; E^{l-2m+1}_{2,\beta_1}(\mathcal{K}) \times E^{l-\underline{\mu}+1/2}_{2,\beta_1}(\partial\mathcal{K}) \\ &\subset V^{l-2m}_{2,\beta_2}(\mathcal{K}) \times V^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{K}).\end{aligned}$$

Consequently, it follows from Theorem 6.1.4 that $(u,\underline{u}) \in V^l_{2,\beta_2}(\mathcal{K}) \times V^{l+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{K})$. Since, moreover, $u \in E^l_{2,\beta_1}(\mathcal{K})$ and $\underline{u} \in E^{l+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{K})$, we obtain the assertion of the lemma.

Now let $\beta_1 < \beta_2 \leq \beta_1 + 1/2$. In this case the spaces $E^l_{2,\beta_1}(\mathcal{K})$, $E^{l+\tau_j-1/2}_{2,\beta_1}(\partial\mathcal{K})$ are continuously imbedded into $E^0_{2,\beta_2-l}(\mathcal{K})$ and $E^0_{2,\beta_2-l-\tau_j+1/2}(\partial\mathcal{K})$, respectively. Using Lemma 6.5.1, we get

$$\begin{aligned}\|\zeta_k u\|^2_{E^l_{2,\beta_2}(\mathcal{K})} + \|\zeta_k \underline{u}\|^2_{E^{l+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{K})} \;&\leq\; c\Big(\|\eta_k f\|^2_{E^{l-2m}_{2,\beta_2}(\mathcal{K})} + \|\eta_k \underline{g}\|^2_{E^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{K})} \\ &\quad + \|\eta_k u\|^2_{E^0_{2,\beta_2-l}(\mathcal{K})} + \sum_{j=1}^{J} \|\eta_k u_j\|^2_{E^0_{2,\beta_2-l-\tau_j+1/2}(\partial\mathcal{K})}\Big),\end{aligned}$$

where ζ_k, η_k are the same functions as in the proof of Lemma 6.5.2 and c is a constant independent of k. Summing up over all integer k, we obtain the expression

$$\|f\|^2_{E^{l-2m}_{2,\beta_2}(\mathcal{K})} + \|\underline{g}\|^2_{E^{l-\underline{\mu}-1/2}_{2,\beta_2}(\partial\mathcal{K})} + \|u\|^2_{E^0_{2,\beta_2-l}(\mathcal{K})} + \sum_{j=1}^{J} \|u_j\|^2_{E^0_{2,\beta_2-l-\tau_j+1/2}(\partial\mathcal{K})}$$

on the right hand-hand side. Hence the series

$$\sum_{k=-\infty}^{+\infty} \Big(\|\zeta_k u\|^2_{E^l_{2,\beta_2}(\mathcal{K})} + \|\zeta_k \underline{u}\|^2_{E^{l+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{K})}\Big)$$

is convergent, i.e., $u \in E^l_{2,\beta_2}(\mathcal{K})$, $\underline{u} \in E^{l+\underline{\tau}-1/2}_{2,\beta_2}(\partial\mathcal{K})$.

Thus, the assertion of the lemma is proved for $\beta_1 - 1 \le \beta_2 \le \beta_1 + 1/2$. For $\beta_2 < \beta_1 - 1$ and $\beta_2 > \beta_1 + 1/2$ the assertion follows by induction. ■

THEOREM 6.5.3. *If the operator (6.5.11) is an isomorphism for $\mu = \pm i$, $\beta = \beta_0$, $l = l_0$, then this operator is an isomorphism for $\mu = \pm i$ and for all numbers l and β satisfying the inequalities $\lambda_- < -\beta + l - n/2 < \lambda_+$.*

Proof: First we show the uniqueness of the solution. Let $(u, \underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ be a solution of the equation $\mathcal{A}(\pm i)\,(u, \underline{u}) = 0$. Then it follows from Lemma 6.5.3 that $u \in E^{l_0}_{2,\beta_0}(\mathcal{K})$, $\underline{u} \in E^{l_0+\underline{\tau}-1/2}_{2,\beta_0}(\partial\mathcal{K})$ and, consequently, $(u, \underline{u}) = 0$. Hence the kernel of $\mathcal{A}(\pm i)$ in $E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ is trivial. Using Theorem 6.5.1, we obtain the estimate

$$\|u\|_{E^l_{2,\beta}(\mathcal{K})} + \|\underline{u}\|_{E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \le c\left(\|f\|_{E^{l-2m}_{2,\beta}(\mathcal{K})} + \|\underline{g}\|_{E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})}\right) \tag{6.5.19}$$

for every solution $(u, \underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ of problem (6.5.12), (6.5.13).

We prove the existence of a solution $(u, \underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ for given functions $f \in E^{l-2m}_{2,\beta}(\mathcal{K})$, $\underline{g} \in E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$. Let $f^{(j)} \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$, $\underline{g}^{(j)} \in C_0^\infty(\partial\mathcal{K}\backslash\{0\})^{m+J}$ be functions which converge to f and $\underline{g}$ in the norms of the spaces $E^{l-2m}_{2,\beta}(\mathcal{K})$ and $E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{K})$, respectively. Then by our assumption, there exist solutions $(u^{(j)}, \underline{u}^{(j)}) \in E^{l_0}_{2,\beta_0}(\mathcal{K}) \times E^{l_0+\underline{\tau}-1/2}_{2,\beta_0}(\partial\mathcal{K})$ of the equations

$$\mathcal{A}(\pm i)\,(u^{(j)}, \underline{u}^{(j)}) = (f^{(j)}, \underline{g}^{(j)}).$$

By Lemma 6.5.3, these solutions belong to $E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$. Furthermore, by (6.5.19), the sequence $\{(u^{(j)}, \underline{u}^{(j)})\}$ converges in $E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$. The limit $(u, \underline{u})$ is a solution of the equation $\mathcal{A}(\pm i)\,(u, \underline{u}) = (f, \underline{g})$. This completes the proof. ■

Bijectivity of the operator $\mathcal{A}(it)$. By means of the coordinate change $x = |\mu|y$, we can extend the assertions of Theorem 6.5.3 to arbitrary purely imaginary μ.

First we give some equivalent norms in $E^l_{2,\beta}(\mathcal{K})$ and $E^{l-1/2}_{2,\beta}(\partial\mathcal{K})$.

LEMMA 6.5.4. *The norm in $E^l_{2,\beta}(\mathcal{K})$ is equivalent to*

$$\|u\| = \|u\|_{V^l_{2,\beta}(\mathcal{K})} + \|u\|_{V^0_{2,\beta}(\mathcal{K})}, \tag{6.5.20}$$

while the norm in $E^{l-1/2}_{2,\beta}(\partial\mathcal{K})$, $l \ge 1$, is equivalent to

$$\|u\| = \|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K})} + \|r^\beta u\|_{L_2(\partial\mathcal{K})}. \tag{6.5.21}$$

Proof: Let u be an arbitrary function from $E^l_{2,\beta}(\mathcal{K})$, $v(x,t) = e^{i2^k t}\,u(2^k x)$, and $\chi_k(x) = \zeta_k(2^k x)$, where ζ_k is an infinitely differentiable function satisfying the conditions (6.1.3). Then, analogously to the proof of Lemma 6.5.1, there exist positive constants c_1, c_2 independent of u and k such that

$$c_1\,\|\zeta_k u\|^2_{E^l_{2,\beta}(\mathcal{K})} \le 2^{k(n+2\beta-2l)}\,\|\chi_k v\|^2_{W^l_2(\mathcal{K}\times(-1,1))} \le c_2\,\|\zeta_k u\|^2_{E^l_{2,\beta}(\mathcal{K})}\,.$$

Using the equivalence of the norm in $W_2^l(\mathcal{K}\times(-1,1))$ to

$$\|v\| = \Big(\|v\|^2_{L_2(\mathcal{K}\times(-1,1))} + \|\partial_t^l v\|^2_{L_2(\mathcal{K}\times(-1,1))} + \sum_{|\alpha|\le l} \|\partial_x^\alpha v\|^2_{L_2(\mathcal{K}\times(-1,1))}\Big)^{1/2},$$

we can prove in the same way that

$$c_1\left(\|\zeta_k u\|^2_{V^l_{2,\beta}(\mathcal{K})} + \|\zeta_k u\|^2_{V^0_{2,\beta}(\mathcal{K})}\right) \le 2^{k(n+2\beta-2l)}\, \|\chi_k v\|^2_{W_2^l(\mathcal{K}\times(-1,1))}$$
$$\le c_2\left(\|\zeta_k u\|^2_{V^l_{2,\beta}(\mathcal{K})} + \|\zeta_k u\|^2_{V^0_{2,\beta}(\mathcal{K})}\right).$$

Consequently,

$$c_1\, \|\zeta_k u\|^2_{E^l_{2,\beta}(\mathcal{K})} \le \|\zeta_k u\|^2_{V^l_{2,\beta}(\mathcal{K})} + \|\zeta_k u\|^2_{V^0_{2,\beta}(\mathcal{K})} \le c_2\, \|\zeta_k u\|^2_{E^l_{2,\beta}(\mathcal{K})}\,.$$

Summing up over all integer k, we obtain the equivalence of the norm in $E^l_{2,\beta}(\mathcal{K})$ to the norm (6.5.20). Similarly, it can be shown that the norm in $E^{l-1/2}_{2,\beta}(\partial\mathcal{K})$ is equivalent to (6.5.21). ■

Lemma 6.5.4 is used in the proof of the following assertions.

LEMMA 6.5.5. *There exists a positive constant c independent of μ such that*

$$\|u\|_{V^l_{2,\beta}(\mathcal{K})} + |\mu|^l\, \|u\|_{V^0_{2,\beta}(\mathcal{K})} \le \sum_{j=0}^{l} |\mu|^j\, \|u\|_{V^{l-j}_{2,\beta}(\mathcal{K})} \le c\left(\|u\|_{V^l_{2,\beta}(\mathcal{K})} + |\mu|^l\, \|u\|_{V^0_{2,\beta}(\mathcal{K})}\right)$$

for all $u \in E^l_{2,\beta}(\mathcal{K})$ and

$$\|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K})} + |\mu|^{l-1/2}\, \|r^\beta u\|_{L_2(\partial\mathcal{K})}$$
$$\le \sum_{j=0}^{l-1} |\mu|^j\, \|u\|_{V^{l-j-1/2}_{2,\beta}(\mathcal{K})} + |\mu|^{l-1/2}\, \|r^\beta u\|_{L_2(\partial\mathcal{K})}$$
$$\le c\left(\|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K})} + |\mu|^{l-1/2}\, \|r^\beta u\|_{L_2(\partial\mathcal{K})}\right)$$

for all $u \in E^{l-1/2}_{2,\beta}(\partial\mathcal{K})$, $l \ge 1$.

Proof: Let u be an arbitrary function from $E^l_{2,\beta}(\mathcal{K})$. We set $v(x) = u(x/|\mu|)$. Then

$$|\mu|^{n+2(\beta-l)} \sum_{j=0}^{l} |\mu|^{2j}\, \|u\|^2_{V^{l-j}_{2,\beta}(\mathcal{K})} = \int_{\mathcal{K}} \sum_{j+|\alpha|\le l} r^{2(\beta-l+j+|\alpha|)} |D_x^\alpha v|^2\, dx \le c_1\, \|v\|^2_{E^l_{2,\beta}(\mathcal{K})}$$
$$\le c_2\left(\|v\|^2_{V^l_{2,\beta}(\mathcal{K})} + \|v\|^2_{V^0_{2,\beta}(\mathcal{K})}\right) = c_2\, |\mu|^{n+2(\beta-l)}\left(\|u\|^2_{V^l_{2,\beta}(\mathcal{K})} + |\mu|^{2l}\, \|v\|^2_{V^0_{2,\beta}(\mathcal{K})}\right).$$

This proves the first inequality. Analogously, the second estimate holds. ■

We introduce the following parameter-dependent norms in the spaces $E^l_{2,\beta}(\mathcal{K})$ and $E^{l-1/2}_{2,\beta}(\partial\mathcal{K})$:

$$\|u\|_{V^l_{2,\beta}(\mathcal{K},|\mu|)} = \|u\|_{V^l_{2,\beta}(\mathcal{K})} + |\mu|^l\, \|u\|_{V^0_{2,\beta}(\mathcal{K})}\,,$$
$$\|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K},|\mu|)} = \|u\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{K})} + |\mu|^{l-1/2}\, \|r^\beta u\|_{L_2(\partial\mathcal{K})}\,.$$

THEOREM 6.5.4. *Let the operator (6.5.11) be an isomorphism for $\mu = \pm i$, $\beta = \beta_0$, $l = l_0$. Then the operator (6.5.11) is an isomorphism for arbitrary purely imaginary $\mu \neq 0$, $\lambda_- < -\beta + l - n/2 < \lambda_+$.*

Moreover, every solution $(u, \underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ of problem (6.5.1), (6.5.2) satisfies the estimate

$$
(6.5.22) \qquad \|u\|_{V^l_{2,\beta}(\mathcal{K},|\mu|)} + \sum_{j=1}^{J} \|u_j\|_{V^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{K},|\mu|)} \leq c \Big(\|f\|_{V^{l-2m}_{2,\beta}(\mathcal{K},|\mu|)} + \sum_{k=1}^{m+J} \|g_k\|_{V^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{K},|\mu|)} \Big),
$$

where the constant c is independent of $(u, \underline{u})$ and μ.

Proof: Let $(u, \underline{u}) \in E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ be a solution of problem (6.5.1), (6.5.2) with purely imaginary μ. We set $v(x) = u(x/|\mu|)$ and $v_j(x) = |\mu|^{\tau_j}\, u_j(x/|\mu|)$ for $j = 1, \dots, J$. Then

$$
\|u\|_{V^l_{2,\beta}(\mathcal{K},|\mu|)} + \sum_{j=1}^{J} \|u_j\|_{V^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{K},|\mu|)} \leq c\,|\mu|^{l-\beta-n/2} \Big(\|v\|_{E^l_{2,\beta}(\mathcal{K})} + \|\underline{v}\|_{E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \Big).
$$

Furthermore, $(v, \underline{v})$ satisfies the equations

$$
L(x, \partial_x, \mu/|\mu|)\, v(x) = |\mu|^{-2m}\, f(x/|\mu|)\,, \quad x \in \mathcal{K},
$$

$$
B_k(x, \partial_x, \mu/|\mu|)\, v(x) + \sum_{j=1}^{J} C_{k,j}(x, \partial_x, \mu/|\mu|)\, v_j(x) = |\mu|^{-\mu_k} g_k(x/|\mu|), \;\; x \in \partial\mathcal{K}\backslash\{0\},
$$

$k = 1, \dots, m+J$. By Theorem 6.5.3, this problem is uniquely solvable in $E^l_{2,\beta}(\mathcal{K}) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})$ for all $f \in E^{l-2m}_{2,\beta}(\mathcal{K})$, $g_k \in E^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{K})$, and the solution satisfies the estimate

$$
\begin{aligned}
&\|v\|_{E^l_{2,\beta}(\mathcal{K})} + \|\underline{v}\|_{E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{K})} \\
&\leq c \Big(|\mu|^{-2m}\, \|f(\cdot/|\mu|)\|_{E^{l-2m}_{2,\beta}(\mathcal{K})} + \sum_{k=1}^{m+J} |\mu|^{-\mu_k}\, \|g_k(\cdot/|\mu|)\|_{E^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{K})} \Big) \\
&\leq c\,|\mu|^{\beta-l+n/2} \Big(\|f\|_{V^l_{2,\beta}(\mathcal{K},|\mu|)} + \sum_{k=1}^{m+J} \|g_k\|_{V^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{K},|\mu|)} \Big)\,.
\end{aligned}
$$

This proves the theorem. ■

6.5.4. Parameter-depending boundary value problems in bounded domains with conical points. Now we consider parameter-depending boundary value problems in a bounded domain $\mathcal{G} \subset \mathbb{R}^n$. As in the previous sections, we suppose that there exists a finite set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$ of points on the boundary $\partial\mathcal{G}$ such that $\partial\mathcal{G}\backslash\mathcal{S}$ is smooth and for each of the points $x^{(\tau)}$, $\tau = 1, \dots, d$, there exists a neighbourhood $\mathcal{U}_\tau$ such that $\mathcal{G} \cap \mathcal{U}_\tau = \mathcal{K}_\tau \cap \mathcal{U}_\tau$, where $\mathcal{K}_\tau$ is an infinite cone with the vertex $x^{(\tau)}$. The intersection of the unit sphere with center in $x^{(\tau)}$ and the cone $\mathcal{K}_\tau$ is denoted by Ω_τ.

Admissible parameter-dependent operators. Analogously to Definition 6.2.1, the operator

$$P(x, \partial_x, \mu) = \sum_{|\alpha|+j\le k} p_{\alpha,j}(x)\, \partial_x^\alpha\, \mu^j$$

is said to be an *admissible operator of order k with the parameter μ* in a neighbourhood of the conical point $x^{(\tau)}$ if the coefficients $p_{\alpha,j}$ have the form

$$p_{\alpha,j}(x) = r^{|\alpha|+j-k}\, p^{(0)}_{\alpha,j}(\omega, r) \tag{6.5.23}$$

in this neighbourhood, where the functions $p^{(0)}_{\alpha,j}$ are infinitely differentiable in $\overline{\Omega} \times \mathbb{R}_+$, continuous in $\overline{\Omega} \times \overline{\mathbb{R}}_+$, and

$$(r\partial_r)^j\, \partial_\omega^\gamma\, \big(p^{(0)}_{\alpha,j}(\omega, r) - p^{(0)}_{\alpha,j}(\omega, 0)\big) \to 0 \quad \text{as } r \to 0 \tag{6.5.24}$$

uniformly with respect to $\omega \in \overline{\Omega}_\tau$. Here again ω are coordinates on the unit sphere with center in $x^{(\tau)}$ and $r = |x - x^{(\tau)}|$ denotes the distance to $x^{(\tau)}$. Analogously, we define admissible operators with parameter on $\partial\mathcal{G}\backslash\mathcal{S}$.

If $P(x, \partial_x, \mu)$ is an admissible operator of order k with the coefficients (6.5.23), then the operator

$$\begin{aligned} P^{(\tau)}(x, \partial_x, \mu) &= \sum_{|\alpha|\le k} r^{|\alpha|+j-k}\, p^{(0)}_{\alpha,j}(\omega, 0)\, \partial_x^\alpha\, \mu^j \\ &= r^{-k} \sum_{j+|\alpha|\le k} p_{\alpha,j}(\omega, 0)\, (r\mu)^j\, r^{|\alpha|}\, \partial_x^\alpha \end{aligned}$$

is called the *parameter-dependent leading part of P at the point $x^{(\tau)}$*. Obviously, $P^{(\tau)}(x, \partial_x, \mu)$ is a parameter-depending model operator of order k in the cone $\mathcal{K}_\tau$.

REMARK 6.5.1. If $P(x, \partial_x, \mu)$ is an admissible operator of order k with parameter, then $P(x, \partial_x, \mu)$ is an admissible operator of order k in the sense of Definition 6.2.1 for arbitrary fixed μ. The leading part (in the sense of Definition 6.2.1) of this operator at the point $x^{(\tau)}$ coincides with the operator $P^{(\tau)}(x, \partial_x, 0)$.

Indeed, let the coefficients of P have the form (6.5.23). Then P can be written in the form

$$P(x, \partial_x, \mu) = \sum_{|\alpha|\le k} r^{|\alpha|-k} \Big(\sum_{j=0}^{k-|\alpha|} (\mu^j r)^j\, p^{(0)}_{\alpha,j}(\omega, r) \Big)\, \partial_x^\alpha\, .$$

It follows from (6.5.24) that

$$(r\partial_r)^q\, \partial_\omega^\gamma \Big(\sum_{j=0}^{k-|\alpha|} (\mu r)^j\, p^{(0)}_{\alpha,j}(\omega, r) - p_{\alpha,0}(\omega, 0) \Big) \to 0$$

as $r \to 0$ uniformly with respect to $\omega \in \overline{\Omega}_\tau$. Hence $P(x, \partial_x, \mu)$ is admissible and the leading part of this operator (in the sense of Definition 6.2.1) coincides with

$$P^{(\tau)}(x, \partial_x, 0) = \sum_{|\alpha|\le k} r^{|\alpha|-k}\, p^{(0)}_{\alpha,0}(\omega, 0)\, \partial_x^\alpha\, .$$

The following assertion is an immediate consequence of condition (6.5.24).

LEMMA 6.5.6. *Let P be a parameter-dependent admissible operator of order k in a neighbourhood of the conical point $x^{(\tau)}$ and let ε be a sufficiently small positive real number. Then there exists a constant c_ε independent of μ such that*

$$\|(P(x,\partial_x,\mu) - P^{(\tau)}(x,\partial_x,\mu))\, u\|_{V_{2,\beta}^{l-k}(\mathcal{K}_\tau,|\mu|)} \le c_\varepsilon \, \|u\|_{V_{2,\beta}^{l}(\mathcal{K}_\tau,|\mu|)}$$

for every $u \in E_{2,\beta}^l(\mathcal{K}_\tau)$ equal to zero outside the ball $|x - x^{(\tau)}| < \varepsilon$. The factor c_ε tends to zero as $\varepsilon \to 0$.

We consider the boundary value problem

$$L(x,\partial_x,\mu)\, u = f \quad \text{in } \mathcal{G}, \tag{6.5.25}$$

$$B(x,\partial_x,\mu)\, u + C(x,\partial_x,\mu)\, \underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}, \tag{6.5.26}$$

where L is a parameter-depending admissible operator of order $2m$, B is a vector of parameter-depending admissible operators B_k of order μ_k, and C is a matrix of tangential admissible differential operators $C_{k,j}$ with parameter of order $\mu_k + \tau_j$ in a neighbourhood of each corner $x^{(\tau)}$. Outside a neighbourhood of $\mathcal{S}$ the coefficients of the operators L, B_k, and $C_{k,j}$ are assumed to be smooth. Moreover, we suppose that the maximal order of differentiation in the differential operators B_k is less than $2m$. Then the vector $Bu|_{\partial\mathcal{G}\backslash\mathcal{S}}$ can be written in the form

$$Bu|_{\partial\mathcal{G}\backslash\mathcal{S}} = Q(x,\partial_x,\mu)\cdot \mathcal{D}u|_{\partial\mathcal{G}\backslash\mathcal{S}}\,,$$

where Q is a matrix of parameter-depending tangential differential operators on $\partial\mathcal{G}\backslash\mathcal{S}$.

For every corner $x^{(\tau)}$ we consider the parameter-depending model problem

$$L^{(\tau)}(x,\partial_x,\mu)\, u = f \quad \text{in } \mathcal{K}_\tau, \tag{6.5.27}$$

$$B^{(\tau)}(x,\partial_x,\mu)\, u + C^{(\tau)}(x,\partial_x,\mu)\, \underline{u} = \underline{g} \quad \text{on } \partial\mathcal{K}_\tau\backslash\{x^{(\tau)}\} \tag{6.5.28}$$

formed by the parameter-dependent leading parts of the operators L, B, and C. For $\mu = 0$ this is a model problem without parameter which generates the operator pencil

$$\mathfrak{A}_\tau(\lambda)\, :\, W_2^l(\Omega_\tau)\times W_2^{l+\underline{\tau}-1/2}(\partial\Omega_\tau) \to W_2^{l-2m}(\Omega_\tau)\times W_2^{l-\underline{\mu}-1/2}(\partial\Omega_\tau)$$

(see Section 6.1).

DEFINITION 6.5.2. The boundary value problem (6.5.25), (6.5.26) is said to be *elliptic with parameter* if the problem

$$L(x,\partial_x,\partial_t)\, u = f \quad \text{in } \mathcal{G}\times\mathbb{R},$$
$$B(x,\partial_x,\partial_t)\, u + C(x,\partial_x,\partial_t)\, \underline{u} = \underline{g} \quad \text{on } (\partial\mathcal{G}\backslash\mathcal{S})\times\mathbb{R}$$

is elliptic in $(\overline{\mathcal{G}}\backslash\mathcal{S})\times\mathbb{R}$.

Obviously, the ellipticity with parameter of problem (6.5.25), (6.5.26) implies the ellipticity with parameter of the model problem (6.5.27), (6.5.28). Let $\beta = (\beta_1,\dots,\beta_d)$ be a tuple of real numbers, and let

$$\mathcal{A}_\tau(\mu)\, :\, E_{2,\beta_\tau}^l(\mathcal{K}_\tau)\times E_{2,\beta_\tau}^{l+\underline{\tau}-1/2}(\partial\mathcal{K}_\tau) \to E_{2,\beta_\tau}^{l-2m}(\mathcal{K}_\tau)\times E_{2,\beta_\tau}^{l-\underline{\mu}-1/2}(\partial\mathcal{K}_\tau) \tag{6.5.29}$$

be the operator of the parameter-depending model problem (6.5.27), (6.5.28). We suppose that this operator is an isomorphism for $\mu = \pm i$, $l = l_0$, $\beta_\tau = \beta_\tau^{(0)}$ and denote by $\lambda_\tau^- < \operatorname{Re}\lambda < \lambda_\tau^+$ the widest strip in the complex plane which is free of eigenvalues of the pencil $\mathfrak{A}_\tau$ and contains the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$.

Solvability of the parameter-depending problem. We introduce a parameter-dependent norm in the weighted Sobolev space $V^l_{2,\beta}(\mathcal{G})$. Let ζ_τ $(\tau = 1, \dots, d)$ be infinitely differentiable functions in $\overline{\mathcal{G}}$ equal to one in a neighbourhood of $x^{(\tau)}$ and to zero in $\overline{\mathcal{G}}\backslash\mathcal{U}_\tau$, and let $\zeta_0 = 1 - \zeta_1 - \dots - \zeta_d$. We set

$$\|u\|_{V^l_{2,\beta}(\mathcal{G},|\mu|)} = \|\zeta_0 u\|_{W^l_2(\mathcal{G},|\mu|)} + \sum_{\tau=1}^{d} \|\zeta_\tau u\|_{V^l_{2,\beta_\tau}(\mathcal{K}_\tau,|\mu|)}\,, \tag{6.5.30}$$

where $\|\cdot\|_{V^l_{2,\beta_\tau}(\mathcal{K}_\tau,|\mu|)}$ is the norm introduced before Theorem 6.5.4 and

$$\|u\|_{W^l_2(\mathcal{G},|\mu|)} = \|u\|_{W^l_2(\mathcal{G})} + |\mu|^l\,\|u\|_{L_2(\mathcal{G})}.$$

Analogously, the parameter-dependent norm $\|\cdot\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G},|\mu|)}$ is defined.

THEOREM 6.5.5. *Suppose that the boundary value problem (6.5.25), (6.5.26) is elliptic with parameter, the operators L, B_k, $C_{k,j}$ are admissible, and the operator (6.5.29) is an isomorphism for $\mu = \pm i$, $l = l_0$ $\beta_\tau = \beta_\tau^{(0)}$, $\tau = 1, \dots, d$. Let $\lambda_\tau^- < -\beta_\tau + l - n/2 < \lambda_\tau^+$ for $\tau = 1, \dots, d$. Then there exist positive real constants ρ and δ such that for all $\mu \in \mathbb{C}$ satisfying the conditions*

$$|\mu| > \rho\,, \qquad |\operatorname{Re}\mu| < \delta\,|\operatorname{Im}\mu|$$

problem (6.5.25), (6.5.26) has a unique solution $(u, \underline{u}) \in V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ for arbitrary $f \in V^{l-2m}_{2,\beta}(\mathcal{G})$, $\underline{g} \in V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})$, $l \ge 2m$. This solution satisfies the estimate

$$\begin{aligned} &\|u\|_{V^l_{2,\beta}(\mathcal{G},|\mu|)} + \sum_{j=1}^{J} \|u_j\|_{V^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{G},|\mu|)} \\ &\quad \le c\Big(\|f\|_{V^{l-2m}_{2,\beta}(\mathcal{G},|\mu|)} + \sum_{k=1}^{m+J} \|g_k\|_{V^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{G},|\mu|)}\Big), \end{aligned} \tag{6.5.31}$$

where the constant c is independent of $(u, \underline{u})$ and μ.

Proof: First we prove (6.5.31) for purely imaginary μ, $|\mu| > \rho$. Let ζ_τ be the same functions as in the definition of the norm (6.5.30). Then it follows from Theorem 6.5.4 and Lemma 6.5.6 that

$$\begin{aligned} &\|\zeta_\tau u\|_{V^l_{2,\beta_\tau}(\mathcal{K}_\tau,|\mu|)} + \sum_{j=1}^{J} \|\zeta_\tau u_j\|_{V^{l+\tau_j-1/2}_{2,\beta_\tau}(\partial\mathcal{K}_\tau,|\mu|)} \\ &\quad \le c\Big(\|f^{(\tau)}\|_{V^{l-2m}_{2,\beta_\tau}(\mathcal{K}_\tau,|\mu|)} + \sum_{k=1}^{m+J} \|g_k^{(\tau)}\|_{V^{l-\mu_k-1/2}_{2,\beta_\tau}(\partial\mathcal{G},|\mu|)}\Big) \end{aligned}$$

for $\tau = 1, \dots, d$, where $f^{(\tau)} = L(\zeta_\tau u)$, $g^{(\tau)} = B\,(\zeta_\tau u) + C(\zeta_\tau \underline{u})$. By Theorem 3.6.1, the same inequality with the parameter-dependent norms in $W^l_2(\mathcal{G})$, $W_2^{l+\tau_j-1/2}(\partial\mathcal{G})$, $W_2^{l-2m}(\mathcal{G})$, and $W_2^{l-\mu_k-1/2}(\partial\mathcal{G})$, respectively, is satisfied for $\zeta_0\,(u, \underline{u})$ if ρ is sufficiently large. This implies (6.5.31) for $\operatorname{Re}\mu = 0$, $|\mu| > \rho$. In particular, by (6.5.31), the kernel of the the operator

$$\mathcal{A}(\mu)\ :\ V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \to V^{l-2m}_{2,\beta}(\mathcal{G}) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{6.5.32}$$

corresponding to problem (6.5.25), (6.5.26) is trivial for $\operatorname{Re}\mu = 0$, $|\mu| > \rho$. The same is true for the operator

$$\mathcal{A}^+(-\mu) \,:\, V^q_{2,-\beta+(l-2m+q)\vec{1}}(\mathcal{G}) \times V^{q-2m+\underline{\mu}+1/2}_{2,-\beta+(l-2m+q)\vec{1}}(\partial\mathcal{G})$$
$$\to V^{q-2m}_{2,-\beta+(l-2m+q)\vec{1}}(\mathcal{G}) \times \Big(\prod_{j=1}^{2m} V^{q-2m+j-1/2}_{2,-\beta+(l-2m+q)\vec{1}}(\partial\mathcal{G})\Big) \times V^{q-2m-\underline{\tau}+1/2}_{2,-\beta+(l-2m+q)\vec{1}}(\partial\mathcal{G})$$

of the formally adjoint problem. Consequently, by Theorem 6.3.3, the operator (6.5.32) is an isomorphism for $\operatorname{Re}\mu = 0$, $|\mu| > \rho$. Thus, the theorem is proved for purely imaginary μ.

Arguing as in the proof of Theorem 3.6.1, we obtain the assertions of the theorem for $|\mu| > \rho$, $|\operatorname{Re}\mu| < \delta\,|\operatorname{Im}\mu|$ if ρ is sufficiently large and δ is sufficiently small. ■

Spectral properties of the pencil $\mathcal{A}$. Under the assumptions of the foregoing theorem, the pencil $\mathcal{A}(\mu)$ is Fredholm in the sense of Definition 5.1.1. Indeed, the operator (6.5.32) is invertible for purely imaginary μ with sufficiently large modulus. Furthermore, since $\mathcal{A}(\mu_1) - \mathcal{A}(\mu_2)$ is a compact operator, it follows that the operator (6.5.32) is Fredholm for arbitrary μ. Thus, as a consequence of Lemma 5.1.1, the following assertion holds.

COROLLARY 6.5.1. *Let the conditions of Theorem 6.5.5 be satisfied. Then the spectrum of the pencil (6.5.32) consists of isolated eigenvalues with finite algebraic multiplicities. All eigenvalues, with the possible exception of finitely many, are situated in a double angle*

$$\{\mu \in \mathbb{C} : |\operatorname{Re}\mu| > \delta\,|\operatorname{Im}\mu|\}\,, \qquad \delta > 0,$$

of the complex plane containing the real axis.

REMARK 6.5.2. Under the assumptions of Theorem 6.5.5, the spectrum of the pencil (6.5.32) does not depend on the choice of the numbers l, β satisfying the inequality $\lambda_- < -\beta + l - n/2 < \lambda_+$.

Indeed, let μ_0 be an eigenvalue of the pencil (6.5.32), and let $(u, \underline{u}) \in V^l_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ be an eigenvector corresponding to this eigenvalue. Then it follows from Corollary 6.3.2 that $(u, \underline{u}) \in V^{l_1}_{2,\beta_1}(\mathcal{G}) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{G})$, where l_1, β_1 are arbitrary numbers satisfying the inequality $\lambda_- < -\beta_1 + l_1 - n/2 < \lambda_+$. Hence μ_0 is also an eigenvalue of the pencil

$$\mathcal{A}(\mu) \,:\, V^{l_1}_{2,\beta_1}(\mathcal{G}) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta_1}(\partial\mathcal{G}) \to V^{l_1-2m}_{2,\beta_1}(\mathcal{G}) \times V^{l_1-\underline{\mu}-1/2}_{2,\beta_1}(\partial\mathcal{G})\,.$$

6.6. Examples

6.6.1. The Dirichlet problem for the Laplace operator in a plane domain with angular points. Let $\mathcal{G}$ be a plane domain with boundary $\partial\mathcal{G}$ which is smooth outside the set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$. Here $x^{(1)}, \dots, x^{(d)}$ are angular points on the boundary, and it is assumed that $\mathcal{G}$ coincides with a plane angle $\mathcal{K}_\tau$ with aperture $\alpha_\tau \in (0, 2\pi)$ in a neighbourhood of $x^{(\tau)}$ for $\tau = 1, \dots, d$.

Solvability of the Dirichlet problem. We consider the problem

$$-\Delta u = f \quad \text{in } \mathcal{G}\,, \qquad u = g \quad \text{on } \partial\mathcal{G} \tag{6.6.1}$$

and denote the operator of this problem by $\mathcal{A}$. From the results of this chapter it follows that

1) *the operator*

$$\mathcal{A} \,:\, V^l_{2,\beta}(\mathcal{G}) \to V^{l-2}_{2,\beta}(\mathcal{G}) \times V^{l-1/2}_{2,\beta}(\partial\mathcal{G}), \tag{6.6.2}$$

where $l \geq 2$ *and* $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$, *is Fredholm if and only if the numbers* $(l-1-\beta_\tau)\alpha_\tau/\pi$ *are noninteger or equal to zero for* $\tau = 1, \dots, d$,

2) *the operator* $\mathcal{A}$ *or, more precisely, the operator*

$$\tilde{V}^{2,2}_{2,\beta-(l-2)\vec{1}}(\mathcal{G}) \ni \big(u, \mathcal{D}^{(2)}u\big) \to \big(-\Delta u, u|_{\partial\mathcal{G}\backslash\mathcal{S}}\big) \in V^0_{2,\beta-(l-2)\vec{1}}(\mathcal{G}) \times V^{3/2}_{2,\beta-(l-2)\vec{1}}(\partial\mathcal{G})$$

can be uniquely extended to a continuous operator

$$\mathcal{A} \,:\, \tilde{V}^{l,2}_{2,\beta}(\mathcal{G}) \to \tilde{V}^{l-2,0}_{2,\beta}(\mathcal{G}) \times V^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{6.6.3}$$

with arbitrary integer $l \leq 1$. *This operator is also Fredholm if and only if the numbers* $(l-1-\beta_\tau)\alpha_\tau/\pi$ *are noninteger or equal to zero for* $\tau = 1, \dots, d$.

We show that the mappings (6.6.2) an (6.6.3) are isomorphisms if and only if $-\pi/\alpha_\tau < l-1-\beta_\tau < \pi/\alpha_\tau$ for $\tau = 1, \dots, d$. For this we need the following lemma.

LEMMA 6.6.1. *The sets* $\overset{\circ}{W}{}^1_2(\mathcal{G})$ *and* $\{u \in V^1_{2,0}(\mathcal{G}) \,:\, u|_{\partial\mathcal{G}} = 0\}$ *coincide. On this sets the* $W^1_2(\mathcal{G})$*- and* $V^1_{2,0}(\mathcal{G})$*-norms are equivalent.*

Proof: It suffices to show that the $W^1_2(\mathcal{G})$- and $V^1_{2,0}(\mathcal{G})$-norms are equivalent on $C_0^\infty(\mathcal{G})$. With no loss of generality, we may assume that the boundary $\partial\mathcal{G}$ contains only one angular point $x^{(1)}$ which lies in the origin. Obviously, the $V^1_{2,0}(\mathcal{G})$-norm can be estimated from below by the $W^1_2(\mathcal{G})$-norm. Furthermore, for arbitrary $\varepsilon > 0$ there exists a constant c such that

$$\int\limits_{\substack{\mathcal{G}\\|x|>\varepsilon}} r^{-2}\,|u|^2\,dx \leq c\,\|u\|^2_{W^1_2(\mathcal{G})}$$

for all $u \in C_0^\infty(\mathcal{G})$. We suppose that $\mathcal{G}$ coincides with the wedge $\mathcal{K} = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, \theta = \arg(x_1 + ix_2) \in \Omega\}$ near the angular point, where Ω is an interval with length less than 2π. Then by Friedrichs' inequality, we have

$$\int\limits_{\substack{\mathcal{G}\\|x|<\varepsilon}} r^{-2}\,|u|^2\,dx = \int\limits_0^\varepsilon \int\limits_\Omega r^{-1}\,|u|^2\,d\theta\,dr \leq c \int\limits_0^\varepsilon \int\limits_\Omega r^{-1}\,|\partial_\theta u|^2\,d\theta\,dr \leq c \int\limits_{\mathcal{G}} |\nabla u|^2\,dx.$$

This proves the equivalence of the $W^1_2(\mathcal{G})$- and $V^1_{2,0}(\mathcal{G})$-norms on $C_0^\infty(\mathcal{G})$. ■

We consider the operator (6.6.3) for $l = 1$, $\beta = 0$. In this case the functional $(F, \underline{g}) = \mathcal{A}(u, \underline{\phi}) \in V^1_{2,0}(\mathcal{G})^* \times V^{1/2}_{2,0}(\mathcal{G})$ is determined for arbitrary $(u, \underline{\phi}) = (u, \phi_1, \phi_2) \in \tilde{V}^{1,2}_{2,0}(\mathcal{G})$ by the equalities

$$\int\limits_{\mathcal{G}} \nabla u \cdot \nabla \overline{v}\,dx - i\,(\phi_2, v)_{\partial\mathcal{G}} = (F, v)_{\mathcal{G}} \quad \text{for all } v \in V^1_{2,0}(\mathcal{G}) \tag{6.6.4}$$

$$u = \phi_1 = g \quad \text{on } \partial\mathcal{G} \tag{6.6.5}$$

(see Section 4.3.1).

LEMMA 6.6.2. *The operator (6.6.3) is an isomorphism for $l = 1$, $\beta = 0$.*

Proof: As we have noted above, the operator (6.6.3) is Fredholm for $l = 1$, $\beta = 0$. Let $(u, \phi_1, \phi_2) \in \tilde{V}^1_{2,0}(\mathcal{G})$ be an element of the kernel of this operator. Then $u \in \overset{\circ}{W}{}^1_2(\mathcal{G})$ and

$$\int\limits_{\mathcal{G}} \nabla u \cdot \nabla \overline{v}\, dx - i\,(\phi_2, v)_{\partial\mathcal{G}} = 0 \qquad \text{for all } v \in V^1_{2,0}(\mathcal{G})$$

Setting $v = u$, we get $\int\limits_{\mathcal{G}} |\nabla u|^2\, dx = 0$, i.e., $u = 0$. This implies $\phi_1 = \phi_2 = 0$. Consequently, the kernel of the operator (6.6.3) is trivial. Since the boundary value problem (6.6.1) is formally adjoint to itself, we can conclude from Theorem 6.3.3, that the cokernel is also trivial. This proves the lemma. ■

As a consequence of the Lemma 6.6.2, the following statement holds.

THEOREM 6.6.1. *The operators (6.6.2) and (6.6.3) are isomorphisms if and only if $-\pi/\alpha_\tau < l - 1 - \beta_\tau < \pi/\alpha_\tau$ for $\tau = 1, \dots, d$.*

Proof: By Corollary 6.3.3 and Lemma 6.6.2, the operators (6.6.2) and (6.6.3) are isomorphisms, if the condition of the theorem on l and β is satisfied. The necessity of this condition follows from Lemma 6.3.3 and Theorem 6.3.7. ■

Asymptotics of the solution $u \in W^1_2(\mathcal{G})$. Let $f \in V^0_{2,\vec{1}}(\mathcal{G})$, $g \in V^{3/2}_{2,\vec{1}}(\partial\mathcal{G})$. Then by Theorem 6.6.1, there exists a unique solution $u \in V^2_{2,\vec{1}}(\mathcal{G})$ of problem (6.6.1). Note that this solution coincides with the unique solution $u \in W^1_2(\mathcal{G})$ of the variational problem

$$\int\limits_{\mathcal{G}} \nabla u \cdot \nabla \overline{v}\, dx = \int\limits_{\mathcal{G}} f \cdot \overline{v}\, dx \qquad \text{for all } v \in \overset{\circ}{W}{}^1_2(\mathcal{G}), \tag{6.6.6}$$

$$u = g \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}. \tag{6.6.7}$$

Indeed, let $u \in V^2_{2,\vec{1}}(\mathcal{G})$ be a solution of problem (6.6.1). Integrating by parts, we obtain

$$\int\limits_{\mathcal{G}} f \cdot \overline{v}\, dx = -\int\limits_{\mathcal{G}} \Delta u \cdot \overline{v}\, dx = \int\limits_{\mathcal{G}} \nabla u \cdot \nabla \overline{v}\, dx - \int\limits_{\partial\mathcal{G}} \frac{\partial u}{\partial \nu} \cdot \overline{v}\, dx$$

for arbitrary $v \in V^1_{2,0}(\mathcal{G})$. Since $\overset{\circ}{W}{}^1_2(\mathcal{G}) \subset V^1_{2,0}(\mathcal{G})$, this implies (6.6.6).

We consider the asymptotics of the solution $u \in W^1_2(\mathcal{G})$, if the functions f and g belong to the spaces $V^{l-2}_{2,\beta}(\mathcal{G})$ and $V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$, respectively, where $l \geq 2$, $l - \beta_\tau \geq 1$ for $\tau = 1, \dots, d$. Suppose that $(l-1-\beta_1)\alpha/\pi$ is noninteger and $\mathcal{G}$ coincides with the plane wedge $\mathcal{K} = \{x \in \mathbb{R}^2 \, : \, r > 0, \ 0 < \theta < \alpha\}$ in a neighbourhood of the angular point $x^{(1)} = 0$. (Here r, θ denote the polar coordinates of the point x.) From what has been said in 6.1.7 it follows that the solution u admits the decomposition

$$u = \sum_j c_j\, r^{j\pi/\alpha} \sin \frac{j\pi\theta}{\alpha} + w \tag{6.6.8}$$

in a neighbourhood of $x^{(1)}$, where $w \in V^l_{2,\beta}(\mathcal{G})$ and the summation is extended over all positive integer j less than $(l - 1 - \beta_1)\alpha/\pi$.

By Theorem 6.4.3, the coefficients c_j in (6.6.8) satisfy the equality

$$c_j = -\big(\Delta(\zeta u), v_j\big)_{\mathcal{G}} - \big(\zeta g, \frac{\partial v_j}{\partial \nu}\big)_{\partial\mathcal{G}}, \tag{6.6.9}$$

where

$$v_j = \frac{1}{j\pi} r^{-j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha}.$$

This formula has the disadvantage that it contains the solution u of the boundary value problem. Another formula holds by means of Theorem 6.4.4. Since problem (6.6.1) is uniquely solvable in $V^2_{2,\vec{1}}(\mathcal{G})$, the number q in Theorem 6.4.4 equals zero. Thus, Theorem 6.4.4 yields

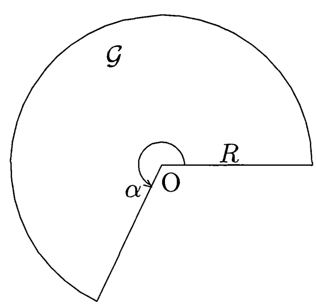
$$c_j = \big(f, v_j\big)_{\mathcal{G}} - \big(g, \frac{\partial v_j}{\partial \nu}\big)_{\partial\mathcal{G}}, \tag{6.6.10}$$

where v_j are solutions of the homogeneous problem (6.6.1) having the representation

$$v_j = \frac{1}{j\pi} r^{-j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha} + v_j^{(1)}, \qquad v_j^{(1)} \in V^2_{2,\vec{1}}(\mathcal{G}).$$

The functions $v_j^{(1)}$ are uniquely determined.

Let, for example, $\mathcal{G}$ be the sector $\{x = (x_1, x_2) : 0 < r < R,\ 0 < \theta < \alpha\}$.



Then we have

$$v_j = \frac{1}{j\pi}\left(r^{-j\pi/\alpha} - R^{-2j\pi/\alpha} r^{j\pi/\alpha}\right) \sin\frac{j\pi\theta}{\alpha}.$$

6.6.2. The Neumann problem for the Laplace operator in a plane domain with angular points. Let $\mathcal{G}$ be the same domain as before. We consider the Neumann problem

$$-\Delta u = f \quad \text{in } \mathcal{G}, \qquad \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}, \tag{6.6.11}$$

where $f \in V^{l-2}_{2,\beta}(\mathcal{G})$, $g \in V^{l-3/2}_{2,\beta}(\partial\mathcal{G})$, $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$, $l \ge 2$, $l - \beta_\tau > 1$ for $\tau = 1, \dots, d$.

If f and g satisfy the solvability condition

$$\int_{\mathcal{G}} f\, dx + \int_{\partial\mathcal{G}} g\, d\sigma = 0, \tag{6.6.12}$$

then there exists a solution $u \in W_2^1(\mathcal{G})$ of the variational problem

$$\int\limits_{\mathcal{G}} \nabla u \cdot \nabla \overline{v}\, dx = \int\limits_{\mathcal{G}} f \cdot \overline{v}\, dx + \int\limits_{\partial\mathcal{G}} g \cdot \overline{v}\, d\sigma \quad \text{for all } v \in W_2^1(\mathcal{G}) \tag{6.6.13}$$

corresponding to the Neumann problem (6.6.11). This solution is uniquely determined up to a constant.

By means of Hardy's inequality (see Chapter 7), it can be easily shown that $W_2^1(\mathcal{G})$ is continuously imbedded into the space $V^1_{2,\varepsilon\vec{1}}(\mathcal{G})$ for arbitrary $\varepsilon > 0$. Let $u \in W_2^1(\mathcal{G})$ be a solution of problem (6.6.13), $\phi_1 = u|_{\partial\mathcal{G}}$, and $\phi_2 = -ig$. Then $(u, \underline{\phi}) = (u, \phi_1, \phi_2)$ is an element of the space $\tilde{V}^{1,2}_{2,\varepsilon\vec{1}}(\mathcal{G})$.

We denote the extension of the operator of the boundary value problem (6.6.11) to the space $\tilde{V}^{1,2}_{2,\varepsilon\vec{1}}(\mathcal{G})$ by $\mathcal{A}$. The mapping

$$V^{1,2}_{2,\varepsilon\vec{1}}(\mathcal{G}) \ni (u, \underline{\phi}) \to (f, \underline{g}) = \mathcal{A}(u, \underline{\phi}) \in V^1_{2,-\varepsilon\vec{1}}(\mathcal{G})^* \times V^{-1/2}_{2,\varepsilon\vec{1}}(\partial\mathcal{G})$$

is definded as follows:

$$(f, v)_\Omega = \int\limits_{\mathcal{G}} \nabla u \cdot \nabla \overline{v}\, dx - i\,(\phi_2, v)_{\partial\mathcal{G}} \quad \text{for all } v \in V^1_{2,-\varepsilon\vec{1}}(\mathcal{G}),$$

$$g = i\,\phi_2.$$

Hence the solution $u \in W_2^1(\mathcal{G})$ of the variational problem (6.6.13) and the vector-function $\underline{\phi} = \big(u|_{\partial\mathcal{G}\setminus\mathcal{S}}, -ig\big)$ form a solution of the equation $\mathcal{A}\,(u, \underline{\phi}) = (f, g)$. Using Lemma 6.3.1, we obtain $(u, \underline{\phi}) \in \tilde{V}^{2,2}_{2,(1+\varepsilon)\vec{1}}(\mathcal{G})$ and, in particular, $u \in V^2_{2,(1+\varepsilon)\vec{1}}(\mathcal{G})$.

Asymptotics of the solution $u \in W_2^1(\mathcal{G})$. We consider the asymptotics of the solution $u \in W_2^1(\mathcal{G}) \cap V^2_{2,(1+\varepsilon)\vec{1}}(\mathcal{G})$ near the angular point $x^{(1)} = 0$. Again let $\mathcal{G}$ coincide with the plane wedge $\mathcal{K} = \{x \in \mathbb{R}^2 : r > 0,\ 0 < \theta < \alpha\}$ in a neighbourhood of $x^{(1)}$ and let $(l - 1 - \beta_1)\alpha/\pi$ be noninteger. Then u admits the decomposition

$$u = c_0 + \sum_j c_j\, r^{j\pi/\alpha} \cos\frac{j\pi\theta}{\alpha} + w \tag{6.6.14}$$

in a neighbourhood of $x^{(1)}$, where $w \in V^l_{2,\beta_1}(\mathcal{K})$ and the summation is extended over all positive integer j less than $(l - 1 - \beta_1)\alpha/\pi$. The logarithmic term does not appear in (6.6.14), since it is not a W_2^1-function.

The coefficients c_j satisfy the equality

$$c_j = -\big(\Delta(\zeta u), v_j\big)_{\mathcal{G}} + \big(\frac{\partial(\zeta u)}{\partial\nu}\,, v_j\big)_{\partial\mathcal{G}}\,,$$

where ζ is a smooth cut-off function equal to one near $x^{(1)}$ and

$$v_j = \frac{1}{j\pi}\, r^{-j\pi/\alpha} \cos\frac{j\pi\theta}{\alpha}\,.$$

6.6.3. The Dirichlet problem for the Laplace operator in a higher-dimensional domain. Now let $\mathcal{G}$ be a domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{G}$ which is smooth outside the set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$ of conical points and coincides with a cone $\mathcal{K}_\tau$ in a neighbourhood of the point $x^{(\tau)}$ for $\tau = 1, \dots, d$. We consider the problem

$$-\Delta u = f \quad \text{in } \mathcal{G}, \qquad u = g \quad \text{on } \partial\mathcal{G}. \tag{6.6.15}$$

As in the plane case, we denote by $\mathcal{A}$ the extension or restriction of the operator $\tilde{V}^{2,2}_{2,\beta-(l-2)\vec{1}}(\mathcal{G}) \ni (u, \mathcal{D}^{(2)}u) \to (-\Delta u, u|_{\partial\mathcal{G}\backslash\mathcal{S}}) \in V^{0}_{2,\beta-(l-2)\vec{1}}(\mathcal{G}) \times V^{3/2}_{2,\beta-(l-2)\vec{1}}(\partial\mathcal{G})$ to the space $\tilde{V}^{l,2}_{2,\beta}(\mathcal{G})$. The so-defined operator

$$\mathcal{A} \,:\, \tilde{V}^{l,2}_{2,\beta}(\mathcal{G}) \to \tilde{V}^{l-2,0}_{2,\beta}(\mathcal{G}) \times V^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{6.6.16}$$

is continuous for arbitrary integer l.

Let $\mathfrak{A}_\tau(\lambda)$ be the operator pencil generated by the model problem

$$-\Delta u = f \quad \text{in } \mathcal{K}\tau\,, \qquad u = g \quad \text{on } \partial\mathcal{K}_\tau$$

(see Section 6.1). From the representation

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\delta$$

for the Laplace operator, where $r = |x - x^{(\tau)}|$ and δ is the Laplace-Beltrami operator on the shere $|x - x^{(\tau)}| = 1$, it follows that $\mathfrak{A}_\tau(\lambda)$ is the operator of the problem

$$-\delta u - \lambda\,(\lambda + n - 2)\,u = f \quad \text{in } \Omega_\tau\,, \qquad u = 0 \quad \text{on } \partial\Omega_\tau. \tag{6.6.17}$$

Here Ω_τ denotes the intersection of the cone $\mathcal{K}_\tau$ with the sphere $|x - x^{(\tau)}| = 1$. Therefore, the number λ_0 is an eigenvalue of $\mathfrak{A}_\tau(\lambda)$ if and only if $\lambda_0(\lambda_0 + n - 2)$ is an eigenvalue of the operator $-\delta$ in Ω_τ with the Dirichlet boundary conditions. Consequently, if $\Lambda_{\tau,j}$ are the eigenvalues of $-\delta$ in Ω_τ, then the eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ are given by the equality

$$\lambda^{\pm}_{\tau,j} = 1 - \frac{n}{2} \pm \sqrt{\Big(1 - \frac{n}{2}\Big)^2 + \Lambda_{\tau,j}}\,. \tag{6.6.18}$$

Let σ_τ be the greatest number such that the strip $-\sigma_\tau + 1 - n/2 < \operatorname{Re}\lambda < \sigma_\tau + 1 - n/2$ does not contain eigenvalues of $\mathfrak{A}_\tau(\lambda)$. By (6.6.18), the number σ_τ is equal to $\sqrt{(1-n/2)^2 + \Lambda_{\tau,1}}$, where $\Lambda_{\tau,1}$ is the first eigenvalue of $-\delta$ in Ω_τ. Analogously to Theorem 6.6.1, the following statement holds.

THEOREM 6.6.2. *The operator (6.6.16) is an isomorphism if and only if* $-\sigma_\tau < l - 1 - \beta_\tau < \sigma_\tau$ *for* $\tau = 1, \dots, d$.

Estimation of the number σ_τ *for a rotational cone.* Let $\mathcal{K}_1$ be the rotational cone $\{x : x_1/r > \cos\alpha\}$. Furthermore, let $\theta_1, \dots, \theta_{n-1}$ be the spherical coordinates defined by the relations

$$\begin{aligned} x_1 &= r\,\cos\theta_1\,, \\ x_2 &= r\,\sin\theta_1\,\cos\theta_2\,, \\ &\vdots \\ x_{n-1} &= r\,\sin\theta_1\,\sin\theta_2\cdots\sin\theta_{n-2}\,\cos\theta_{n-1}\,, \\ x_n &= r\,\sin\theta_1\,\sin\theta_2\cdots\sin\theta_{n-2}\,\sin\theta_{n-1}\,. \end{aligned}$$

Then the Laplace-Beltrami operator has the form

$$\delta = \sum_{j=1}^{n-1} \frac{1}{q_j\,\sin^{n-j-1}\theta_j}\,\frac{\partial}{\partial\theta_j}\Big(\sin^{n-j-1}\theta_j\,\frac{\partial}{\partial\theta_j}\Big)\,,$$

where

$$q_1 = 1, \qquad q_j = \big(\sin\theta_1\,\sin\theta_2\cdots\sin\theta_{j-1}\big)^2 \quad \text{for } j = 2, \dots, n$$

(see [**165**, Ch.11, 2]).

LEMMA 6.6.3. *Let $\mathcal{K}_1$ be the cone $\{x : x_1/r > \cos\alpha\}$. Furthermore, let λ_1 be the smallest positive root of the function $C_\lambda^{(n-2)/2}(\cos\alpha)$, where $C_\lambda^{(n-2)/2}$ denotes the Gegenbauer function. Then $\Lambda_1 = \lambda_1(\lambda_1 + n - 2)$ is the first eigenvalue of the operator $-\delta$ in Ω_1 with the Dirichlet boundary conditions.*

Proof: We show that there exists a positive function $u = u(\theta_1)$ satisfying the equation $-\delta u = \Lambda_1 u$ for $\theta_1 < \alpha$ and the Dirichlet boundary condition $u(\alpha) = 0$. For functions u depending only on the variable θ_1, the homogeneous problem (6.6.17) takes the form

$$u''(\theta_1) + \frac{(n-2)\,\cos\theta_1}{\sin\theta_1}\,u'(\theta_1) + \lambda(\lambda+n-2)\,u(\theta_1) = 0 \quad \text{for } 0 \le \theta_1 < \alpha,$$
$$u(\alpha) = 0.$$

Substituting $\cos\theta_1 = t$, we obtain the equations

$$v''(t) + \frac{(n-1)t}{t^2-1}\,v'(t) - \frac{\lambda(\lambda+n-2)}{t^2-1}\,v(t) = 0, \qquad v(\cos\alpha) = 0$$

for the function $v(t) = u(\theta_1) = u(\arccos t)$. This problem has the positive solution $v(t) = C_{\lambda_1}^{(n-2)/2}(t)$, if λ_1 is the smallest positive root of the function $C_\lambda^{(n-2)/2}(\cos\alpha)$. Hence $u(\theta_1) = C_{\lambda_1}^{(n-2)/2}(\cos\theta_1)$ is an eigenfunction of $-\delta$ to the eigenvalue $\Lambda_1 = \lambda_1(\lambda_1 + n - 2)$. Since this eigenfunction is positive, Λ_1 is the first eigenvalue of the operator $-\delta$. The Lemma is proved. ∎

COROLLARY 6.6.1. *Suppose that $\mathcal{K}_\tau$ are rotational cones with aperture $2\alpha_\tau$. Then the operator (6.6.16) is an isomorphism if and only if*

$$\big|l - 1 - \beta_\tau\big| < \lambda_{\tau,1} - 1 + n/2\,,$$

where $\lambda_{\tau,1}$ are the smallest positive roots of the function $C_\lambda^{(n-2)/2}(\cos\alpha_\tau)$.

CHAPTER 7

Elliptic boundary value problems in weighted Sobolev spaces with nonhomogeneous norms

In Chapter 6 we have considered elliptic boundary value problems in domains with conical points in the class of weighted Sobolev spaces $V^l_{2,\beta}$ with homogeneous norms. Unfortunately, this class does not contain the usual Sobolev spaces without weight. For this reason, we introduce another class of weighted Sobolev spaces $W^l_{2,\beta}$ which coincide with the usual Sobolev spaces W^l_2 for $\beta = 0$. The goal of this chapter is to extend the main results of Chapter 6 to these weighted Sobolev spaces $W^l_{2,\beta}$. This requires to study the relations between the spaces $V^l_{2,\beta}$ and $W^l_{2,\beta}$. Here different cases have to be considered.

7.1. Relations between weighted Sobolev spaces

In the first section of this chapter we study the weighted Sobolev spaces $W^l_{2,\beta}(\mathcal{G})$ and the corresponding trace spaces in the cases $\beta \le -n/2$, $\beta > l - n/2$, and $s - n/2 < \beta < s + 1 - n/2$, where s is a nonnegative integer less than l. We show that in these cases the space $W^l_{2,\beta}(\mathcal{G})$ either coincides with the space $V^l_{2,\beta}(\mathcal{G})$ or is the direct sum of $V^l_{2,\beta}(\mathcal{G})$ and a set of polynomials. An analogous results holds for the trace spaces.

7.1.1. Auxiliary inequalities. Let $\mathcal{K}$ be an open cone in $\mathbb{R}^n$ with vertex at the origin having the representation

$$\mathcal{K} = \{x \in \mathbb{R}^n \, : \, 0 < r < \infty, \ \omega \in \Omega\}$$

in spherical coordinates r, ω, where Ω is a domain on the unit sphere S^{n-1} with smooth boundary $\partial\Omega$. In the sequel, let u be a function in $\mathcal{K}$ from the space $W^l_{2,loc}(\overline{\mathcal{K}}\backslash\{0\})$, i.e., $D^\alpha_x u$ belongs to the space $L_2(S)$ for every multi-index α, $|\alpha| \le l$, and every compact set $S \subset \overline{\mathcal{K}}\backslash\{0\}$.

The following lemma is based on *Hardy's inequality*

$$(7.1.1) \qquad \|r^\delta u\|_{L_2(\mathbb{R}_+)} \le \frac{2}{|2\delta + 1|} \, \|r^{\delta+1}\partial_r u\|_{L_2(\mathbb{R}_+)}\,, \qquad u \in C^\infty([0,\infty)),$$

which is satisfied

$$\text{for } \delta < -1/2 \quad \text{if } u(0) = 0,$$
$$\text{for } \delta > -1/2 \quad \text{if } u(\infty) = 0$$

(see, e.g., [**85**, Theorem 330]).

LEMMA 7.1.1. *Suppose that $u \in W^l_{2,loc}(\overline{\mathcal{K}}\backslash\{0\})$ is a function in $\mathcal{K}$ with compact support such that $r^\beta\, D^\alpha u \in L_2(\mathcal{K})$ for $|\alpha| = l$, where $\beta > l - n/2$. Then*

$$\|r^{\beta-l} u\|_{L_2(\mathcal{K})} \le c \sum_{|\alpha|=l} \|r^\beta\, D^\alpha u\|_{L_2(\mathcal{K})} \tag{7.1.2}$$

with a constant c independent of u.

Proof: From (7.1.1) it follows that

$$\|r^{\beta-l} u\|_{L_2(\mathcal{K})} \le c\, \|r^\beta\, \partial_r^l u\|_{L_2(\mathcal{K})}.$$

Since $\partial_r^l = \sum_{|\alpha|=l} a_\alpha(\omega)\, D_x^\alpha$, where a_α are smooth functions on S^{n-1}, this yields the desired inequality. ■

LEMMA 7.1.2. *Let u be a function on the interval $(0,1)$ such that $u \in L_2(\frac12, 1)$ and $r^{\delta+1}\, \partial_r u \in L_2(0,1)$, where $\delta < -1/2$. Then $u \in C[0,1]$. Furthermore,*

$$|u(0)|^2 \le c \left(\|r^{\delta+1}\, \partial_r u\|^2_{L_2(0,1)} + \|u\|^2_{L_2(\frac12,1)} \right).$$

Proof: Using the equality

$$u(0) = u(r) - \int_0^r u'(t)\, dt,$$

we get

$$|u(0)|^2 \le c \left(|u(r)|^2 + \int_0^1 t^{2(\delta+1)}\, |u'(t)|^2\, dt \right)$$

for $r \le 1$. Integrating this inequality with respect to r over the interval $(\frac12, 1)$, we obtain the desired inequality. ■

Let u be a function in $\mathcal{K}$. We use the same letter u if we consider this function as a function of the spherical coordinates r, ω and denote by

$$\overset{\circ}{u}(r) = \frac{1}{|\Omega|} \int_\Omega u(\omega, r)\, d\omega$$

the average of $u(\cdot, r)$ in Ω.

LEMMA 7.1.3. *Let $u \in W^1_{2,loc}(\overline{\mathcal{K}}\backslash\{0\})$ be a function in $\mathcal{K}$ such that $r^\beta \nabla u \in L_2(\mathcal{K})$, $\beta < 1 - n/2$. Then the limit $u(0) = \lim_{r\to 0} \overset{\circ}{u}(r)$ exists and*

$$\|r^{\beta-1}(u - u(0))\|_{L_2(\mathcal{K})} \le c\, \|r^\beta \nabla u\|_{L_2(\mathcal{K})} \tag{7.1.3}$$

$$|u(0)| \le c \left(\|r^\beta \nabla u\|_{L_2(\mathcal{K})} + \|u\|_{L_2(\mathcal{K}')} \right), \tag{7.1.4}$$

where $\mathcal{K}' = \{x \in \mathcal{K} :\ \frac12 < |x| < 1\}$.

Proof: Applying *Poincaré's inequality*

$$\|v\|_{L_2(\Omega)} \le c \left(\|\nabla_\omega v\|_{L_2(\Omega)} + \Big| \int_\Omega v(\omega)\, d\omega \Big| \right)$$

to the function $v(\omega) = u(\omega, r) - \overset{\circ}{u}(r)$, we get

$$\|u(\cdot, r) - \overset{\circ}{u}(r)\|_{L_2(\Omega)} \le c\, \|\nabla_\omega u\|_{L_2(\Omega)} .$$

Since $|\nabla_\omega u| \le c\, r\, |\nabla_x u|$, this implies

$$\|r^{\beta-1}(u - \overset{\circ}{u}(r))\|_{L_2(\mathcal{K})} \le c\, \|r^\beta\, \nabla_x u\|_{L_2(\mathcal{K})} . \tag{7.1.5}$$

Furthermore, from Lemma 7.1.2 it follows that $\overset{\circ}{u}(\cdot)$ is continuous in $[0,1]$ and

$$|\, \overset{\circ}{u}\,(0)|^2 \le c \Big(\|r^{\beta+(n-1)/2}\, \partial_r\, \overset{\circ}{u}\|^2_{L_2(0,1)} + \|\, \overset{\circ}{u}\|^2_{L_2(1/2,1)} \Big) .$$

Integrating this inequality over Ω, we get

$$|\, \overset{\circ}{u}(0)|^2 \le c \Big(\|r^\beta \nabla_x u\|^p_{L_2(\mathcal{K})} + \|u\|^2_{L_2(\mathcal{K}')} \Big)$$

This yields (7.1.4). Moreover, according to (7.1.1), we have

$$\Big\|r^{\beta-1+(n-1)/2}\, \big(\, \overset{\circ}{u}(\cdot) - \overset{\circ}{u}(0)\big)\Big\|_{L_2(0,\infty)} \le c\, \|r^{\beta+(n-1)/2}\, \partial_r\, \overset{\circ}{u}\|_{L_2(0,\infty)}$$

and, consequently,

$$\|r^{\beta-1}(\overset{\circ}{u} - \overset{\circ}{u}(0))\|_{L_2(\mathcal{K})} \le c\, \|r^\beta\, \nabla_x u\|_{L_2(\mathcal{K})} .$$

Using (7.1.5) and the last inequality, we obtain (7.1.3). ■

Now let u be a function in $\mathcal{K}$ satisfying the condition

$$\|r^\beta\, D^\alpha u\|_{L_2(\mathcal{K})} < \infty \qquad \text{for } |\alpha| \le l, \tag{7.1.6}$$

where $\beta < 1 - n/2$. Then by Lemma 7.1.3, the limits

$$(\partial_x^\alpha u)(0) = \lim_{r\to 0} \frac{1}{|\Omega|} \int\limits_\Omega (\partial_x^\alpha u)(\omega, r)\, d\omega$$

exist for $|\alpha| \le l - 1$ and satisfy the inequality

$$\sum_{|\alpha|\le l-1} |(\partial_x^\alpha u)(0)| \le c \sum_{|\alpha|\le l} \|r^\beta\, \partial_x^\alpha u\|_{L_2(\mathcal{K})} . \tag{7.1.7}$$

In the sequel,

$$p_{l-1}(u) = p_{l-1}(u; x) = \sum_{|\alpha|\le l-1} \frac{1}{\alpha!}\, (\partial_x^\alpha u)(0)\, x^\alpha$$

denotes the Taylor polynomial of degree $l-1$ corresponding to the function u. For integer $l \le 0$ we set $p_{l-1}(u; x) = 0$.

LEMMA 7.1.4. *Suppose that u satisfies condition (7.1.6), where $\beta < 1 - n/2$. Then*

$$\big\|r^{\beta-l}\big(u - p_{l-1}(u)\big)\big\|_{L_2(\mathcal{K})} \le c \sum_{|\alpha|=l} \|r^\beta\, D^\alpha u\|_{L_2(\mathcal{K})} .$$

Proof: We set $v = u - p_{l-1}(u)$. Since $\partial_x^\alpha p_{l-1}(u) = p_{l-1-|\alpha|}(\partial_x^\alpha u)$ for $|\alpha| = l-1$, it follows that $(\partial_x^\alpha v)(0) = 0$ for $|\alpha| \le l-1$. Hence (7.1.3) yields

$$\begin{aligned} \|r^{\beta-l} v\|_{L_2(\mathcal{K})} &\le c \sum_{|\alpha|=1} \|r^{\beta-l+1}\, D^\alpha v\|_{L_2(\mathcal{K})} \le \dots \\ &\le c \sum_{|\alpha|=l} \|r^\beta\, D^\alpha v\|_{L_2(\mathcal{K})} = c \sum_{|\alpha|=l} \|r^\beta\, D^\alpha u\|_{L_2(\mathcal{K})}\,. \end{aligned}$$

This proves the lemma. ■

Note that $p_{l-1}(u;\cdot) \equiv 0$ if u satisfies (7.1.6) with $\beta \le -n/2$.

7.1.2. Weighted Sobolev spaces with nonhomogeneous norms. Let $\mathcal{G}$ be a bounded domain in $\mathbb{R}^n$. For the sake of simplicity, we suppose that there is only one conical point on the boundary $\partial\mathcal{G}$ which lies in the origin, $\partial\mathcal{G}\backslash\{0\}$ is smooth and $\mathcal{G}$ coincides with the cone $\mathcal{K}$ in a neighbourhood of the conical point.

We define the space $W^l_{2,\beta}(\mathcal{G})$ as the set of all functions in $\mathcal{G}$ such that $r^\beta D^\alpha u \in L_2(\mathcal{G})$ for $|\alpha| \le l$ and provide this space with the norm

$$\|u\| = \Big(\int_{\mathcal{G}} r^{2\beta} \sum_{|\alpha|\le l} |D_x^\alpha u|^2\, dx \Big)^{1/2}. \tag{7.1.8}$$

Furthermore, we denote the set of the polynomials (in x) of degree not greater than s by Π_s.

Obviously, the space $V^l_{2,\beta}(\mathcal{G})$ is continuously imbedded into $W^l_{2,\beta}(\mathcal{G})$, since $r = r(x)$ is bounded in $\mathcal{G}$. As a consequence of Lemma 7.1.1, the following assertion holds.

LEMMA 7.1.5. *If $\beta > s - n/2$, $s \in \{1, \dots, l-1\}$, then*

$$W^l_{2,\beta}(\mathcal{G}) \subset W^{l-1}_{2,\beta-1}(\mathcal{G}) \subset \dots \subset W^{l-s}_{2,\beta-s}(\mathcal{G}). \tag{7.1.9}$$

All these imbeddings are continuous.

Proof: It suffices to prove the imbedding $W^l_{2,\beta}(\mathcal{G}) \subset W^{l-s}_{2,\beta-s}(\mathcal{G})$. Let u be an arbitrary function from $W^l_{2,\beta}(\mathcal{G})$ and let ζ be a smooth cut-off function with sufficiently small support which is equal to one near the conical point. Then by means of Lemma 7.1.1, we obtain

$$\begin{aligned} \|u\|_{W^{l-s}_{2,\beta-s}(\mathcal{G})} &\le \|\zeta u\|_{W^{l-s}_{2,\beta-s}(\mathcal{G})} + \|(1-\zeta)\, u\|_{W^{l-s}_{2,\beta-s}(\mathcal{G})} \\ &\le c \Big(\sum_{|\alpha|\le l-s} \|r^{\beta-s} D^\alpha(\zeta u)\|_{L_2(\mathcal{G})} + \|(1-\zeta)\, u\|_{W^l_{2,\beta}(\mathcal{G})} \Big) \\ &\le c \Big(\sum_{|\alpha|\le l} \|r^{\beta} D^\alpha(\zeta u)\|_{L_2(\mathcal{G})} + \|(1-\zeta)\, u\|_{W^l_{2,\beta}(\mathcal{G})} \Big) \le c\, \|u\|_{W^l_{2,\beta}(\mathcal{G})}\,. \end{aligned}$$

This proves the lemma. ■

In particular, it follows from Lemma 7.1.5 that $W^l_{2,\beta}(\mathcal{G}) = V^l_{2,\beta}(\mathcal{G})$ if $\beta > l - n/2$. Moreover, the following statement holds.

THEOREM 7.1.1. 1) *If $\beta \le -n/2$ or $\beta > l - n/2$, then the spaces $V^l_{2,\beta}(\mathcal{G})$ and $W^l_{2,\beta}(\mathcal{G})$ coincide and the norms in $V^l_{2,\beta}(\mathcal{G})$ and $W^l_{2,\beta}(\mathcal{G})$ are equivalent.*

2) *Suppose that* β *satisfies the inequality*

$$s - \frac{n}{2} < \beta < s + 1 - \frac{n}{2}$$

for a certain $s \in \{0, 1, \dots, l-1\}$. *Then the space* $W^l_{2,\beta}(\mathcal{G})$ *is the direct sum* $V^l_{2,\beta}(\mathcal{G}) \oplus \Pi_{l-s-1}$. *The projection of* $u \in W^l_{2,\beta}(\mathcal{G})$ *into* Π_{l-s-1} *is the polynomial* $p_{l-s-1}(u)$. *Furthermore, the norm in* $W^l_{2,\beta}(\mathcal{G})$ *is equivalent to*

$$\|u - p_{l-s-1}(u)\|_{V^l_{2,\beta}(\mathcal{G})} + \sum_{|\alpha| \le l-s-1} |(\partial_x^\alpha u)(0)| \tag{7.1.10}$$

Proof: 1) In the case $\beta > l - n/2$ the assertion of the theorem follows from Lemma 7.1.5. Let $\beta \le -n/2$ and let u be a function from $W^l_{2,\beta}(\mathcal{G})$ with support in a sufficiently small neighbourhood of the conical point. Then by Lemma 7.1.4, we have

$$\|r^{\beta-l+|\alpha|} (\partial_x^\alpha u - p_{l-1-|\alpha|}(\partial_x^\alpha u)\|_{L_2(\mathcal{K})} \le c\, \|u\|_{W^l_{2,\beta}(\mathcal{K})} . \tag{7.1.11}$$

for $|\alpha| \le l - 1$. Since $r^\beta \partial_x^\alpha u \in L_2(\mathcal{K})$, this implies

$$r^\beta\, p_{l-1-|\alpha|}(\partial_x^\alpha u) \in L_2(\mathcal{K}). \tag{7.1.12}$$

In particular, for $|\alpha| = l - 1$ it holds $r^\beta (\partial_x^\alpha u)(0) \in L_2(\mathcal{K})$, i.e., $(\partial_x^\alpha u)(0) = 0$. From this and from (7.1.12) we conclude by induction that $(\partial_x^\alpha u)(0) = 0$ for $|\alpha| \le l - 2$. This means, the polynomial $p_{l-1-|\alpha|}(\partial_x^\alpha u)$ vanishes for $|\alpha| \le l - 1$. Consequently, by (7.1.12), we have

$$\|u\|_{V^l_{2,\beta}(\mathcal{G})} \le c\, \|u\|_{W^l_{2,\beta}(\mathcal{G})} . \tag{7.1.13}$$

For functions $u \in W^l_{2,\beta}(\mathcal{G})$ with support outside a neighbourhood of the conical point this estimate is obvious. Hence (7.1.13) is valid for arbitrary functions $u \in W^l_{2,\beta}(\mathcal{G})$, $\beta \le -n/2$.

2) Let $s - n/2 < \beta < s+1-n/2$. We show that the intersection of Π_{l-s-1} with the space $V^l_{2,\beta}(\mathcal{G})$ consists only of the polynomial $p = 0$. Let the polynomial

$$p(x) = \sum_{|\alpha| \le l-s-1} c_\alpha \frac{x^\alpha}{\alpha!}$$

be an element of the space $V^l_{2,\beta}(\mathcal{G})$. Then

$$\partial_x^\gamma p(x) = \sum_{|\alpha| \le l-s-1-|\gamma|} c_{\alpha+\gamma} \frac{x^\alpha}{\alpha!} \in V^{l-|\gamma|}_{2,\beta}(\mathcal{G}) \subset V^0_{2,\beta-s-1}(\mathcal{G}) \tag{7.1.14}$$

for $|\gamma| \le l - s - 1$. Obviously, all terms with $|\alpha| \ge 1$ in the sum of (7.1.14) belong to the space $V^0_{2,\beta-s-1}(\mathcal{G})$, since $\beta - s > -n/2$. Consequently, we obtain $c_\gamma \in V^0_{2,\beta-s-1}(\mathcal{G})$, i.e., $c_\gamma = 0$. Thus, $p = 0$ is the only polynomial of degree $l - s - 1$ which belongs to the space $V^l_{2,\beta}(\mathcal{G})$.

Next we show that the difference $u - p_{l-s-1}(u)$ belongs to the space $V^l_{2,\beta}(\mathcal{G})$ for every $u \in W^l_{2,\beta}(\mathcal{G})$ and there exists a constant c independent of u such that

$$\|u - p_{l-s-1}(u)\|_{V^l_{2,\beta}(\mathcal{G})} \le c\, \|u\|_{W^l_{2,\beta}(\mathcal{G})} . \tag{7.1.15}$$

For functions with support outside a given neighbourhood of the conical point the inequality (7.1.15) is obvious. Let u be a function with support in a neighbourhood of the conical point such that $\operatorname{supp} u \cap \mathcal{G} = \operatorname{supp} u \cap \mathcal{K}$. Since $\partial_x^\alpha p_{l-s-1}(u) = 0$ for $|\alpha| \ge l-s$, by means of Lemma 7.1.1, we get

$$\|r^{\beta-l+|\alpha|}\, \partial_x^\alpha (u - p_{l-s-1}(u))\|_{L_2(\mathcal{G})} \le c\, \|u\|_{W^l_{2,\beta}(\mathcal{G})}$$

for $|\alpha| \ge l-s$. If $|\alpha| \le l-s-1$, then Lemma 7.1.4 yields

$$\begin{aligned} \|r^{\beta-l+|\alpha|}\, \partial_x^\alpha (u - p_{l-s-1}(u))\|_{L_2(\mathcal{G})} &= \|r^{\beta-l+|\alpha|}\, (\partial_x^\alpha u - p_{l-s-1-|\alpha|}(\partial_x^\alpha u))\|_{L_2(\mathcal{G})} \\ &\le \sum_{|\gamma|=l-s} \|r^{\beta-s}\, D^\gamma u\|_{L_2(\mathcal{G})} \le c\, \|u\|_{W^l_{2,\beta}(\mathcal{G})}\,. \end{aligned}$$

This proves (7.1.15). Therefore, $W^l_{2,\beta}(\mathcal{G})$ is the direct sum $V^l_{2,\beta}(\mathcal{G}) \oplus \Pi_{l-s-1}$.

Moreover, by Lemma 7.1.3, we have

$$\sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha u)(0)| \le c\, \|u\|_{W^l_{2,\beta}(\mathcal{G})}\,. \tag{7.1.16}$$

Hence the norm in $W^l_{2,\beta}(\mathcal{G})$ can be estimated from below by the norm (7.1.10). On the other hand,

$$\begin{aligned} \|u\|_{W^l_{2,\beta}(\mathcal{G})} &\le \|u - p_{l-s-1}(u)\|_{W^l_{2,\beta}(\mathcal{G})} + \|p_{l-s-1}(u)\|_{W^l_{2,\beta}(\mathcal{G})} \\ &\le c\Big(\|u - p_{l-s-1}(u)\|_{V^l_{2,\beta}(\mathcal{G})} + \sum_{|\alpha|\le l-s-1} |(\partial^\alpha u)(0)|\Big). \end{aligned}$$

Thus, the norms (7.1.8) and (7.1.10) are equivalent. The proof is complete. ■

LEMMA 7.1.6. *Suppose that* $s - n/2 < \beta < s+1-n/2$ *for a certain* $s \in \{0,1,\dots,l-1\}$. *Then for all* $u \in W^l_{2,\beta}(\mathcal{G})$ *and* $|\alpha| = l-s-1$ *there is the estimate*

$$|(\partial_x^\alpha u)(0)| \le \varepsilon\, \|u\|_{W^l_{2,\beta}(\mathcal{G})} + c_\varepsilon\, \|u\|_{W^{l-1}_{2,\beta}(\mathcal{G})}\,,$$

where ε *is an arbitrary constant,* $0 < \varepsilon < 1$, *and the constant* c_ε *depends only on* ε.

Proof: Let v be an arbitrary function in $\mathcal{K}$ satisfying condition (7.1.6). Then by (7.1.7) and Lemma 7.1.1, we have

$$|(\partial_x^\alpha v)(0)| \le c \sum_{|\gamma|\le l-s} \|r^{\beta-s}\, \partial_x^\gamma v\|_{L_2(\mathcal{K})} \le c \sum_{|\gamma|\le l} \|r^\beta\, \partial_x^\gamma v\|_{L_2(\mathcal{K})}\,.$$

for $|\alpha| \le l-s-1$. Setting $v(x) = w(\varepsilon x)$, we get

$$\varepsilon^{|\alpha|} |(\partial_x^\alpha w)(0)| \le c \sum_{|\gamma|\le l} \varepsilon^{|\gamma|-\beta-n/2}\, \|r^\beta\, \partial_x^\gamma w\|_{L_2(\mathcal{K})}\,.$$

In particular, for $|\alpha| = l-s-1$ it holds

$$\begin{aligned} |(\partial_x^\alpha w)(0)| &\le c\,\varepsilon^{s+1-n/2-\beta} \sum_{|\gamma|=l} \|r^\beta\, \partial_x^\gamma w\|_{L_2(\mathcal{K})} \\ &\quad + c\,\varepsilon^{s+1-n/2-\beta-l} \sum_{|\gamma|\le l-1} \|r^\beta\, \partial_x^\gamma w\|_{L_2(\mathcal{K})}\,, \end{aligned} \tag{7.1.17}$$

where the constant c is independent of ε and w. Now let u be an arbitrary function from $W^l_{2,\beta}(\mathcal{G})$ and ζ a smooth cut-off function equal to one near the point $x=0$ such that $\mathcal{K} \cap \operatorname{supp}\zeta = \mathcal{G} \cap \operatorname{supp}\zeta$. Then ζu can be considered as a function in $\mathcal{K}$. Applying the estimate (7.1.17) to the function ζu and replacing $c\,\varepsilon^{s+1-n/2-\beta}$ by ε, we get the desired inequality for u. ■

Finally, we consider the case when the boundary of $\mathcal{G}$ contains d conical points $x^{(1)}, x^{(2)}, \dots, x^{(d)}$. We suppose that there exist neighbourhoods $\mathcal{U}_\tau$ of $x^{(\tau)}$, $\tau = 1, \dots, d$, such that $\mathcal{G} \cap \mathcal{U}_\tau = \mathcal{K}_\tau \cap \mathcal{U}_\tau$, where $\mathcal{K}_\tau$ are infinite cones with vertices $x^{(\tau)}$.

Let ζ_τ be infinitely differentiable functions in $\overline{\mathcal{G}}$ equal to one in a neighbourhood of $x^{(\tau)}$ and to zero in $\mathcal{G}\backslash\mathcal{U}_\tau$, $\tau = 1, \dots, d$. Furthermore, let $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$ and l a nonnegative integer. We set $\zeta_0 = 1 - \zeta_1 - \dots - \zeta_d$ and define the space $W^l_{2,\beta}(\mathcal{G})$ as the set of all functions u on $\mathcal{G}$ such that $\zeta_0 u \in W^l_2(\mathcal{G})$ and $\zeta_\tau\, |x - x^{(\tau)}|^{\beta_\tau}\, D^\alpha u \in L_2(\mathcal{G})$ for $\tau = 1, \dots, d$, $|\alpha| \le l$. Equipped with the norm

$$\|u\|_{W^l_{2,\beta}(\mathcal{G})} = \|\zeta_0 u\|_{W^l_2(\mathcal{G})} + \sum_{\tau=1}^{d} \sum_{|\alpha| \le l} \|\zeta_\tau\, |x - x^{(\tau)}|^{\beta_\tau}\, D^\alpha u\|_{L_2(\mathcal{G})}\,,$$

the space $W^l_{2,\beta}(\mathcal{G})$ is complete.

From Theorem 7.1.1 it follows that

1. $W^l_{2,\beta}(\mathcal{G}) = V^l_{2,\beta}(\mathcal{G})$ if $\beta_\tau \le -n/2$ or $\beta_\tau > l - n/2$ for $\tau = 1, \dots, d$.
2. $W^l_{2,\beta}(\mathcal{G}) = V^l_{2,\beta}(\mathcal{G}) \oplus \zeta_1\, \Pi_{l-s_1-1} \oplus \dots \oplus \zeta_d\, \Pi_{l-s_d-1}$ if $s_\tau - n/2 < \beta_\tau < s_\tau + 1 - n/2$ for $\tau = 1, \dots, d$ (s_τ integer, $s_\tau \ge 0$).

Here $\zeta_\tau\, \Pi_{l-s-1}$ denotes the set of all functions of the form $\zeta_\tau p(\cdot)$, where $p(\cdot)$ is a polynomial of degree not greater than $l - s - 1$, $\zeta_\tau\, \Pi_{l-s-1} = \{0\}$ if $l - s - 1 < 0$.

7.1.3. The trace spaces. We suppose again that $\partial\mathcal{G}$ contains only one conical point. For positive integer l and real β we define $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ as the space of traces of functions from $W^l_{2,\beta}(\mathcal{G})$ on $\partial\mathcal{G}\backslash\{0\}$ equipped with the norm

$$\|u\|_{W^{l-1/2}_{2,\beta}(\partial\mathcal{G})} = \inf\Big\{\|v\|_{W^l_{2,\beta}(\mathcal{G})} \,:\, v \in W^l_{2,\beta}(\mathcal{G}),\ v|_{\partial\mathcal{G}\backslash\{0\}} = u\Big\}. \tag{7.1.18}$$

By Theorem 7.1.1, the space $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ coincides with the space $V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ if $\beta \le -n/2$ or $\beta > l - n/2$. Furthermore, for $\beta > s - n/2$, $s \in \{0, 1, \dots, l-1\}$ there are the imbeddings

$$W^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \subset W^{l-3/2}_{2,\beta-1}(\partial\mathcal{G}) \subset \dots \subset W^{l-s-1/2}_{2,\beta-s}(\partial\mathcal{G}).$$

(cf. (7.1.9)). We show that $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ is the direct sum of $V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ and a finite-dimensional space of polynomials if $s - n/2 < \beta < s + 1 - n/2$. For this we need the following lemma.

LEMMA 7.1.7. *Let p be a polynomial of degree not greater than $l - s - 1$ such that $p|_{\partial\mathcal{G}} \in V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$, where $s - n/2 < \beta < s + 1 - n/2$, $s \in \{0, 1, \dots, l-1\}$. Then $p|_{\partial\mathcal{K}} = 0$.*

Proof: Since $V^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \subset V^{l-s-1/2}_{2,\beta-s}(\partial\mathcal{G})$, it suffices to prove the lemma for $s = 0$, i.e., $-n/2 < \beta < 1 - n/2$. Let $p(x) = \sum_{|\alpha| \le l-1} a_\alpha\, x^\alpha$ be a polynomial such that $p|_{\partial\mathcal{G}} \in V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$. Then by Lemma 6.1.2,

$$\int_{\partial\mathcal{G}} r^{2(\beta-l)+1}\, |p(x)|^2\, dx < \infty. \tag{7.1.19}$$

Using (7.1.19) and the obvious inequality

$$\int_{\partial\mathcal{G}} r^{-n+1-2k} \Big| \sum_{|\alpha| \ge k+1} a_\alpha\, x^\alpha \Big|^2\, dx < \infty,$$

we can easily show by induction that

$$\int_{\partial\mathcal{G}} r^{-n+1-2k} \Big| \sum_{|\alpha|=k} a_\alpha\, x^\alpha \Big|^2 dx < \infty$$

for $k = 0, 1, \dots, l-1$ and, consequently,

$$\int_0^1 r^{-1} \Big(\int_{\partial\Omega} \Big| \sum_{|\alpha|=k} a_\alpha \cdot \big(\frac{x}{|x|}\big)^\alpha \Big|^2 d\omega \Big)\, dr < \infty.$$

This is only possible if $\sum_{|\alpha|=k} a_\alpha\, (x/|x|)^\alpha = 0$ for $x/|x| \in \partial\Omega$ or, what is the same, $\sum_{|\alpha|=k} a_\alpha\, x^\alpha \big|_{\partial\mathcal{K}} = 0$ for $k = 0, 1, \dots, l-1$. The lemma is proved. ∎

We denote by Ψ_{l-s-1} the linear subset of all polynomials $p(\cdot) \in \Pi_{l-s-1}$ such that $p|_{\partial\mathcal{K}} = 0$. Then Π_{l-s-1} can be written as the direct sum

$$\Pi_{l-s-1} = \Psi_{l-s-1} \oplus \Upsilon_{l-s-1}\,.$$

Let π be the projection of Π_{l-s-1} onto the subspace Υ_{l-s-1}. Furthermore, let u be a function from $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$, $s - n/2 < \beta < s + 1 - n/2$, $s \in \{0, 1, \dots, l-1\}$. The polynomial $\pi p_{l-s-1}(v; \cdot)$, where $v \in W^l_{2,\beta}(\mathcal{G})$ is an arbitrary extension of u, does not depend on this extension. Indeed, if v_1 and v_2 are extensions of u, then $v_j - p_{l-s-1}(v_j) \in V^l_{2,\beta}(\mathcal{G})$ for $j = 1$, $j = 2$ and $v_1 - v_2 = 0$ on $\partial\mathcal{G}$. Consequently, $p_{l-s-1}(v_1 - v_2)|_{\partial\mathcal{G}} \in V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ and Lemma 7.1.7 yields $p_{l-s-1}(v_1 - v_2)|_{\partial\mathcal{K}} = 0$, i.e., $\pi p_{l-s-1}(v_1) = \pi p_{l-s-1}(v_2)$.

Hence the polynomial $\pi p_{l-s-1}(v)$ depends only on u and can be denoted by $\mathfrak{p}_{l-s-1}(u)$. Analogously to Theorem 7.1.1, the following assertions hold.

THEOREM 7.1.2. *Let β be a real number satisfying the inequality $s - n/2 < \beta < s + 1 - n/2$ for a certain $s \in \{0, 1, \dots, l-1\}$. Then*

$$W^{l-1/2}_{2,\beta}(\partial\mathcal{G}) = V^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \oplus \Upsilon_{l-s-1}.$$

The projection of $u \in W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ into Υ_{l-s-1} is the polynomial $\mathfrak{p}_{l-s-1}(u)$. The norm in $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ is equivalent to

$$\|u - \mathfrak{p}_{l-s-1}(u)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} + \max_{\partial\mathcal{G}} |\mathfrak{p}_{l-s-1}(u)|\,. \tag{7.1.20}$$

Proof: The representation $W^{l-1/2}_{2,\beta}(\partial\mathcal{G}) = V^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \oplus \Upsilon_{l-s-1}$ follows immediately from Theorem 7.1.1 and Lemma 7.1.7. We prove the equivalence of the norms (7.1.18) and (7.1.20).

Since Υ_{l-s-1} is a finite-dimensional space, the norm (7.1.18) and the C-norm are equivalent on Υ_{l-s-1}. Hence we have

$$\|u\|_{W^{l-1/2}_{2,\beta}(\partial\mathcal{G})} \le \|u - \mathfrak{p}_{l-s-1}(u)\|_{W^{l-1/2}_{2,\beta}(\partial\mathcal{G})} + c \max_{\partial\mathcal{G}} |\mathfrak{p}_{l-s-1}(u)|.$$

Here $u - \mathfrak{p}_{l-s-1}(u)$ belongs to the space $V^{l-1/2}_{2,\beta}(\partial\mathcal{G})$. Therefore, there exists an extension $v \in V^l_{2,\beta}(\mathcal{G})$ of $u - \mathfrak{p}_{l-s-1}(u)$ such that

$$\|v\|_{V^l_{2,\beta}(\mathcal{G})} \le 2\, \|u - \mathfrak{p}_{l-s-1}(u)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})}\,.$$

Using the fact that $p_{l-s-1}(v) = 0$ and the norms (7.1.8) and (7.1.10) are equivalent, we get

$$\begin{aligned}\|u - \mathfrak{p}_{l-s-1}(u)\|_{W^{l-1/2}_{2,\beta}(\partial\mathcal{G})} &\le \|v\|_{W^l_{2,\beta}(\mathcal{G})} \le c\,\|v\|_{V^l_{2,\beta}(\mathcal{G})} \\ &\le 2c\,\|u - \mathfrak{p}_{l-s-1}(u)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})}\,.\end{aligned}$$

Thus, the norm in $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ is majorized by the norm (7.1.20).

We show the reverse estimate. Let u be an arbitrary function from $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ and let $v \in W^l_{2,\beta}(\mathcal{G})$ be an extension of u such that

$$\|v\|_{W^l_{2,\beta}(\mathcal{G})} \le 2\,\|u\|_{W^{l-1/2}_{2,\beta}(\partial\mathcal{G})}\,.$$

Then $\mathfrak{p}_{l-s-1}(u) = \pi p_{l-s-1}(v)$ and

$$\max_{\partial\mathcal{G}} |\mathfrak{p}_{l-s-1}(u)| = \max_{\partial\mathcal{G}} |\pi p_{l-s-1}(v)| \le c \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha v)(0)|\,.$$

Furthermore,

$$\begin{aligned}&\|u - \mathfrak{p}_{l-s-1}(u)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} \\ &\le \|u - p_{l-s-1}(v)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} + \|(1-\pi)p_{l-s-1}(v)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} \\ &\le \|v - p_{l-s-1}(v)\|_{V^l_{2,\beta}(\mathcal{G})} + \|(1-\pi)p_{l-s-1}(v)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})}\,.\end{aligned}$$

Since $(1-\pi)p_{l-s-1}(v) = 0$ on $\partial\mathcal{K}$, we get

$$\|(1-\pi)p_{l-s-1}(v)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} \le c \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha v)(0)|\,.$$

Consequently, Theorem 7.1.1 yields

$$\begin{aligned}&\|u - \mathfrak{p}_{l-s-1}(u)\|_{V^{l-1/2}_{2,\beta}(\partial\mathcal{G})} + \max_{\partial\mathcal{G}} |\mathfrak{p}_{l-s-1}(u)| \\ &\le \|v - p_{l-s-1}(v)\|_{V^l_{2,\beta}(\mathcal{G})} + c \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha v)(0)| \\ &\le c'\,\|v\|_{W^l_{2,\beta}(\mathcal{G})} \le 2c'\,\|u\|_{W^{l-1/2}_{2,\beta}(\partial\mathcal{G})}\,.\end{aligned}$$

The proof is complete. ■

Note that an analogous result holds if $\partial\mathcal{G}$ contains more than one conical point.

7.1.4. The case n=2. First we give a description of the subspaces Ψ_l and Υ_l which were used in the formulation of Theorem 7.1.2 for the case $n = 2$. Without loss of generality, we may assume that the angle $\mathcal{K}$ is bounded by the lines $x_2 = 0$ and $x_2 = \alpha x_1$. Then every polynomial

(7.1.21)
$$p_{j,k}(x_1, x_2) = x_2^j - \alpha^{j-k}\, x_1^{j-k}\, x_2^k\,, \qquad j = 1, 2, \dots, l,\ k = 1, 2, \dots, j-1,$$

belongs to the set Ψ_l. We set

(7.1.22) $$p_{j,j}(x_1, x_2) = \sum_{k=0}^{j} \alpha^{j-k}\, x_1^{j-k}\, x_2^k\,, \quad p_{j,j+1}(x_1, x_2) = x_1^j \quad \text{for } j = 1, \dots, l.$$

It can be easily seen that an arbitrary nonvanishing linear combination $c_1\, p_{j,j} + c_2\, p_{j,j+1}$ is not simultaneously equal to zero on the lines $x_2 = 0$ and $x_2 = \alpha x_1$

if $\alpha \neq 0$. Furthermore, the polynomials $p_{j,1}, p_{j,2}, \dots, p_{j,j+1}$ form a basis in the subspace of all homogeneous polynomials of degree j. Hence for $\alpha \neq 0$ there is the representation

$$\Pi_l = \Psi_l \oplus \Upsilon_l \,,$$

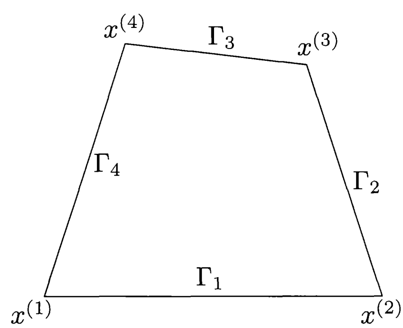
where Ψ_l is spanned by the polynomials (7.1.21) and Υ_l is spanned by the polynomials (7.1.22) and $p_0 \equiv 1$.

If $\alpha = 0$, then the set Ψ_l consists of all polynomials of the form

$$p(x) = x_2\, q(x),$$

where $q(\cdot)$ is a polynomial of degree not greater than $l-1$, while Υ_l is the linear closure of the functions $1, x_1, \dots, x_1^l$.

Now we suppose that there are $d \geq 2$ angular points $x^{(1)}, \dots, x^{(d)}$ on the boundary $\partial\mathcal{G}$. In contrast to the case $n \geq 3$, the set $\partial\mathcal{G}$ is not connected for $n = 2$. Let Γ_τ be the side connecting the points $x^{(\tau)}$ and $x^{(\tau+1)}$, $\tau = 1, \dots, d$. Here we have set $x^{(d+1)} = x^{(1)}$.



For integer $l \geq 1$ and $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$ let $V_{2,\beta}^{l-1/2}(\Gamma_\tau)$, $W_{2,\beta}^{l-1/2}(\Gamma_\tau)$ be the spaces of traces of functions from $V_{2,\beta}^l(\mathcal{G})$, $W_{2,\beta}^l(\mathcal{G})$ on Γ_τ equipped with the norms

$$\|u\|_{V_{2,\beta}^{l-1/2}(\Gamma_\tau)} = \inf\left\{ \|v\|_{V_{2,\beta}^l(\mathcal{G})} \;:\; v \in V_{2,\beta}^l(\mathcal{G}),\ v|_{\Gamma_\tau} = u \right\}$$

and

$$\|u\|_{W_{2,\beta}^{l-1/2}(\Gamma_\tau)} = \inf\left\{ \|v\|_{W_{2,\beta}^l(\mathcal{G})} \;:\; v \in W_{2,\beta}^l(\mathcal{G}),\ v|_{\Gamma_\tau} = u \right\},$$

respectively. Clearly, the spaces $V_{2,\beta}^{l-1/2}(\Gamma_\tau)$ and $W_{2,\beta}^{l-1/2}(\Gamma_\tau)$ depend only on the components β_τ, $\beta_{\tau+1}$ of the vector β.

Let $r_\tau(x) = |x - x^{(\tau)}|$ be the distance of the point $x \in \Gamma_\tau$ to $x^{(\tau)}$. In a neighbourhood of $x^{(\tau)}$ any function on Γ_τ can be considered as a function of the variable r_τ. We denote by ∂_r the derivative with respect to this variable. If $u \in W_{2,\beta}^{l-1/2}(\Gamma_\tau)$, where $\beta_\tau < s$ and s is an integer, $0 \leq s \leq l-1$, then the derivatives $(\partial_r^j u)(x^{(\tau)})$ exist for $j = 0, 1, \dots, l-s-1$. We set

$$\tilde{p}_{l-s-1}^{(\tau)}(u; x) = \sum_{j=0}^{l-s-1} \frac{1}{j!}\, (\partial_r^j u)(x^{(\tau)})\, |x - x^{(\tau)}|^j \,.$$

LEMMA 7.1.8. *Let $\tilde{p}$ be a function of the form*

$$\tilde{p}(x) = \sum_{j=0}^{l-s-1} c_j \, |x - x^{(\tau)}|^j \,, \tag{7.1.23}$$

where $s \in \{0, 1, \dots, l-1\}$ and $s - 1 < \beta_\tau < s$. Furthermore, let ζ_τ be an infinitely differentiable function with small support which is equal to one near $x^{(\tau)}$. Then

1) $\zeta_\tau \, \tilde{p}\big|_{\Gamma_\tau} \in W_{2,\beta}^{l-1/2}(\Gamma_\tau)$,
2) $\zeta_\tau \, \tilde{p}\big|_{\Gamma_\tau} \in V_{2,\beta}^{l-1/2}(\Gamma_\tau)$ *if and only if* $\tilde{p} \equiv 0$.

Proof: Without loss of generality, we may assume that $x^{(\tau)}$ lies in the origin and Γ_τ coincides with the x_1-axis on the support of the function ζ_τ.

1) Obviously, the trace of $\zeta_\tau \tilde{p}$ coincides with the trace of the function

$$u(x_1, x_2) = \zeta_\tau \sum_{j=0}^{l-s-1} c_j \, x_1^j \in W_{2,\beta}^l(\mathcal{G})$$

on Γ_τ. This proves the first assertion.

2) If $\zeta_\tau \tilde{p}\big|_{\Gamma_\tau} \in V_{2,\beta}^{l-1/2}(\Gamma_\tau)$, then

$$\int_0^1 x_1^{2(\beta_\tau - l)+1} \Big| \sum_{j=0}^{l-s-1} c_j \, x_1^j \Big|^2 \, dx_1 < \infty$$

(see Lemma 6.1.2). Since $\beta_\tau < s$, this is only possible if $c_0 = \ldots = c_{l-s-1} = 0$. ■

We define $\tilde{\Pi}_{l-s-1}^{(\tau)}$ as the set of all functions of the form (7.1.23). In the case $l - s - 1 < 0$ we set $\tilde{\Pi}_{l-s-1}^{(\tau)} = \{0\}$.

THEOREM 7.1.3. *1) If every of the components β_τ, $\beta_{\tau+1}$ is not greater than -1 or greater than $l-1$, then the spaces $V_{2,\beta}^{l-1/2}(\Gamma_\tau)$ and $W_{2,\beta}^{l-1/2}(\Gamma_\tau)$ coincide.*

2) Let $s_\tau - 1 < \beta_\tau < s_\tau$ and $s_{\tau+1} - 1 < \beta_{\tau+1} < s_{\tau+1}$, where s_τ, $s_{\tau+1}$ are nonnegative integers. Then

$$W_{2,\beta}^{l-1/2}(\Gamma_\tau) = V_{2,\beta}^{l-1/2}(\Gamma_\tau) \oplus \zeta_\tau \tilde{\Pi}_{l-s_\tau-1}^{(\tau)} \oplus \zeta_{\tau+1} \tilde{\Pi}_{l-s_{\tau+1}-1}^{(\tau+1)}$$

Here ζ_τ is the cut-off function introduced in Lemma 7.1.8.

7.2. Elliptic problems in spaces with nonhomogeneous norms

Now we consider solutions of elliptic boundary value problems in the class of the spaces $W_{2,\beta}^l$ introduced in the foregoing section. Using the relations between the spaces $V_{2,\beta}^l$ and $W_{2,\beta}^l$ and the results of Chapter 6, we prove the Fredholm property of the operator of the boundary value problem and describe the asymptotics of the solutions near the conical points. Throughout this section, we assume, for the sake of simplicity, that the boundary $\partial\mathcal{G}$ contains only one conical point $x^{(1)} = 0$. All results of this section (with obvious changes) are also true in the case of $d \geq 2$ conical points.

7.2.1. Continuity of differential operators. Let $P(x, \partial_x)$ be an admissible operator of order k in the domain $\mathcal{G}$, i.e., in a neighbourhood of the origin there is a representation

$$P(x,\partial_x) = r^{-k} \sum_{|\alpha|+j\le k} p_{\alpha,j}(\omega, r)\, \partial_\omega^\alpha\, (r\partial_r)^j$$

with infinitely differentiable coefficients $p_{\alpha,j}$ in $\overline{\Omega} \times \mathbb{R}_+$ satisfying the condition (6.2.7). Using the fact that P realizes a continuous mapping $V^l_{2,\beta}(\mathcal{G}) \to V^{l-k}_{2,\beta}(\mathcal{G})$ for arbitrary $l \ge k$, $\beta \in \mathbb{R}$, we get the following assertion.

LEMMA 7.2.1. *Suppose that P is an admissible operator of order $k \le l$.*

1) *If $\beta \le -n/2$ or $\beta > l - n/2$, then the operator P continuously maps the space $W^l_{2,\beta}(\mathcal{G})$ into $W^{l-k}_{2,\beta}(\mathcal{G})$.*

2) *If for a certain $s \in \{0, 1, \dots, l-1\}$ the inequalities $s - n/2 < \beta < s + 1 - n/2$ are satisfied, then the inclusion $P W^l_{2,\beta}(\mathcal{G}) \subset W^{l-k}_{2,\beta}(\mathcal{G})$ holds if and only if $P\Pi_{l-s-1} \subset W^{l-k}_{2,\beta}(\mathcal{G})$. If the last condition is satisfied, then the operator $P : W^l_{2,\beta}(\mathcal{G}) \to W^{l-k}_{2,\beta}(\mathcal{G})$ is continuous.*

Proof: The first assertion follows immediately from part 1) of Theorem 7.1.1. We show the second. Let $s - n/2 < \beta < s + 1 - n/2$. Then by the second part of Theorem 7.1.1, $u - p_{l-s-1}(u)$ belongs to $V^l_{2,\beta}(\mathcal{G})$ for every $u \in W^l_{2,\beta}(\mathcal{G})$. Consequently, $P(u - p_{l-s-1}(u)) \in V^{l-k}_{2,\beta}(\mathcal{G}) \subset W^{l-k}_{2,\beta}(\mathcal{G})$. Hence Pu belongs to $W^{l-k}_{2,\beta}(\mathcal{G})$ for every $u \in W^l_{2,\beta}(\mathcal{G})$ if and only if $P p_{l-s-1}(u) \in W^{l-k}_{2,\beta}(\mathcal{G})$ for every $u \in W^l_{2,\beta}(\mathcal{G})$, i.e. $P\Pi_{l-s-1} \subset W^{l-k}_{2,\beta}(\mathcal{G})$. If the last condition is satisfied, then the estimate

$$\begin{aligned}\|Pu\|_{W^{l-k}_{2,\beta}(\mathcal{G})} &\le \|P(u - p_{l-s-1}(u))\|_{V^{l-k}_{2,\beta}(\mathcal{G})} + \|P p_{l-s-1}\|_{W^{l-k}_{2,\beta}(\mathcal{G})} \\ &\le c\Big(\|u - p_{l-s-1}\|_{V^l_{2,\beta}(\mathcal{G})} + \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha u)(0)|\Big)\end{aligned}$$

holds for all $u \in W^l_{2,\beta}(\mathcal{G})$. By Theorem 7.1.1, the last expression defines a norm in $W^l_{2,\beta}(\mathcal{G})$ which is equivalent to the norm (7.1.8). This proves the continuity of the operator P. ■

COROLLARY 7.2.1. *Let P be an admissible operator of order $k \le l - 1$.*

1) *If $\beta \le -n/2$ or $\beta > l - n/2$, then the mapping $u \to Pu|_{\partial\mathcal{G}\backslash\{0\}}$ is continuous from $W^l_{2,\beta}(\mathcal{G})$ into $W^{l-k-1/2}_{2,\beta}(\partial\mathcal{G})$.*

2) *If $s - n/2 < \beta < s + 1 - n/2$ for some $s \in \{0, 1, \dots, l-1\}$, then $Pu|_{\partial\mathcal{G}} \in W^{l-k-1/2}_{2,\beta}(\partial\mathcal{G})$ for all $u \in W^l_{2,\beta}(\mathcal{G})$ if and only if $P\Pi_{l-s-1}|_{\partial\mathcal{G}} \subset W^{l-k-1/2}_{2,\beta}(\mathcal{G})$. Under the last condition, the operator $u \to Pu|_{\partial\mathcal{G}}$ is continuous from $W^l_{2,\beta}(\mathcal{G})$ into $W^{l-k-1/2}_{2,\beta}(\partial\mathcal{G})$.*

An analogous result holds for tangential differential operators on $\partial\mathcal{G}\backslash\{0\}$. Let P be a tangential admissible operator of order $k \le l-1$ on $\partial\mathcal{G}\backslash\{0\}$. If $\beta \le -n/2$ or $\beta > l - n/2$, then P continuously maps $W^{l-1/2}_{2,\beta}(\partial\mathcal{G})$ into $W^{l-k-1/2}_{2,\beta}(\partial\mathcal{G})$. In the case $s - n/2 < \beta < s + 1 - n/2$, $s \in \{0, 1, \dots l-1\}$, the operator P realizes a continuous mapping $W^{l-1/2}_{2,\beta}(\partial\mathcal{G}) \to W^{l-k-1/2}_{2,\beta}(\partial\mathcal{G})$ if and only if $P\Upsilon_{l-s-1} \subset W^{l-k-1/2}_{2,\beta}(\partial\mathcal{G})$.

Examples: 1) Let $P(x,\partial_x) = \sum_{|\alpha|\le k} a_\alpha(x)\,\partial_x^\alpha$ be a differential operator of order k with coefficients $a_\alpha \in C^\infty(\overline{\mathcal{G}})$. Then $P(x,\partial_x)\,p \in W_{2,\beta}^{l-k}(\mathcal{G})$ for every polynomial p and $\beta > -n/2$. Hence the mapping $u \to P(x,\partial_x)\,u$ is continuous from $W_{2,\beta}^l(\mathcal{G})$ into $W_{2,\beta}^{l-k}(\mathcal{G})$ and the mapping $u \to P(x,\partial_x)\,u|_{\partial\mathcal{G}}$ is continuous from $W_{2,\beta}^l(\mathcal{G})$ into $W_{2,\beta}^{l-k-1/2}(\partial\mathcal{G})$ if $\beta \notin \{1-n/2,\dots,l-n/2\}$.

2) We show that the operator

$$u \to Pu|_{\partial\mathcal{G}\backslash\{0\}} = \frac{\partial u}{\partial\nu}\Big|_{\partial\mathcal{G}\backslash\{0\}} \tag{7.2.1}$$

does not map $W_{2,\beta}^l(\mathcal{G})$ into $W_{2,\beta}^{l-3/2}(\partial\mathcal{G})$ if $l \ge 2$, $-n/2 < \beta < l-1-n/2$, and the cone $\mathcal{K}$ is not a half-space.

Obviously, the functions $p_j(x) = x_j$, $j = 1,\dots,n$, belong to $W_{2,\beta}^l(\mathcal{G})$. Suppose that the operator (7.2.1) maps $W_{2,\beta}^l(\mathcal{G})$ into $W_{2,\beta}^{l-3/2}(\partial\mathcal{G})$. Then there exist functions $v_j \in W_{2,\beta}^{l-1}(\mathcal{G})$ such that

$$v_j|_{\partial\mathcal{G}\backslash\{0\}} = \frac{\partial p_j}{\partial\nu}\Big|_{\partial\mathcal{G}\backslash\{0\}} = \cos(\nu(x), x_j)$$

for $j = 1,\dots,n$. By Lemma 7.1.3, the limits $v_j(0) = \lim_{x\to 0} v_j(x)$ exist. Since $\nu(x) = \nu(\varepsilon x)$ for $\varepsilon < 1$ and for sufficiently small $|x|$, it follows that $\cos(\nu(x),x_j) = c_j = const$ for $j = 1,\dots,n$ in a neighbourhood of the conical point. Hence the normal $\nu(x)$ is independent of x in a neighbourhood of the conical point, i.e., $\partial\mathcal{K}$ is a hyperplane. This contradicts our assumption.

3) If $n = 2$ and the boundary of $\mathcal{G}$ contains $d \ge 2$ conical points $x^{(1)},\dots,x^{(d)}$, then the mapping

$$u \to Pu|_{\Gamma_\tau} = \frac{\partial u}{\partial\nu}\Big|_{\Gamma_\tau}$$

(Γ_τ denotes the line of $\partial\mathcal{G}$ which connects the points $x^{(\tau)}$ and $x^{(\tau+1)}$) is continuous from $W_{2,\beta}^l(\mathcal{G})$ into $W_{2,\beta}^{l-3/2}(\Gamma_\tau)$ for $l \ge 2$, $\beta \notin \{0,1,\dots,l-1\}$.

7.2.2. Solvability of elliptic boundary value problems. We consider the boundary value problem

$$Lu = f \quad \text{in } \mathcal{G}, \tag{7.2.2}$$

$$Bu + C\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}\backslash\{0\}, \tag{7.2.3}$$

where L is a differential operator of order $2m$, B is a vector of differential operators B_k, $\operatorname{ord} B_k \le \mu_k$, and C is a matrix of tangential differential operators $C_{k,j}$ on $\partial\mathcal{G}\backslash\{0\}$, $\operatorname{ord} C_{k,j} \le \mu_k + \tau_j$, $j = 1,\dots,J$, $k = 1,\dots,m+J$. For the sake of brevity, we set

$$W_{2,\beta}^{l+\vec{\tau}-1/2}(\partial\mathcal{G}) \stackrel{def}{=} \prod_{j=1}^{J} W_{2,\beta}^{l+\tau_j-1/2}(\partial\mathcal{G}), \quad W_{2,\beta}^{l-\vec{\mu}-1/2}(\partial\mathcal{G}) \stackrel{def}{=} \prod_{k=1}^{m+J} W_{2,\beta}^{l-\mu_k-1/2}(\partial\mathcal{G}),$$

and

$$\Upsilon_{l+\vec{\tau}-1} \stackrel{def}{=} \prod_{j=1}^{J} \Upsilon_{l+\tau_j-1}\,.$$

Using Lemma 7.2.1 and Corollary 7.2.1, we get the following theorem.

THEOREM 7.2.1. *Let L, $B_{k,j}$, and $C_{k,j}$ be admissible operators of order $2m$, μ_k and $\mu_k+\tau_j$, respectively. Furthermore, let l be an integer, $l \ge 2m$, $l > \max \mu_k$.*

1) *If $\beta \le -n/2$ or $\beta > l+\max(\tau_j,0)-n/2$, then the operator $\mathcal{A}$ of the boundary value problem (7.2.2), (7.2.3) is a continuous mapping from*

$$W^{l}_{2,\beta}(\mathcal{G}) \times W^{l+\vec{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{7.2.4}$$

into

$$W^{l-2m}_{2,\beta}(\mathcal{G}) \times W^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}). \tag{7.2.5}$$

2) *If $s-n/2 < \beta < s+1-n/2$, where $s \in \{0,1,\dots,l+\max(\tau_j,0)-1\}$, then the operator $\mathcal{A}$ maps (7.2.4) into (7.2.5) if and only if*

$$\mathcal{A}\Big(\Pi_{l-s-1} \times \Upsilon_{l+\vec{\tau}-s-1}\Big) \subset W^{l-2m}_{2,\beta}(\mathcal{G}) \times W^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}).$$

Under this assumption, the operator $\mathcal{A}$ is continuous.

Let $\mathring{\Pi}_{l-s-1}$ be the subspace of Π_{l-s-1} which consists of all polynomials $p(\cdot)$ of degree not greater than $l-s-1$ such that

$$Lp \in W^{l-2m}_{2,\beta}(\mathcal{G}) \quad \text{and} \quad B_k p \in W^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{G}) \ \text{ for } k=1,\dots,m+J.$$

Furthermore, we define $\mathring{\Upsilon}_{l+\vec{\tau}-s-1}$ as the set of all vector-polynomials $\underline{p} \in \Upsilon_{l+\vec{\tau}-s-1}$ such that $C\underline{p} \in W^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})$. Then in the case $s-n/2 < \beta < s+1-n/2$, $s \in \{0,1,\dots,l+\max(\tau_j,0)\}$, the operator $\mathcal{A}$ continuously maps the space

$$(V^{l}_{2,\beta}(\mathcal{G}) \oplus \mathring{\Pi}_{l-s-1}) \times (V^{l+\vec{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \oplus \mathring{\Upsilon}_{l+\vec{\tau}-s-1}) \tag{7.2.6}$$

into (7.2.5).

In accordance with the notation in Chapter 6, let $\mathfrak{A}(\lambda)$ be the operator pencil of the problem

$$\mathcal{L}(\omega,\partial_\omega,\lambda)\,u = f \qquad \text{in } \Omega,$$
$$\mathcal{B}_k(\omega,\partial_\omega,\lambda)\,u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(\omega,\partial_\omega,\lambda+\tau_j)\,u_j = g_k \quad \text{on } \partial\Omega, \quad k=1,\dots,m+J,$$

where

$$\mathcal{L}(\omega,\partial_\omega,r\partial_r) = r^{2m}\,L^{(0)}(x,\partial_x)\,, \qquad \mathcal{B}_k(\omega,\partial_\omega,r\partial_r) = r^{\mu_k}\,B^{(0)}_k(x,\partial_x),$$
$$\mathcal{C}_{k,j}(\omega,\partial_\omega,r\partial_r) = r^{\mu_k+\tau_j}\,C^{(0)}_{k,j}(x,\partial_x)$$

and $L^{(0)}(x,\partial_x)$, $B^{(0)}_k(x,\partial_x)$, $C^{(0)}_{k,j}(x,\partial_x)$ are the leading parts of L, B_k and $C_{k,j}$ at the conical point $x=0$ (see Definition 6.2.1).

As a consequence of Theorems 6.3.1 and 6.3.3, the following result holds.

THEOREM 7.2.2. *Let L, B_k, and $C_{k,j}$ be admissible operators of order $2m$, μ_k, and $\mu_k + \tau_j$, respectively. We suppose that the boundary value problem (7.2.2), (7.2.3) is elliptic in $\overline{\mathcal{G}}\backslash\{0\}$ and there are no eigenvalues of $\mathfrak{A}(\lambda)$ on the line $\mathrm{Re}\,\lambda = -\beta + l - n/2$. Furthermore, we assume that $\beta + n/2 \notin \{1, 2, \dots, l + \max(\tau_j, 0) - 1\}$. Then*

1) *for every element $(u, \underline{u})$ of the space (7.2.4) such that $Lu \in W^{l-2m}_{2,\beta}(\mathcal{G})$, $Bu|_{\partial\mathcal{G}} + C\underline{u} \in W^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})$ there is the estimate*

$$\begin{aligned} \|u\|_{W^l_{2,\beta}(\mathcal{G})} &+ \|\underline{u}\|_{W^{l+\vec{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})} \\ &\le c\Big(\|Lu\|_{W^{l-2m}_{2,\beta}(\mathcal{G})} + \|Bu + C\underline{u}\|_{W^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})} + \|u\|_{W^{l-1}_{2,\beta}(\mathcal{G})} \\ &\quad + \sum_{j=1}^{J} \|u_j\|_{W^{\max(l+\tau_j-3/2,0)}_{2,\beta}(\partial\mathcal{G})}\Big) \end{aligned} \tag{7.2.7}$$

(here the norm of u_j in $W^0_{2,\beta}(\partial\mathcal{G})$ is defined as the L_2-norm of $r^\beta u_j$),

2) *$\mathcal{A}$ is a Fredholm operator from (7.2.6) into (7.2.5).*

Proof: 1) In the cases $\beta \le -n/2$, $\beta > l + \max(\tau_j, 0) - n/2$ the assertions of the theorem are obvious. We assume that $s - n/2 < \beta < s + 1 - n/2$, where $s \in \{0, 1, \dots, l + \max(\tau_j, 0) - 1\}$. By Theorems 7.1.1 and 7.1.2, there are the decompositions

$$u = v + p_{l-s-1}(u), \qquad u_j = v_j + \mathfrak{p}_{l+\tau_j-s-1}(u_j)$$

for the function u and the components u_j of the vector-function $\underline{u}$, where $v \in V^l_{2,\beta}(\mathcal{G})$, $v_j \in V^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{G})$,

$$\|u\|_{W^l_{2,\beta}(\mathcal{G})} \le c\Big(\|v\|_{V^l_{2,\beta}(\mathcal{G})} + \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha u)(0)|\Big),$$

$$\|u_j\|_{W^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{G})} \le c\Big(\|v_j\|_{V^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{G})} + \max_{\partial\mathcal{G}} |\mathfrak{p}_{l+\tau_j-s-1}(u_j)|\Big).$$

Furthermore, from Theorem 6.3.1 it follows that

$$\begin{aligned} \|v\|_{V^l_{2,\beta}(\mathcal{G})} + \|\underline{v}\|_{V^{l+\vec{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})} \;&\le\; c\Big(\|Lv\|_{V^{l-2m}_{2,\beta}(\mathcal{G})} + \|Bv + C\underline{v}\|_{V^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})} \\ &\quad + \|v\|_{V^{l-1}_{2,\beta}(\mathcal{G})} + \|\underline{v}\|_{V^{l+\vec{\tau}-3/2}_{2,\beta}(\partial\mathcal{G})}\Big). \end{aligned}$$

Since the $V^{l-2m}_{2,\beta}(\mathcal{G})$- and $W^{l-2m}_{2,\beta}(\mathcal{G})$-norms are equivalent on $V^{l-2m}_{2,\beta}(\mathcal{G})$, we have

$$\begin{aligned} \|Lv\|_{V^{l-2m}_{2,\beta}(\mathcal{G})} \;&\le\; c\,\|Lv\|_{W^{l-2m}_{2,\beta}(\mathcal{G})} \le c\Big(\|Lu\|_{W^{l-2m}_{2,\beta}(\mathcal{G})} + \|L\,p_{l-s-1}(u)\|_{W^{l-2m}_{2,\beta}(\mathcal{G})}\Big) \\ &\le\; c\Big(\|Lu\|_{W^{l-2m}_{2,\beta}(\mathcal{G})} + \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha u)(0)|\Big). \end{aligned}$$

Analogously,

$$\begin{aligned} \|Bv + C\underline{v}\|_{V^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})} \;&\le\; c\Big(\|Bu + C\underline{u}\|_{W^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})} \\ &\quad + \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha u)(0)| + \sum_{j=1}^{J} \max_{\partial\mathcal{G}} |\mathfrak{p}_{l+\tau_j-s-1}(u_j)|\Big). \end{aligned}$$

Hence we get

$$\|u\|_{W^l_{2,\beta}(\mathcal{G})} + \|\underline{u}\|_{W^{l+\vec{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})} \le c\Big(\|Lu\|_{W^{l-2m}_{2,\beta}(\mathcal{G})} + \|Bu + C\underline{u}\|_{W^{l-\vec{\mu}-1/2}_{2,\beta}(\partial\mathcal{G})}$$
$$+\|v\|_{V^{l-1}_{2,\beta}(\mathcal{G})} + \|\underline{v}\|_{V^{l+\vec{\tau}-3/2}_{2,\beta}(\partial\mathcal{G})} + \sum_{|\alpha|\le l-s-1} |(\partial_x^\alpha u)(0)| + \sum_{j=1}^{J} \max_{\partial\mathcal{G}} |\mathfrak{p}_{l+\tau_j-s-1}(u_j)|\Big).$$

Using Lemma 7.1.6 and an analogous estimate for functions on the boundary, we obtain (7.2.7).

2) From (7.2.7) it follows that the range of the operator $\mathcal{A}$ is closed and the kernel is finite-dimensional. The finiteness of the dimension of the cokernel follows from the Fredholm property for the operator of the boundary value problem (7.2.2), (7.2.3) in corresponding spaces of the class $V^l_{2,\beta}$ and from the finiteness of the dimensions of the spaces $\Pi_{l-2m-s-1}$, $\Upsilon_{l-\mu_k-s-1}$. ■

7.2.3. Asymptotics of the solution. Now we study the behaviour of the solution

$$(u, \underline{u}) \in W^l_{2,\beta}(\mathcal{G}) \times W^{l+\vec{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$$

of the boundary value problem (7.2.2), (7.2.3) in a neighbourhood of the conical point if the right-hand side $(f, \underline{g})$ belongs to the space

$$W^{l_1-2m}_{2,\beta_1}(\mathcal{G}) \times W^{l_1+\vec{\mu}-1/2}_{2,\beta_1}(\partial\mathcal{G}), \tag{7.2.8}$$

where $\beta_1 - l_1 < \beta - l$. To this end, we impose similar assumptions on the coefficients of the operators L, B_k, and $C_{k,j}$ as in Section 6.4. We suppose that the coefficients a_α of L have the representation

$$a_\alpha(x) = r^{|\alpha|-2m}\Big(a_\alpha^{(0)}(\omega) + \sum_{\iota=1}^{\sigma(\alpha)} r^{\delta_\iota}\, a_\alpha^{(\iota)}(\omega, \log r) + r^{\delta_{\sigma(\alpha)+1}} a_\alpha^{(\sigma(\alpha)+1)}(\omega, r)\Big) \tag{7.2.9}$$

in a neighbourhood $\mathcal{U}$ of the conical point, where $\delta_1, \delta_2, \dots$ is a given sequence of complex numbers such that

$$0 < \operatorname{Re}\delta_1 \le \operatorname{Re}\delta_2 \le \dots$$

$a_\alpha^{(0)} \in C^\infty(\overline{\Omega})$, $a_\alpha^{(1)}, \dots, a_\alpha^{(\sigma(\alpha))}$ are polynomials of $\log r$ with coefficients from $C^\infty(\overline{\Omega})$, and the function $a_\alpha^{(\sigma(\alpha)+1)} \in C^\infty(\overline{\Omega} \times \mathbb{R}_+)$ satisfies the condition

$$|\,(r\partial_r)^\mu\, \partial_\omega^\gamma\, a_\alpha^{(\sigma(\alpha)+1)}(\omega, r)| < c_{\mu,\gamma} \tag{7.2.10}$$

for every integer $\mu \ge 0$ and every multi-index γ, where the constants $c_{\mu,\gamma}$ are independent of ω and r.

Furthermore, we assume that the coefficients $b_{k;\alpha}$, $c_{k,j;\alpha}$ of B_k and $C_{k,j}$ in a neighbourhood of the conical point have the form

$$(7.2.11) \qquad b_{k;\alpha}(x) = r^{|\alpha|-\mu_k}\Big(b^{(0)}_{k;\alpha}(\omega) + \sum_{\iota=1}^{\sigma(\alpha)} r^{\delta_\iota}\, b^{(\iota)}_{k;\alpha}(\omega,\log r) + r^{\delta_{\sigma(\alpha)+1}} b^{(\sigma(\alpha)+1)}_{k;\alpha}(\omega,r)\Big)$$

$$(7.2.12) \qquad c_{k,j;\alpha}(x) = r^{|\alpha|-\mu_k-\tau_j}\Big(c^{(0)}_{k,j;\alpha}(\omega) + \sum_{\iota=1}^{\sigma(\alpha,j)} r^{\delta_\iota}\, c^{(\iota)}_{k,j;\alpha}(\omega,\log r) + r^{\delta_{\sigma(\alpha,j)+1}} c^{(\sigma(\alpha,j)+1)}_{k,j;\alpha}(\omega,r)\Big),$$

where $b^{(\iota)}_{k;\alpha}$, $c^{(\iota)}_{k,j;\alpha}$ satisfy the same conditions as $a^{(\iota)}_\alpha$.

Using the regularity assertions for solutions of elliptic boundary value problems in the class of the spaces $V^l_{2,\beta}$, we can prove the following theorem.

THEOREM 7.2.3. *Let* $(u,\underline{u}) \in W^l_{2,\beta}(\mathcal{G}) \times W^{l+\vec{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ *be a solution of the boundary value problem (7.2.2), (7.2.3), where*

$$(f,\underline{g}) \in W^{l-2m+k}_{2,\beta+k}(\mathcal{G}) \times W^{l+k-\vec{\mu}-1/2}_{2,\beta+k}(\partial\mathcal{G}),$$

$l \ge 2m$, $l > \max \mu_k$, $\beta + n/2 \notin \{1, \dots, l + \max(\tau_j, 0)\}$. *Suppose that the boundary value problem (7.2.2), (7.2.3) is elliptic in* $\overline{\mathcal{G}}\backslash\{0\}$ *and the coefficients of the operators* L, B_k, $C_{k,j}$ *have the representations (7.2.9)–(7.2.12), where*

$$\operatorname{Re}\delta_{\sigma(\alpha)+1} + |\alpha| > -\beta + l - n/2,$$
$$\operatorname{Re}\delta_{\sigma(\alpha,j)+1} + |\alpha| > -\beta + l + \tau_j - n/2.$$

Then $(u,\underline{u}) \in W^{l+k}_{2,\beta+k}(\mathcal{G}) \times W^{l+k+\vec{\tau}-1/2}_{2,\beta+k}(\partial\mathcal{G})$.

Proof: For $\beta \le -n/2$ and $\beta > l + \max(\tau_j,0) - n/2$ the Sobolev spaces in the formulation of the theorem coincide with the corresponding spaces of the class $V^l_{2,\beta}$ and the assertion of the theorem is an immediate consequence of Lemma 6.3.1.

We assume that $s - n/2 < \beta < s + 1 - n/2$, $s \in \{0, 1, \dots, l - 1 + \max(\tau_j, 0)\}$. Then there are the decompositions

$$u = v + p_{l-s-1}(u), \qquad f = \varphi + p_{l-2m-s-1}(f),$$

where $v \in V^l_{2,\beta}(\mathcal{G})$, $\varphi \in V^{l+k-2m}_{2,\beta+k}(\mathcal{G})$. From the equation $Lu = f$ it follows that

$$Lv = \varphi + p_{l-2m-s-1}(f) - L\,p_{l-s-1}(u).$$

Here $p_{l-2m-s-1}(f) - L\,p_{l-s-1}(u)$ can be written in the form

$$p_{l-2m-s-1}(f) - \sum_{|\alpha|\le 2m} a_\alpha(x)\, p_{l-s-1-|\alpha|}(\partial_x^\alpha u)$$
$$= \sum_{|\gamma|\le l-2m-s-1} \frac{1}{\gamma!}(\partial_x^\gamma f)(0)\, x^\gamma - \sum_{\substack{|\alpha|+|\gamma|\le l-s-1\\ |\alpha|\le 2m}} \frac{1}{\gamma!} a_\alpha(x)\, (\partial_x^{\alpha+\gamma} u)(0)\, x^\gamma = F_1 + F_2\,,$$

where $F_1 \in V^{l+k-2m}_{2,\beta+k}(\mathcal{G})$ and F_2 is a finite sum of terms of the form

$$a(\omega)\, r^\sigma \quad \text{and} \quad a(\omega)\,(\log r)^j\, r^{\sigma+\delta_\iota}$$

with integer j, σ, $j \geq 0$, $\sigma + \delta_\iota \leq l - 2m - \beta - n/2$, $a \in C^\infty(\overline{\Omega})$. Since

$$p_{l-2m-s-1}(f) - L\, p_{l-s-1}(u) = Lv - \varphi \in V_{2,\beta}^{l-2m}(\mathcal{G}),$$

we get $F_2 = 0$. This implies $Lv \in V_{2,\beta+k}^{l+k-2m}(\mathcal{G})$.

Let $\underline{v} = (v_1, \dots, v_j) \in V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G})$ be the vector-function defined by the equalities

$$u_j = \mathfrak{p}_{l+\tau_j-s-1}(u_j) + v_j\,.$$

Then in the same way as above, we obtain $Bv|_{\partial\mathcal{G}\setminus\{0\}} + C\underline{v} \in V_{2,\beta+k}^{l+k-\vec{\mu}-1/2}(\partial\mathcal{G})$, and from Lemma 6.3.1 we conclude that $v \in V_{2,\beta+k}^{l+k}(\mathcal{G})$ and $\underline{v} \in V_{2,\beta+k}^{l+k+\vec{\tau}-1/2}(\partial\mathcal{G})$. Now the assertion of the theorem follows from Theorems 7.1.1 and 7.1.2. ■

Let l_1 and β_1 be given numbers. For every complex number λ_0 we denote by $\Lambda(\lambda_0)$ the set of all sums

$$\delta = \delta_{\iota_1} + \cdots + \delta_{\iota_k}$$

formed by the numbers δ_ι in (7.2.9) - (7.2.12) such that $\operatorname{Re}(\lambda_0+\delta) \leq -\beta_1+l_1-n/2$.

THEOREM 7.2.4. *Suppose that the boundary value problem (7.2.2), (7.2.3) is elliptic, the coefficients of* L, B_k, $C_{k,j}$ *have the representations (7.2.9)–(7.2.12) in a neighbourhood of the conical point, where*

$$0 < \operatorname{Re}\delta_1 \leq \operatorname{Re}\delta_2 \leq \dots$$

and, in addition,

$$\operatorname{Re}\delta_{\sigma(\alpha)+1} > (\beta - l) - (\beta_1 - l_1), \qquad \operatorname{Re}\delta_{\sigma(\alpha,j)+1} > (\beta - l) - (\beta_1 - l_1),$$
$$\operatorname{Re}\delta_{\sigma(\alpha)+1} + |\alpha| > -\beta_1 + l_1 - n/2\,, \quad \operatorname{Re}\delta_{\sigma(\alpha,j)+1} + |\alpha| > -\beta_1 + l_1 + \tau_j - n/2.$$

Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}(\lambda)$ *on the lines* $\operatorname{Re}\lambda = -\beta+l-n/2$ *and* $\operatorname{Re}\lambda = -\beta_1+l_1-n/2$, *where* $l, l_1 \geq 2m$, $l, l_1 > \max \mu_k$, $\beta+n/2 \notin \{1, 2, \dots, l+\max(\tau_j, 0)\}$, $\beta_1+n/2 \notin \{1, 2, \dots, l_1 - \min(2m, \mu_k)\}$.

If $(u, \underline{u}) \in W_{2,\beta}^{l}(\mathcal{G}) \times W_{2,\beta}^{l+\vec{\tau}-1/2}(\partial\mathcal{G})$ *is a solution of the boundary value problem (7.2.2), (7.2.3) with right-hand sides* $f \in W_{2,\beta_1}^{l_1-2m}(\mathcal{G})$ *and* $\underline{g} \in W_{2,\beta_1}^{l_1-\vec{\mu}-1/1}(\partial\mathcal{G})$, *then* $(u, \underline{u}) = (u, u_1, \dots, u_J)$ *admits the decomposition*

$$(7.2.13) \qquad u = p_{l-s-1}(u) + \sum_{\mu=1}^{N} \sum_{\delta\in\Lambda(\lambda_\mu)} r^{\lambda_\mu+\delta}\, u_{\mu,\delta}^{(1)}(\omega, \log r) + \sum_{\mu=0}^{l_1-s_1-1} \sum_{\delta\in\Lambda(\mu)} r^{\mu+\delta}\, u_{\mu,\delta}^{(2)}(\omega, \log r) + w\,,$$

$$(7.2.14) \qquad u_j = \mathfrak{p}_{l+\tau_j-s-1}(u_j) + \sum_{\mu=1}^{N} \sum_{\delta\in\Lambda(\lambda_\mu)} r^{\lambda_\mu+\tau_j+\delta}\, u_{j;\mu,\delta}^{(1)}(\omega, \log r) + \sum_{\mu=0}^{l_1-s_1-1} \sum_{\delta\in\Lambda(\mu)} r^{\mu+\tau_j+\delta}\, u_{j;\mu,\delta}^{(2)}(\omega, \log r) + w_j$$

in a neighbourhood of the conical point. Here $\lambda_1, \dots, \lambda_N$ *are the eigenvalues of* $\mathfrak{A}(\lambda)$ *lying in the strip* $-\beta + l - n/2 < \operatorname{Re}\lambda < -\beta_1 + l_1 - n/2$, $w \in V_{2,\beta_1}^{l_1}(\mathcal{K})$,

$w_j \in V_{2,\beta_1}^{l_1+\tau_j-1/2}(\partial\mathcal{K})$, *and* $u_{\mu,\delta}^{(k)}$, $u_{j;\mu,\delta}^{(k)}$ *are polynomials of* $\log r$ *with coefficients from* $C^\infty(\overline{\Omega})$ *and* $C^\infty(\partial\Omega)$, *respectively.*

Proof: Let $s - n/2 < \beta < s + 1 - n/2$, $s_1 - n/2 < \beta_1 < s_1 + 1 - n/2$, where $s \in \{0, 1, \dots, l + \max(\tau_j, 0) - 1\}$, $s_1 \in \{0, 1, \dots, l_1 - \max(2m, \mu_k) - 1\}$. Note that under the assumptions of the theorem, $l - s$ does not exceed $l_1 - s_1$. By Theorem 7.1.1, the functions u and f can be written in the form

$$u = v + p_{l-s-1}(u)\,, \qquad f = \varphi + p_{l_1-2m-s_1-1}(f),$$

where $v \in V_{2,\beta}^l(\mathcal{G})$, $\varphi \in V_{2,\beta_1}^{l_1-2m}(\mathcal{G})$. Then the equation $Lu = f$ yields

$$Lv = \varphi + p_{l_1-2m-s_1-1}(f) - L\,p_{l-s-1}(u).$$

Here

$$p_{l_1-2m-s_1-1}(f) - L\,p_{l-s-1}(u) = \sum_{|\gamma|\le l_1-2m-s_1-1} \frac{1}{\gamma!}(\partial_x^\gamma f)(0)\,x^\gamma - \sum_{\substack{|\alpha|+|\gamma|\le l-s-1\\ |\alpha|\le 2m}} \frac{1}{\gamma!}a_\alpha(x)\,(\partial_x^{\alpha+\gamma}u)(0)\,x^\gamma$$

is a finite sum of terms of the form

$$r^{\nu-2m}\,a(\omega)\,, \qquad 0 \le \nu \le l_1 - s_1 - 1, \tag{7.2.15}$$

$$r^{\delta_\iota+\nu-2m}\,a_1(\omega, \log r)\,, \qquad 0 \le \nu \le l - s - 1, \tag{7.2.16}$$

$$r^{\delta_{\sigma(\alpha)+1}+\nu-2m}\,a_2(\omega, r)\,, \qquad |\alpha| \le \nu \le l - s - 1, \tag{7.2.17}$$

where $a \in C^\infty(\overline{\Omega})$, a_1 is a polynomial of $\log r$ with coefficients from $C^\infty(\overline{\Omega})$, and a_2 satisfies the condition (7.2.10). The term (7.2.17) belongs to $V_{2,\beta_1}^{l_1-2m}(\mathcal{G})$. Since

$$p_{l_1-2m-s_1-1}(f) - L\,p_{l-s-1}(u) = Lv - \varphi \in V_{2,\beta}^{l-2m}(\mathcal{G}),$$

all terms of the form (7.2.15) with $\nu \le l - \beta - n/2$ and all terms of the form (7.2.16) with $\delta_\iota + \nu \le l - \beta - n/2$ vanish. Hence Lv is the sum of a function from $V_{2,\beta_1}^{l_1-2m}(\mathcal{G})$, of finitely many functions of the form (7.2.15) with $l - s \le \nu \le l_1 - s_1 - 1$, and of finitely many functions of the form (7.2.16) with $\max(0, -\delta_\iota + l - \beta - n/2) < \nu \le l - s - 1$.

An analogous representation holds for

$$B_k\,v + \sum_{j=1}^J C_{k,j}\,\big(u_j - \mathfrak{p}_{l+\tau_j-s-1}(u_j)\big).$$

Applying Theorem 6.4.2 to the functions v and $v_j = u_j - \mathfrak{p}_{l+\tau_j-s-1}(u_j)$, we get (7.2.13) and (7.2.14). ∎

REMARK 7.2.1. Analogous assertions to Theorems 7.2.1 - 7.2.4 are valid in the case $n = 2$ if $\partial\mathcal{G}$ contains $d \ge 2$ conical points and we consider the operator of the boundary value problem (7.2.2), (7.2.3) as a mapping from

$$W_{2,\beta}^l(\mathcal{G}) \times \prod_{\tau=1}^d \prod_{j=1}^J W_{2,\beta}^{l+\tau_j-1/2}(\Gamma_\tau)$$

into

$$W_{2,\beta}^{l-2m}(\mathcal{G}) \times \prod_{\tau=1}^d \prod_{k=1}^{m+J} W_{2,\beta}^{l-\mu_k-1/2}(\Gamma_\tau).$$

Then different boundary conditions on the lines Γ_τ of $\partial\mathcal{G}$ are allowed.

7.3. Weighted Sobolev spaces with critical value of the weight parameter

In Section 7.1 we have studied the relations between the space $W^l_{2,\beta}(\mathcal{G})$ and its subspace $V^l_{2,\beta}(\mathcal{G})$. Here the cases $\beta + n/2 = 1, 2, \dots, l$ were excluded. We have shown that $W^l_{2,\beta}(\mathcal{G})$ is the direct sum of the space $V^l_{2,\beta}(\mathcal{G})$ and a finite-dimensional space of polynomials if $\beta + n/2 \notin \{1, 2, \dots, l\}$. This is not true if $\beta + n/2$ is an integer between one and l.

The goal of this section is to study the relations between the spaces $V^l_{2,\beta}(\mathcal{G})$ and $W^l_{2,\beta}(\mathcal{G})$ and between the corresponding trace spaces for the "critical" weight parameters $\beta = 1 - n/2, \dots, l - n/2$. We show that in this case every function $u \in W^l_{2,\beta}(\mathcal{G})$ can be decomposed into a "quasipolynomial" and a function from $V^l_{2,\beta}(\mathcal{G})$, where the coefficients of the quasipolynomial are functions of the distance r to the conical point. An analogous result holds for the trace spaces.

As in Section 7.2, we give some applications of these results to elliptic boundary value problems.

7.3.1. Relations between weighted Sobolev spaces with homogeneous and nonhomogeneous norms. For the sake of simplicity, we suppose again that the domain $\mathcal{G}$ contains only one conical point O and coincides with the cone $\mathcal{K} = \{x : x/|x| \in \Omega\}$ for $|x| < \varepsilon$, where Ω is a smooth domain on the sphere S^{n-1}. For an arbitrary function u on $\mathcal{G}$ we set

$$\mathring{u}(r) = \frac{1}{|\Omega|} \int_\Omega u(\omega, r)\, d\omega, \qquad 0 < r < \varepsilon. \tag{7.3.1}$$

Weighted Sobolev spaces on the half-axis. We define the weighted Sobolev spaces $V^l_{2,\beta}(\mathbb{R}_+)$ and $W^l_{2,\beta}(\mathbb{R}_+)$ for nonnegative integer l and real β as the set of all functions u on the interval $(0, +\infty)$ with finite norms

$$\|u\|_{V^l_{2,\beta}(\mathbb{R}_+)} = \Big(\int_0^\infty r^{2(\beta - l + j)} \sum_{j=0}^{l} |u^{(j)}(r)|^2\, dr \Big)^{1/2} \tag{7.3.2}$$

and

$$\|u\|_{W^l_{2,\beta}(\mathbb{R}_+)} = \Big(\int_0^\infty r^{2\beta} \sum_{j=0}^{l} |u^{(j)}(r)|^2\, dr \Big)^{1/2}, \tag{7.3.3}$$

respectively. Replacing the upper limit ∞ in the integrals in (7.3.2) and (7.3.3) by ε, we obtain the norms in the space $V^l_{2,\beta}((0,\varepsilon))$ and $W^l_{2,\beta}((0,\varepsilon))$, respectively.

LEMMA 7.3.1. 1) *If $u \in W^l_{2,\beta}(\mathcal{G})$, then $\mathring{u} \in W^l_{2,\beta+(n-1)/2}(\mathbb{R}_+)$.*

2) *Let $f \in W^1_{2,\beta+(n-1)/2}(\mathbb{R}_+)$ and let ζ be a smooth function on $\mathbb{R}_+$ with support in $[0, \varepsilon)$. Then the function $x \to \zeta(|x|)\, f(|x|)$ belongs to the space $W^1_{2,\beta}(\mathcal{G})$.*

3) *Let ζ be an arbitrary smooth function on $\mathbb{R}_+$ with support in $[0, \varepsilon)$ equal to one in $(0, \varepsilon/2)$. Then there exists a constant c such that*

$$\int_{\mathcal{G}} r^{2(\beta-1)} \big| u(x) - \zeta(|x|)\ \mathring{u}(|x|) \big|^2 dx \le c\, \|u\|^2_{W^1_{2,\beta}(\mathcal{G})}$$

for all $u \in W^1_{2,\beta}(\mathcal{G})$.

Proof: Using the inequality

$$\sum_{j=0}^{l} |\partial_r^j \overset{\circ}{u}(r)|^2 = \frac{1}{|\Omega|^2} \sum_{j=0}^{l} \Big| \int_\Omega \partial_r^j u(\omega, r)\, d\omega \Big|^2 \le c \sum_{|\alpha|\le l} \int_\Omega \big|(\partial_x^\alpha u)(\omega, r)\big|^2\, d\omega,$$

we obtain

$$\| \overset{\circ}{u} \|^2_{W^l_{2,\beta+(n-1)/2}((0,\varepsilon))} \le c \sum_{|\alpha|\le l} \int_0^\varepsilon \int_\Omega r^{2\beta+n-1}\, |D_x^\alpha u|^2\, d\omega\, dr \le c\, \|u\|^2_{W^l_{2,\beta}(\mathcal{G})} .$$

This proves the first assertion.

Assertion 2) follows from the inequality $|\partial_{x_j} f(|x|)| \le c\, |f'(r)|$. Furthermore, the inequality

$$\int_\Omega \int_\Omega \big|u(\omega, r) - u(\omega', r)\big|^2\, d\omega\, d\omega' \le c\, r^2 \int_\Omega \big|(\nabla_x u)(\omega, r)\big|^2\, d\omega$$

implies

$$\begin{aligned} &\int\limits_{\substack{\mathcal{G} \\ |x|<\varepsilon/2}} r^{2(\beta-1)}\, \big|u(x) - \zeta(|x|)\, \overset{\circ}{u}(|x|)\big|^2\, dx \\ &= \int_0^{\varepsilon/2} r^{2\beta+n-3} \int_\Omega \int_\Omega \big|u(\omega, r) - u(\omega', r)\big|^2\, d\omega\, d\omega'\, dr \\ &\le c \int_0^{\varepsilon/2} \int_\Omega r^{2\beta+n-1}\, |\nabla_x u|^2\, d\omega\, dr \le c\, \|u\|^2_{W^1_{2,\beta}(\mathcal{G})} \end{aligned}$$

This proves the third assertion. ■

LEMMA 7.3.2. *The space $W^1_{2,1/2}((0,\varepsilon))$ is continuously imbedded into the space $W_2^{1/2}((0,\varepsilon))$.*

Proof: We have to prove that the norm in $W^1_{2,1/2}((0,\varepsilon))$ can be estimated from below by the norm

$$\|u\|_{W_2^{1/2}((0,\varepsilon))} = \Big(\|u\|^2_{L_2((0,\varepsilon))} + \int_0^\varepsilon \int_0^\varepsilon \Big| \frac{u(r) - u(\rho)}{r - \rho} \Big|^2\, dr\, d\rho \Big)^{1/2}. \tag{7.3.4}$$

Obviously, the norm (7.3.4) is equivalent to the norm

$$\|u\| = \Big(\|u\|^2_{L_2((0,\varepsilon))} + \int_0^\varepsilon \int_{\rho/2}^\rho \Big| \frac{u(r) - u(\rho)}{r - \rho} \Big|^2\, dr\, d\rho \Big)^{1/2}.$$

Using Hardy's inequality, we can easily show that

$$\|u\|_{L_2((0,\varepsilon))} \le c\, \|u\|_{W^1_{2,1/2}((0,\varepsilon))} .$$

Furthermore, by means of the equality

$$\frac{u(r)-u(\rho)}{r-\rho} = \int_0^1 u'\big(r+t(\rho-r)\big)\,dt,$$

we obtain

$$\int_0^\varepsilon \int_{\rho/2}^{\rho} \Big|\frac{u(r)-u(\rho)}{r-\rho}\Big|^2 dr\,d\rho \le \int_0^\varepsilon \int_0^1 \int_{\rho/2}^{\rho} |u'(r+t(\rho-r))|^2\,dr\,dt\,d\rho$$

$$= \int_0^\varepsilon \int_0^1 \int_{(1+t)\rho/2}^{\rho} |u'(r)|^2\,dr\,dt\,d\rho = \int_0^\varepsilon \int_{\rho/2}^{\rho} \int_0^{2r/\rho-1} |u'(r)|^2\,dt\,dr\,d\rho$$

$$= \int_0^\varepsilon \int_{\rho/2}^{\rho} \Big(\frac{2r}{\rho}-1\Big)\,|u'(r)|^2\,dr\,d\rho \le \int_0^\varepsilon \int_r^{2r} \Big(\frac{2r}{\rho}-1\Big)\,|u'(r)|^2\,d\rho\,dr$$

$$= (2\log 2 - 1)\int_0^\varepsilon r\,|u'(r)|^2\,dr.$$

This proves the lemma. ■

Let ζ be a smooth cut-off function on $\mathbb{R}_+$ with support in $[0,\varepsilon)$ which is equal to one in $[0,\varepsilon/2)$, and let $\psi \in C_0^\infty((\frac{1}{2},1))$ be a function satisfying the condition

$$\int_{1/2}^{1} \psi(r)\,dr = 1.$$

We introduce the following integral operator on the space $W_2^{1/2}((0,\varepsilon))$:

$$(Kf)(r) = \zeta(r)\int_{1/2}^{1} f(tr)\,\psi(t)\,dt, \qquad 0 < r < \varepsilon. \tag{7.3.5}$$

For $r > \varepsilon$ we set $(Kf)(r) = 0$.

LEMMA 7.3.3. 1) *The operator (7.3.5) realizes a continuous mapping*

$$W_2^{1/2}((0,\varepsilon)) \to W_{2,l-1/2}^{l}(\mathbb{R}_+)$$

for arbitrary integer $l \ge 1$.

2) *There exists a constant c such that*

$$\int_0^\varepsilon r^{-1}\,\big|f(r)-(Kf)(r)\big|^2\,dr \le c\,\|f\|^2_{W_2^{1/2}((0,\varepsilon))} \tag{7.3.6}$$

for arbitrary $f \in W_2^{1/2}((0,\varepsilon))$.

Proof: 1) Obviously,

$$(7.3.7)\quad \int\limits_0^\infty r\,\big|(Kf)(r)\big|^2\,dr \;\le\; \int\limits_0^\varepsilon r\Big|\int\limits_{1/2}^1 f(tr)\,\psi(t)\,dt\Big|^2\,dr$$
$$\le\; \frac{1}{2}\int\limits_0^\varepsilon\int\limits_{r/2}^r |f(t)|^2\,|\psi(t/r)|^2\,dt\,dr \le c\int\limits_0^\varepsilon |f(t)|^2\,dt.$$

We estimate the L_2 norms of $r^{k-1/2}d^kKf/dr^k$ for $k\ge 1$. Since

$$\int\limits_{r/2}^r \frac{d^k}{dr^k}\,\frac{1}{r}\,\psi(t/r)\,dt = \frac{d^k}{dr^k}\int\limits_{r/2}^r \frac{1}{r}\,\psi(t/r)\,dt = \frac{d^k}{dr^k}\int\limits_{1/2}^1 \psi(t)\,dt = 0\,,$$

we have

$$\frac{d^k}{dr^k}\,(Kf)(r) = \sum_{j=0}^k \binom{k}{j}\,\zeta^{(k-j)}(r)\,\frac{d^j}{dr^j}\int\limits_{1/2}^1 f(tr)\,\psi(t)\,dt$$
$$= \sum_{j=0}^k \binom{k}{j}\,\zeta^{(k-j)}(r)\int\limits_{r/2}^r f(t)\,\frac{d^j}{dr^j}\,\frac{1}{r}\psi(t/r)\,dt$$
$$= \sum_{j=0}^{k-1} \binom{k}{j}\,\zeta^{(k-j)}(r)\int\limits_{r/2}^r f(t)\,\frac{d^j}{dr^j}\,\frac{\psi(t/r)}{r}\,dt + \zeta(r)\int\limits_{r/2}^r \big(f(t)-f(r)\big)\,\frac{d^k}{dr^k}\,\frac{\psi(t/r)}{r}\,dt$$

for $k\ge 1$. The first term on the right belongs to an arbitrary weighted space $W^0_{2,\beta}(\mathbb{R}_+)$, since $\zeta^{(k)}$ vanishes outside the interval $(\frac{\varepsilon}{2},\varepsilon)$. In order to estimate the last term, we use the inequality

$$\Big|\frac{d^k}{dr^k}\,\frac{1}{r}\,\psi(t/r)\Big| \le c\,r^{-k-1}\,,$$

where c is a positive constant. Then we obtain

$$\int\limits_0^\varepsilon r^{2k-1}\Big|\int\limits_{r/2}^r \big(f(t)-f(r)\big)\,\frac{d^k}{dr^k}\,\frac{1}{r}\psi(t/r)\,dt\Big|^2\,dr$$
$$\le c\int\limits_0^\varepsilon\int\limits_{r/2}^r r^{-2}\,|f(t)-f(r)|^2\,dt\,dr \le c\int\limits_0^\varepsilon\int\limits_{r/2}^r \Big|\frac{f(t)-f(r)}{t-r}\Big|^2\,dt\,dr$$

This proves the first assertion.

2) Analogously, by means of the equality

$$f(r)-(Kf)(r) \;=\; \big(1-\zeta(r)\big)\,f(r) + \zeta(r)\int\limits_{1/2}^1 \big(f(r)-f(tr)\big)\,\psi(t)\,dt$$
$$=\; \big(1-\zeta(r)\big)\,f(r) + \frac{1}{r}\,\zeta(r)\int\limits_{r/2}^r \big(f(r)-f(t)\big)\,\psi(t/r)\,dt,$$

we obtain (7.3.6). The proof is complete. ∎

Relations between the spaces $V^1_{2,1-n/2}(\mathcal{G})$ *and* $W^1_{2,1-n/2}(\mathcal{G})$. In Section 7.1 it was proved that the spaces $W^1_{2,\beta}(\mathcal{G})$ and $V^1_{2,\beta}(\mathcal{G})$ coincide if $\beta \le -n/2$ or $\beta > 1-n/2$, while in the case $-n/2 < \beta < 1-n/2$ we have

$$W^1_{2,\beta}(\mathcal{G} = V^1_{2,\beta}(\mathcal{G}) \oplus \Pi_0\,,$$

where Π_0 is the set of the constant functions. The following theorem describes the relations between the spaces $W^1_{2,\beta}(\mathcal{G})$ and $V^1_{2,\beta}(\mathcal{G})$ in the case $\beta = 1-n/2$.

THEOREM 7.3.1. *Let* u *be an arbitrary function from* $W^1_{2,1-n/2}(\mathcal{G})$. *Then there exists a function* $u_0 \in W^{1/2}_2((0,\varepsilon))$ *such that*

$$u - Ku_0 \in V^1_{2,1-n/2}(\mathcal{G}). \tag{7.3.8}$$

Here by Ku_0 *we mean the function* $x \to (Ku_0)(|x|)$. *The function* u_0 *in (7.3.8) is uniquely determined by the function* u *up to the equivalence relation*

$$f \sim g \quad \overset{def}{\Longleftrightarrow} \quad \int\limits_0^\varepsilon r^{-1}\,|f(r)-g(r)|^2\,dr < \infty. \tag{7.3.9}$$

More precisely, (7.3.8) is valid if and only if $u_0 \sim \overset{\circ}{u}$, *where the function* $\overset{\circ}{u}$ *is defined by (7.3.1).*

Proof: By Lemmas 7.3.1, 7.3.2 and 7.3.3, the function $u_0 = \overset{\circ}{u}$ satisfies (7.3.8). If $u_0 \in W^{1/2}_2((0,\varepsilon))$ is an arbitrary function such that $u_0 \sim \overset{\circ}{u}$, then it follows from Lemma 7.3.3 that $K(u_0 - \overset{\circ}{u}) \in V^1_{2,1/2}(\mathbb{R}_+)$. Consequently, by the second part of Lemma 7.3.1, the function $x \to (Ku_0)(|x|) - (K\overset{\circ}{u})(|x|)$ belongs to $V^1_{2,1-n/2}(\mathcal{G})$. Since $u - K\overset{\circ}{u} \in V^1_{2,1-n/2}(\mathcal{G})$, we get $u - Ku_0 \in V^1_{2,1-n/2}(\mathcal{G})$.

Finally, if $u_0 \in W^{1/2}_2((0,\varepsilon))$ is an arbitrary function such that $u - Ku_0$ belongs to $V^1_{2,1-n/2}(\mathcal{G})$, then we get $K(u_0 - \overset{\circ}{u}) \in V^1_{2,1-n/2}(\mathcal{G})$ and, consequently,

$$\int\limits_0^\varepsilon r^{-1}\,|K(u_0 - \overset{\circ}{u})(r)|^2\,dr = \int\limits_{\substack{\mathcal{G}\\ |x|<\varepsilon}} r^{-n}\,\big|K(u_0 - \overset{\circ}{u})(|x|)\big|^2\,dx < \infty.$$

This together with Lemma 7.3.3 implies $u_0 \sim \overset{\circ}{u}$. The theorem is proved. ∎

Relations between the spaces $V^l_{2,\beta}(\mathcal{G})$ *and* $W^l_{2,\beta}(\mathcal{G})$ *for* $l \ge 1$ *and* $\beta = s-n/2$. By Theorem 7.1.1, the spaces $V^l_{2,\beta}(\mathcal{G})$ and $W^l_{2,\beta}(\mathcal{G})$ coincide if $\beta \le -n/2$ or $\beta > l-n/2$, while in the case $s - n/2 < \beta < s+1-n/2$, $s \in \{0,1,\dots,l-1\}$ there is the representation

$$W^l_{2,\beta}(\mathcal{G}) = V^l_{2,\beta}(\mathcal{G}) \oplus \Pi_{l-s-1}\,.$$

Now we consider the case $\beta = s-n/2$, $s \in \{1,\dots,l\}$. We introduce the functions

$$\overset{\circ}{u}_\alpha(r) = \frac{1}{|\Omega|}\int\limits_\Omega (\partial_x^\alpha u)(\omega, r)\,d\omega, \quad 0 < r < \varepsilon. \tag{7.3.10}$$

If $u \in W^l_{2,s-n/2}(\mathcal{G})$, $s \in \{1,\dots,l\}$, $|\alpha| = l - s$, then, by Lemma 7.1.5,

$$\partial_x^\alpha u \in W^s_{2,s-n/2}(\mathcal{G}) \subset W^1_{2,1-n/2}(\mathcal{G})$$

and, therefore, $\overset{\circ}{u}_\alpha \in W_2^{1/2}((0,\varepsilon))$ (see Lemmas 7.3.1, 7.3.2). Moreover, the functions $\partial_x^\alpha u$ are continuous at the origin for $|\alpha| < l - s - 1$.

LEMMA 7.3.4. *Let $u \in W^l_{2,s-n/2}(\mathcal{G})$, where $s \in \{1,\dots,l\}$. Then*

$$(7.3.11) \qquad \int_{\mathcal{G}} r^{2(s-l+|\alpha|)-n} \Big| \partial_x^\alpha u - \sum_{|\gamma| \le l-s-1-|\alpha|} (\partial_x^{\alpha+\gamma} u)(0) \frac{x^\gamma}{\gamma!} - \sum_{|\gamma| = l-s-|\alpha|} (K \overset{\circ}{u}_{\alpha+\gamma})(r) \frac{x^\gamma}{\gamma!} \Big|^2 dx \le c \, \|u\|^2_{W^l_{2,s-n/2}(\mathcal{G})}$$

for $|\alpha| \le l - s$ and

$$(7.3.12) \qquad \int_{\mathcal{G}} r^{2(s-l+|\alpha|)-n} \left| \partial_x^\alpha u \right|^2 dx \le c \, \|u\|^2_{W^l_{2,s-n/2}(\mathcal{G})}$$

for $l - s < |\alpha| \le l$. The constant c is independent of u.

Proof: If $l - s < |\alpha| \le l$, then $\partial_x^\alpha u \in W^{l-|\alpha|}_{2,s-n/2}(\mathcal{G}) = V^{l-|\alpha|}_{2,s-n/2}(\mathcal{G})$ and the norm of $\partial_x^\alpha u$ in the latter space can be estimated by the norm of u in $W^l_{2,s-n/2}(\mathcal{G})$ (see Theorem 7.1.1). This proves (7.3.12) for $l - s < |\alpha| \le l$.

By Theorem 7.3.1, we have

$$\int_{\mathcal{G}} r^{-n} \left| \partial_x^\alpha u - K \overset{\circ}{u}_\alpha \right|^2 dx \le c \, \|\partial_x^\alpha u\|^2_{W^1_{2,1-n/2}(\mathcal{G})} \le c \, \|u\|^2_{W^l_{2,s-n/2}(\mathcal{G})},$$

for $|\alpha| = l - s$. Thus, (7.3.11) is true for $|\alpha| = l - s$. We assume that (7.3.11) is true for $|\alpha| = k$, where $k \le l - s$, and prove this inequality for $|\alpha| = k - 1$. For $|\alpha| = k - 1 \le l - s - 1$ the function

$$v(x) \overset{def}{=} \partial_x^\alpha u - \sum_{|\gamma| \le l-s-1-|\alpha|} (\partial_x^{\alpha+\gamma} u)(0) \frac{x^\gamma}{\gamma!} - \sum_{|\gamma| = l-s-|\alpha|} (K \overset{\circ}{u}_{\alpha+\gamma})(r) \frac{x^\gamma}{\gamma!}$$

vanishes at the origin. Furthermore,

$$\begin{aligned} \partial_{x_j} v(x) &= \partial_x^{\alpha'} u - \sum_{|\gamma| \le l-s-1-|\alpha'|} (\partial_x^{\alpha'+\gamma} u)(0) \frac{x^\gamma}{\gamma!} - \sum_{|\gamma| = l-s-|\alpha'|} (K \overset{\circ}{u}_{\alpha'+\gamma})(r) \frac{x^\gamma}{\gamma!} \\ &\quad - \sum_{|\gamma| = l-s-|\alpha|} \left(\partial_{x_j} (K \overset{\circ}{u}_{\alpha+\gamma})(r) \right) \frac{x^\gamma}{\gamma!}, \end{aligned}$$

where α' denotes the multi-index $(\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n)$. From our assumption and from Lemma 7.3.3 it follows that

$$\int_{\mathcal{G}} r^{2(s-l+k)-n} |\nabla v|^2 \, dx \le c \, \|u\|^2_{W^l_{2,s-n/2}(\mathcal{G})}.$$

Consequently, by means of Hardy's inequality, we get

$$\int_{\mathcal{G}} r^{2(s-l+k-1)-n} |v(x)|^2 \, dx \le c \int_{\mathcal{G}} r^{2(s-l+k)-n} \, |\nabla v|^2 \, dx \le c \, \|u\|^2_{W^l_{2,s-n/2}(\mathcal{G})}.$$

Thus, (7.3.11) is valid for $|\alpha| = k-1$ and, therefore, for all multi-indices α, $|\alpha| \le l-s$. The lemma is proved. ■

Now we can prove the following generalization of Theorem 7.3.1.

THEOREM 7.3.2. *Let $u \in W^l_{2,s-n/2}(\mathcal{G})$, where $s \in \{1,\dots,l\}$. Furthermore, let c_α, $|\alpha| \le l-s-1$, be complex constants and u_α, $|\alpha| = l-s$, be functions from $W_2^{1/2}((0,\varepsilon))$. Then*

$$u - \sum_{|\alpha|\le l-s-1} c_\alpha \frac{x^\alpha}{\alpha!} - \sum_{|\alpha|=l-s} (Ku_\alpha)(r)\frac{x^\alpha}{\alpha!} \in V^l_{2,s-n/2}(\mathcal{G}). \tag{7.3.13}$$

if and only if

$$c_\alpha = (\partial_x^\alpha u)(0) \quad for\ |\alpha| = l-s-1, \qquad u_\alpha \sim \overset{\circ}{u}_\alpha \quad for\ |\alpha| = l-s. \tag{7.3.14}$$

Proof: By Lemma 7.3.4, the assertion (7.3.13) is valid for $c_\alpha = (\partial_x^\alpha u)(0)$, $u_\alpha \sim \overset{\circ}{u}_\alpha$. In order to prove the implication (7.3.13) $\Rightarrow$ (7.3.14) we have to show that the function

$$v \overset{def}{=} \sum_{|\alpha|\le l-s-1} c_\alpha \frac{x^\alpha}{\alpha!} + \sum_{|\alpha|=l-s} (Ku_\alpha)(r)\frac{x^\alpha}{\alpha!}$$

belongs to $V^l_{2,s-n/2}(\mathcal{G})$ only if $c_\alpha = 0$ for $|\alpha| = l-s-1$ and $u_\alpha \sim 0$ for $|\alpha| = l-s$.

If $v \in V^l_{2,s-n/2}(\mathcal{G})$, then for $|\gamma| = l-s$ we have

$$\partial_x^\gamma v = \partial_x^\gamma \sum_{|\alpha|=l-s} (Ku_\alpha)(r)\frac{x^\alpha}{\alpha!} \in V^s_{2,s-n/2}(\mathcal{G}) \subset V^0_{2,-n/2}(\mathcal{G}).$$

From this and from Lemma 7.3.3 we conclude that $u_\alpha \sim 0$ for $|\alpha| = l-s$. Hence

$$\sum_{|\alpha|=l-s} (Ku_\alpha)(r)\frac{x^\alpha}{\alpha!} \in V^l_{2,s-n/2}(\mathcal{G})$$

and, therefore,

$$\sum_{|\alpha|\le l-s-1} c_\alpha \frac{x^\alpha}{\alpha!} \in V^l_{2,s-n/2}(\mathcal{G}) \subset V^l_{2,s-(n-1)/2}(\mathcal{G}).$$

Applying Theorem 7.1.1, we get $c_\alpha = 0$ for $|\alpha| \le l-s-1$. The theorem is proved. ■

7.3.2. The trace spaces. The relation between the spaces $V^l_{2,s-n/2}(\mathcal{G})$ and $W^l_{2,s-n/2}(\mathcal{G})$ obtained in the previous theorem can be used to get an analogous relation between the trace spaces. First we give the conditions under which the trace of a "quasipolynomial" belongs to the space $V^{l-1/2}_{2,s-n/2}(\partial\mathcal{G})$.

LEMMA 7.3.5. *Let $s \in \{1,\dots,l\}$ and let u be a function of the form*

$$u = p + \sum_{|\alpha|=l-s} (Ku_\alpha)(r)\, x^\alpha ,$$

where p is a polynomial of degree not greater than $l-s-1$ and u_α are functions from $W_2^{1/2}((0,\varepsilon))$. Then $u|_{\partial\mathcal{G}\backslash\{0\}} \in V^{l-1/2}_{2,s-n/2}(\partial\mathcal{G})$ if and only if there exist functions

$v_\alpha \in W_2^{1/2}((0,\varepsilon))$, $v_\alpha \sim 0$, *such that*

$$u|_{\partial\mathcal{K}\backslash\{0\}} = \sum_{|\alpha|=l-s} (Kv_\alpha)(r)\, x^\alpha \Big|_{\partial\mathcal{K}\backslash\{0\}}. \tag{7.3.15}$$

Proof: If $v_\alpha \sim 0$, then the function $(Kv_\alpha)(r)\, x^\alpha$ belongs to $V^l_{2,s-n/2}(\mathcal{G})$. Consequently, if the representation (7.3.15) is valid, we get $u|_{\partial\mathcal{G}\backslash\{0\}} \in V^{l-1/2}_{2,s-n/2}(\partial\mathcal{G})$.

We assume that the restriction of the function u to $\partial\mathcal{G}\backslash\{0\}$ belongs to the space $V^{l-1/2}_{2,s-n/2}(\partial\mathcal{G})$. Obviously, the functions $(Ku_\alpha)(r)\, x^\alpha$ with $|\alpha| = l-s$ belong to the space $V^l_{2,s-(n-1)/2}(\mathcal{G})$. Hence

$$(Ku_\alpha)(r)\, x^\alpha|_{\partial\mathcal{G}\backslash\{0\}} \in V^{l-1/2}_{2,s-(n-1)/2}(\partial\mathcal{G})$$

and, by our assumption on u, we obtain $p|_{\partial\mathcal{G}\backslash\{0\}} \in V^{l-1/2}_{2,s-(n-1)/2}(\partial\mathcal{G})$. Using Lemma 7.1.7, we get $p|_{\partial\mathcal{K}} = 0$. This implies

$$\sum_{|\alpha|=l-s} (Ku_\alpha)(r)\, x^\alpha \Big|_{\partial\mathcal{K}\backslash\{0\}} \in V^{l-1/2}_{2,s-n/2}(\mathcal{G}).$$

In particular, by Lemma 6.1.2,

$$\int_0^\varepsilon \int_{\partial\Omega} r^{-1} \Big| \sum_{|\alpha|=l-s} (Ku_\alpha)(r) \Big(\frac{x}{|x|}\Big)^\alpha \Big|^2 d\omega\, dr < \infty. \tag{7.3.16}$$

Let M be a set of multi-indices α, $|\alpha| = l-s$, such that $\{x^\alpha|_{\partial\mathcal{K}}\}_{\alpha\in M}$ is a basis in $\operatorname{span}\{x^\alpha|_{\partial\mathcal{K}} : |\alpha| = l-s\}$. Then there exist constants $c_{\gamma,\alpha}$ such that

$$x^\gamma - \sum_{\alpha\in M} c_{\gamma,\alpha}\, x^\alpha = 0 \quad \text{on } \partial\mathcal{K}$$

for all multi-indices γ, $|\gamma| = l-s$, $\gamma \notin M$. Consequently,

$$\sum_{|\alpha|=l-s} (Ku_\alpha)(r)\, x^\alpha = \sum_{\alpha\in M} \Big(Ku_\alpha + \sum_{|\gamma|=l-s,\, \gamma\notin M} c_{\gamma,\alpha}\, Ku_\gamma \Big)\, x^\alpha \quad \text{on } \partial\mathcal{K}$$

and (7.3.16) yields

$$\int_0^\varepsilon r^{-1} \Big| \sum_{\alpha\in M} \Big((Ku_\alpha)(r) + \sum_{|\gamma|=l-s,\, \gamma\notin M} c_{\gamma,\alpha}\, (Ku_\gamma)(r) \Big)\, \omega^\alpha \Big|^2 dr < \infty$$

for all $\omega = x/|x| \in \partial\Omega$. Since the functions ω^α, $\alpha \in M$, are linear independent on $\partial\Omega$, this implies

$$\int_0^\varepsilon r^{-1} \Big| Ku_\alpha + \sum_{|\gamma|=l-s,\, \gamma\notin M} c_{\gamma,\alpha} Ku_\gamma \Big|^2 dr < \infty$$

or, what is the same,

$$u_\alpha + \sum_{|\gamma|=l-s,\, \gamma\notin M} c_{\gamma,\alpha}\, u_\gamma \sim 0 \quad \text{for } \alpha \in M.$$

We set $v_\alpha = u_\alpha + \sum_{\gamma \notin M} c_{\gamma,\alpha}\, u_\gamma$ for $\alpha \in M$. Then we have

$$\sum_{|\alpha|=l-s} (Ku_\alpha)(r)\, x^\alpha = \sum_{\alpha\in M} (Kv_\alpha)(r)\, x^\alpha + \sum_{|\gamma|=l-s,\, \gamma\notin M} (Ku_\gamma)(r) \Big(x^\gamma - \sum_{\alpha\in M} c_{\gamma,\alpha}\, x^\alpha \Big), \tag{7.3.17}$$

where $v_\alpha \sim 0$. Since $x^\gamma - \sum_{\alpha\in M} c_{\gamma,\alpha}\, x^\alpha = 0$ for $x \in \partial\mathcal{K}$, the second sum on the right of (7.3.17) vanishes on $\partial\mathcal{K}\backslash\{0\}$. This proves the lemma. ■

As in Section 7.1, we denote the set of all polynomials of degree not greater than $l-s-1$ by Π_{l-s-1} and the subset of all polynomials $p \in \Pi_{l-s-1}$ satisfying the condition $p|_{\partial\mathcal{K}} = 0$ by Ψ_{l-s-1}. Furthermore, we denote the set of all homogeneous polynomials of degree $l-s$ by $\Pi^{(0)}_{l-s}$ and the set of all polynomials $p \in \Pi^{(0)}_{l-s}$ satisfying the condition $p|_{\partial\mathcal{K}} = 0$ by $\Psi^{(0)}_{l-s}$. We write the linear sets Π_{l-s-1} and $\Pi^{(0)}_{l-s}$ as direct sums:

$$\Pi_{l-s-1} = \Psi_{l-s-1} \oplus \Upsilon_{l-s-1}, \qquad \Pi^{(0)}_{l-s} = \Psi^{(0)}_{l-s} \oplus \Upsilon^{(0)}_{l-s}.$$

Let $\{p_1, \dots, p_\mu\}$ be a basis in Υ_{l-s-1} and $\{q_1, \dots, q_\nu\}$ a basis in $\Upsilon^{(0)}_{l-s}$. Then analogously to Theorem 7.1.2, the following statement holds.

THEOREM 7.3.3. *Let u be an arbitrary function from $W^{l-1/2}_{2,s-n/2}(\partial\mathcal{K})$, where s is a positive integer not greater than l. Then there exist numbers $c_1, \dots, c_\mu$ and functions $u_1, \dots, u_\nu \in W_2^{1/2}((0,\varepsilon))$ such that*

$$u - \sum_{j=1}^{\mu} c_j\, p_j(x) - \sum_{j=1}^{\nu} (Ku_j)(r)\, q_j(x) \in V^{l-1/2}_{2,s-n/2}(\partial\mathcal{G}). \tag{7.3.18}$$

The numbers c_j are uniquely determined by the function u, while the functions u_j are uniquely determined up to the equivalence relation (7.3.9).

Proof: Let $v \in W^l_{2,s-n/2}(\mathcal{G})$ be an extension of u to the domain $\mathcal{G}$. Then by Theorem 7.3.2, we have

$$v - \sum_{|\alpha|\le l-s-1} (\partial_x^\alpha u)(0)\, \frac{x^\alpha}{\alpha!} - \sum_{|\alpha|=l-s} (K\, \overset{\circ}{v}_\alpha)(r)\, \frac{x^\alpha}{\alpha!} \in V^l_{2,2-n/2}(\mathcal{G}). \tag{7.3.19}$$

There exist constants c_j and functions $u_j \in W_2^{1/2}((0,\varepsilon))$ such that

$$\sum_{|\alpha|\le l-s-1} (\partial_x^\alpha u)(0)\, \frac{x^\alpha}{\alpha!} - \sum_{j=1}^{\mu} c_j\, p_j(x) \in \Psi_{l-s-1},$$

$$\sum_{|\alpha|=l-s} (K\, \overset{\circ}{v}_\alpha)(r)\, \frac{x^\alpha}{\alpha!} - \sum_{j=1}^{\nu} (Ku_j)(r)\, q_j(x) = 0 \quad \text{on } \partial\mathcal{K}\backslash\{0\},$$

i.e., the restriction of the function (7.3.19) on $\partial\mathcal{G} \cap \{x : |x| < \varepsilon\}$ coincides with

$$u - \sum_{j=1}^{\mu} c_j\, p_j(x) - \sum_{j=1}^{\nu} (Ku_j)(r)\, q_j(x).$$

This proves the validity of (7.3.18).

We prove the uniqueness of the coefficients. Suppose that, additionally to (7.3.18),

$$u - \sum_{j=1}^{\mu} d_j\, p_j(x) - \sum_{j=1}^{\nu} (Kv_j)(r)\, q_j(x) \in V_{2,s-n/2}^{l-1/2}(\partial\mathcal{G}).$$

Then we obtain

$$\sum_{j=1}^{\mu} (c_j - d_j)\, p_j(x) - \sum_{j=1}^{\nu} \big(K(u_j - v_j)\big)(r)\, q_j(x)\Big|_{\partial\mathcal{G}\backslash\{0\}} \in V_{2,s-n/2}^{l-1/2}(\partial\mathcal{G}).$$

According to Lemma 7.3.5, we conclude from this that

$$\sum_{j=1}^{\mu} (c_j - d_j)\, p_j(x) \in \Psi_{l-s-1}$$

and

$$\sum_{j=1}^{\nu} \big(K(u_j - v_j)\big)(r)\, q_j(x) = \sum_{|\alpha|=l-s} (Kw_\alpha)(r)\, x^\alpha \quad \text{on } \partial\mathcal{K}\backslash\{0\}, \tag{7.3.20}$$

where $w_\alpha \in W_2^{1/2}((0,\varepsilon))$, $w_\alpha \sim 0$. Since p_j are linearly independent elements of Υ_{l-s-1}, it holds $c_j = d_j$. Furthermore, the sum on the right of (7.3.20) can be written in the form

$$\sum_{j=1}^{\nu} (Kw_j)\, q_j(x)$$

on $\partial\mathcal{K}\backslash\{0\}$, where the functions w_j are linear combinations of the functions w_α in (7.3.20), i.e., $w_j \sim 0$. Thus, we get

$$\sum_{j=1}^{\nu} \big(K(u_j - v_j - w_j)\big)(r)\, q_j(x) = 0 \quad \text{on } \partial\mathcal{K}\backslash\{0\}.$$

Using the fact, that q_j are linearly independent elements of $\Upsilon_{l-s}^{(0)}$, we obtain $u_j \sim v_j$. The theorem is proved. ■

Now we consider the case $l = 1$. By Theorem 7.3.3, for every $u \in W_{2,1-n/2}^{1/2}(\partial\mathcal{G})$ there exists a function $u_0 \in W_2^{1/2}((0,\varepsilon))$ such that

$$u - (Ku_0)(r) \in V_{2,1-n/2}^{1/2}(\partial\mathcal{G}). \tag{7.3.21}$$

The function u_0 in (7.3.21) is uniquely determined up to the equivalence relation (7.3.9). We prove that $u_0 \sim \overset{\circ}{u}$, where

$$\overset{\circ}{u}(r) = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u(\omega, r)\, d\omega \quad \text{for } 0 < r < \varepsilon.$$

For this we need the following lemma.

LEMMA 7.3.6. *If* $u \in W_{2,1-n/2}^{1/2}(\partial\mathcal{G})$, *then* $\displaystyle\int_{\substack{\partial\mathcal{G}\\ |x|<\varepsilon}} r^{1-n}\, |u(x) - \overset{\circ}{u}(r)|^2\, dx < \infty.$

Proof: Let $v \in W^1_{2,1-n/2}(\mathcal{G})$ be an extension of u such that

$$\|v\|_{W^1_{2,1-n/2}(\mathcal{G})} \le 2\, \|u\|_{W^{1/2}_{2,1-n/2}(\partial\mathcal{G})} \, .$$

By Lemmas 7.3.1, 7.3.3 and Theorem 7.3.1, we have

$$v = \overset{\circ}{v}(r) + w \quad \text{for } |x| < \varepsilon,$$

where $\overset{\circ}{v}(r)$ is the average of $v(\omega, r)$ in Ω (see (7.3.1)) and $w \in V^1_{2,1-n/2}(\mathcal{G})$. Since

$$w(\cdot, r)|_{\partial\Omega} = \big(v(\cdot, r) - \overset{\circ}{v}(r)\big)\big|_{\partial\Omega} = u(\cdot, r) - \overset{\circ}{v}(r)$$

for arbitrary $r < \varepsilon$, the W^1_2-norm of $w(\cdot, r)$ in Ω can be estimated from below by the $W_2^{1/2}$-norm of $u(\cdot, r) - \overset{\circ}{v}(r)$. In particular, it holds

$$\int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|u(\omega, r) - u(\omega', r)|^2}{|\omega - \omega'|^{n-1}}\, d\omega\, d\omega' \le c\, \|w(\cdot, r)\|^2_{W^1_2(\Omega)} \quad \text{for } r < \varepsilon.$$

Consequently, we obtain

$$\int\limits_{\substack{\partial\mathcal{G}\\ |x|<\varepsilon}} r^{1-n}\, |u(x) - \overset{\circ}{u}(r)|^2\, dx = \frac{1}{|\partial\Omega|^2} \int\limits_0^\varepsilon r^{-1} \int\limits_{\partial\Omega} \Big| \int\limits_{\partial\Omega} \big(u(\omega, r) - u(\omega', r)\big)\, d\omega' \Big|^2 d\omega\, dr$$

$$\le c \int\limits_0^\varepsilon r^{-1} \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \big|u(\omega, r) - u(\omega', r)\big|^2 d\omega\, d\omega'\, dr \le c_1 \int\limits_0^\varepsilon r^{-1}\, \|w(\cdot, r)\|^2_{W^1_2(\Omega)}\, dr$$

$$\le c_1\, \|w\|^2_{V^1_{2,1-n/2}(\mathcal{G})} \le c_2\, \|v\|^2_{W^1_{2,1-n/2}(\mathcal{G})} \le 4c_2\, \|u\|^2_{W^{1/2}_{2,1-n/2}(\partial\mathcal{G})} \, .$$

The lemma is proved. ∎

COROLLARY 7.3.1. *Let $u \in W^{1/2}_{2,1-n/2}(\partial\mathcal{G})$ and $u_0 \in W_2^{1/2}((0, \varepsilon))$. Then the inclusion (7.3.21) is true if and only if $u_0 \sim \overset{\circ}{u}$.*

Proof: Suppose that $u_0 \in W_2^{1/2}((0, \varepsilon))$ is a function such that $u - (Ku_0)(r) \in V^{1/2}_{2,1-n/2}(\partial\mathcal{G})$. Then by Lemma 6.1.2, we have $r^{(1-n)/2}(u - Ku_0) \in L_2(\partial\mathcal{G})$, and Lemma 7.3.6 implies

$$\int\limits_0^\varepsilon r^{-1} \big|(Ku_0)(r) - \overset{\circ}{u}(r)\big|^2\, dr < \infty.$$

Using Lemma 7.3.3, we obtain $u_0 \sim \overset{\circ}{u}$. The corollary is proved. ∎

The case $n = 2$. Now let $\mathcal{G}$ be a plane domain with $d \ge 2$ angular points $x^{(1)}, x^{(2)}, \dots, x^{(d)}$ on the boundary. Then the boundary $\partial\mathcal{G}$ consists of the curves $\Gamma_1, \dots, \Gamma_d$, where Γ_τ connects the points $x^{(\tau)}$ and $x^{(\tau+1)}$, $\tau = 1, \dots, d-1$, and Γ_d connects the points $x^{(d)}$ and $x^{(1)}$. Suppose that $\mathcal{G}$ coincides with a plane wedge $\mathcal{K}_\tau$ in the neighbourhood $|x - x^{(\tau)}| < \varepsilon$.

We are interested in the relations between the trace spaces $V^{l-1/2}_{2,\beta}(\Gamma_\tau)$ and $W^{l-1/2}_{2,\beta}(\Gamma_\tau)$, where $\beta = (\beta_1, \dots, \beta_d)$, $\beta_\tau \in \{0, 1, \dots, l-1\}$. In a neighbourhood of the point $x^{(\tau)}$ every function $u \in W^{l-1/2}_{2,\beta}(\Gamma_\tau)$ can be considered as a function of the variable $r = |x - x^{(\tau)}|$. If β_τ is a nonnegative integer less than l, then the values of $u, \partial_r u, \dots, \partial_r^{l-\beta-2} u$ at the point $x^{(\tau)}$ exist, while $\partial_r^{l-\beta-1} u \in W_2^{1/2}((0,\varepsilon))$. This follows from the imbedding $\zeta_\tau W^l_{2,\beta}(\mathcal{G}) \subset W_2^{l-\beta_\tau}(\mathcal{G})$, where ζ_τ is an arbitrary smooth cut-off function with support near $x^{(\tau)}$ (see Lemma 7.1.5).

THEOREM 7.3.4. *Let $u \in W^{l-1/2}_{2,\beta}(\Gamma_\tau)$, where $\beta = (\beta_1, \dots, \beta_d)$ and β_τ is a nonnegative integer less than l. Furthermore, let $c_0, \dots, c_{l-\beta-2}$ be complex numbers, $v \in W_2^{1/2}((0,\varepsilon))$, and ζ_τ a smooth cut-off function equal to one near $x^{(\tau)}$ with sufficiently small support. Then*

$$\zeta_\tau \Big(u - \sum_{k=0}^{l-\beta_\tau-2} c_k \frac{r^k}{k!} - (Kw)(r) \frac{r^{l-\beta_\tau-1}}{(l-\beta_\tau-1)!} \Big) \in V^{l-1/2}_{2,\beta}(\Gamma_\tau) \tag{7.3.22}$$

if and only if $c_k = (\partial_r^k u)(0)$ and $w \sim \partial_r^{l-\beta_\tau-1} u$.

Proof: Without loss of generality, we may assume that $\mathcal{G}$ coincides with the wedge $\mathcal{K}_\tau = \{x : r > 0,\ 0 < \theta < \alpha\}$ near the angular point $x^{(\tau)} = 0$, where r, θ are the polar coordinates of the point x. Then Γ_τ coincides with the x_1-axis near $x^{(\tau)}$. Let $v \in W^l_{2,\beta}(\mathcal{G})$ be an arbitrary extension of u to the domain $\mathcal{G}$. By Theorem 7.3.2, we have

$$\zeta_\tau \Big(v - \sum_{i+j \le l-\beta_\tau-2} (\partial_{x_1}^i \partial_{x_2}^j v)(0) \frac{x_1^i\, x_2^j}{i!\, j!} - \sum_{i+j=l-\beta_\tau-1} (Kw_{i,j})(r) \frac{x_1^i\, x_2^j}{i!\, j!} \Big) \in V^l_{2,\beta}(\mathcal{G})$$

if $w_{i,j} \sim \overset{\circ}{v}_{i,j}$, where $\overset{\circ}{v}_{i,j}(r) = \dfrac{1}{\alpha} \displaystyle\int_0^\alpha (\partial_{x_1}^i \partial_{x_2}^j v)(\theta, r)\, d\theta$. Consequently,

$$\zeta_\tau \Big(u - \sum_{k=0}^{l-\beta_\tau-2} (\partial_{x_1}^k u)(0) \frac{x_1^k}{k!} - (Kw_{l-\beta_\tau-1,0})(r) \frac{x_1^{l-\beta_\tau-1}}{(l-\beta_\tau-1)!} \Big) \in V^{l-1/2}_{2,\beta}(\Gamma_\tau). \tag{7.3.23}$$

Using the inequality

$$\Big| (\partial_{x_1}^{l-\beta_\tau-1} v)(\theta, r) - (\partial_{x_1}^{l-\beta_\tau-1} v)(0, r) \Big|^2 \le \alpha \int_0^\alpha \big| (\partial_\theta \partial_{x_1}^{l-\beta_\tau-1} v)(\theta, r) \big|^2\, d\theta \quad \text{for } \theta < \alpha,$$

we obtain

$$\begin{aligned}
&\int_0^\varepsilon \frac{1}{r} \Big| \overset{\circ}{v}_{l-\beta_\tau-1,0} - \partial_r^{l-\beta_\tau-1} u \Big|^2 dr \\
&= \frac{1}{\alpha^2} \int_0^\varepsilon \frac{1}{r} \Big| \int_0^\alpha (\partial_{x_1}^{l-\beta_\tau-1} v)(\theta, r) - (\partial_{x_1}^{l-\beta_\tau-1} v)(0, r)\, d\theta \Big|^2 dr \\
&\le c \int_0^\varepsilon \int_0^\alpha r \Big(|\partial_{x_1}^{l-\beta_\tau} v|^2 + |\partial_{x_1}^{l-\beta_\tau-1} \partial_{x_2} v|^2 \Big)\, d\theta\, dr \le c\, \|\zeta_\tau v\|^2_{W_2^{l-\beta_\tau}(\mathcal{G})} \le c\, \|v\|^2_{W^l_{2,\beta}(\mathcal{G})}.
\end{aligned}$$

Hence $\partial_{x_1}^{l-\beta_\tau-1} u \sim \overset{\circ}{v}_{l-\beta_\tau-1,0}$ and the function $w_{l-\beta_\tau-1,0}$ in (7.3.23) can be replaced by $\partial_r^{l-\beta_\tau-1} u$.

We prove the uniqueness of the coefficients c_k and w in (7.3.22). For this we have to show that

$$\zeta_\tau \Big(\sum_{k=0}^{l-\beta_\tau-2} c_k \frac{r^k}{k!} - (Kw)(r) \frac{r^{l-\beta_\tau-1}}{(l-\beta_\tau-1)!} \Big) \in V_{2,\beta}^{l-1/2}(\Gamma_\tau) \tag{7.3.24}$$

only if $c_k = 0$ for $k = 0, \dots, l-\beta_\tau-1$ and $w \sim 0$. Suppose that (7.3.24) is true. Then it follows

$$\zeta_\tau \sum_{k=0}^{l-\beta_\tau-2} c_k \frac{r^k}{k!} \in V_{2,\beta+\frac{1}{2}\vec{1}}^{l-1/2}(\Gamma_\tau)$$

and, consequently, $c_k = 0$ for $k = 0, \dots, l-\beta_\tau-1$ (see Lemma 7.1.8). Together with (7.3.24) this implies $(Kw)(r)\, r^{l-\beta_\tau-1} \in V_{2,\beta}^{l-1/2}(\Gamma_\tau)$. Hence $w \sim 0$. The proof is complete. ■

7.3.3. Applications to elliptic boundary value problems. For the proof of the Fredholm property of the operator $\mathcal{A}$ of the boundary value problem (7.2.2), (7.2.3) in Theorem 7.2.2 we have used the Fredholm property of this operator in corresponding weighted Sobolev spaces of the class $V_{2,\beta}^l$ and the finiteness of the dimension of the quotient space $W_{2,\beta}^l / V_{2,\beta}^l$. In the case $\beta = s - n/2$, where s is an integer, $1 \le s \le l + \max(\tau_j, 0)$, the Fredholm property of the operator $\mathcal{A}$ in the class of the spaces $V_{2,\beta}^l$ does not automatically lead to the Fredholm property of this operator in the class of the spaces $W_{2,\beta}^l$. In this case the assertions of Theorem 7.2.2 hold only under additional assumptions on the operators of the boundary value problem. However, regularity assertions and asymptotic decompositions of the solutions can be obtained in the same way as in Section 7.2.

We assume again that the boundary $\partial\mathcal{G}$ contains only one conical point $x^{(1)} = 0$ and consider the boundary value problem (7.2.2), (7.2.3), where L, B, C satisfy the same conditions as in Section 7.2., i.e., L is a differential operator of order $2m$, B is a vector of differential operators B_k, $\operatorname{ord} B_k \le \mu_k$, C is a matrix of tangential differential operators $C_{k,j}$ on $\partial\mathcal{G}\backslash\{0\}$, $\operatorname{ord} C_{k,j} \le \mu_k + \tau_j$ $j = 1, \dots, J$, $k = 1, \dots, m+J$, and the coefficients of L, B_k, and $C_{k,j}$ have the representations (7.2.9), (7.2.11) and (7.2.12), respectively, in a neighbourhood of the conical point. Analogously to Theorem 7.2.3, the following statement holds.

THEOREM 7.3.5. *Let $(u, \underline{u}) \in W_{2,s-n/2}^l(\mathcal{G}) \times W_{2,s-n/2}^{l+\vec{\tau}-1/2}(\partial\mathcal{G})$ be a solution of the boundary value problem (7.2.2), (7.2.3), where*

$$(f, \underline{g}) \in W_{2,s+k-n/2}^{l-2m+k}(\mathcal{G}) \times W_{2,s+k-n/2}^{l+k-\vec{\mu}-1/2}(\partial\mathcal{G}),$$

$l \ge 2m$, $l > \max \mu_k$, $s \in \{1, 2, \dots, l + \max(\tau_j, 0)\}$. Suppose that problem (7.2.2), (7.2.3) is elliptic in and the coefficients of the operators L, B_k, $C_{k,j}$ have the representations (7.2.9)–(7.2.12), where

$$\operatorname{Re} \delta_{\sigma(\alpha)+1} + |\alpha| > l - s, \qquad \operatorname{Re} \delta_{\sigma(\alpha,j)+1} + |\alpha| > l - s + \tau_j.$$

Then $(u, \underline{u}) \in W_{2,s+k-n/2}^{l+k}(\mathcal{G}) \times W_{2,s+k-n/2}^{l+k+\vec{\tau}-1/2}(\partial\mathcal{G})$.

Proof: By Theorem 7.3.2, the function u admits the decomposition

$$u = p_{l-s-1}(u) + \sum_{|\alpha|=l-s} (Ku_\alpha)(r) \frac{x^\alpha}{\alpha!} + v,$$

where $p_{l-s-1}(u)$ denotes the Taylor polynomial of degree $l-s-1$ of u, $u_\alpha \in W_2^{1/2}((0,\varepsilon))$, and $v \in V^l_{2,s-n/2}(\mathcal{G})$. Analogously, the components u_j of the vector-function $\underline{u}$ can be written in the form

$$u_j = \sum_{|\alpha|\le l+\tau_j-s-1} c_{j,\alpha}\, x^\alpha + \sum_{|\alpha|=l+\tau_j-s} (Kw_{j,\alpha})(r)\, x^\alpha + v_j\,,$$

where $c_{j,\alpha}$ are constants, $w_{j,\alpha} \in W_2^{1/2}((0,\varepsilon))$, $v_j \in V^{l+\tau_j-1/2}_{2,s-n/2}(\partial\mathcal{G})$.

From Lemma 7.3.3 it follows that $u - v \in W^{l+k}_{2,s+k-n/2}(\mathcal{G})$ and $u_j - v_j \in W^{l+k+\tau_j-1/2}_{2,s+k-n/2}(\partial\mathcal{G})$ It remains to show that $(v,\underline{v}) \in V^{l+k}_{2,s+k-n/2}(\mathcal{G})\times V^{l+k+\tau_j-1/2}_{2,s+k-n/2}(\partial\mathcal{G})$.

Using the representation

$$f = p_{l-s-2m-1}(f) + \sum_{|\alpha|=l-s-2m} (Kf_\alpha)(r)\, \frac{x^\alpha}{\alpha!} + \varphi\,,$$

where $p_{l-s-2m-1}(f)$ denotes the Taylor polynomial of degree $l-s-2m-1$ of f, $f_\alpha \in W_2^{1/2}((0,\varepsilon))$, and $\varphi \in V^{l+k-2m}_{2,s+k}(\mathcal{G})$, we obtain

$$\begin{aligned}(7.3.25)\quad Lv &= f - L\,p_{l-s-1}(u) - L\sum_{|\alpha|=l-s} (Ku_\alpha)(r)\, \frac{x^\alpha}{\alpha!}\\ &= p_{l-s-2m-1}(f) - L\,p_{l-s-1}(u) + \sum_{|\alpha|=l-s-2m} (Kf_\alpha)(r)\, \frac{x^\alpha}{\alpha!}\\ &\quad - \sum_{|\alpha|=l-s} (Ku_\alpha)(r)\, L^{(0)}(x,\partial_x)\, \frac{x^\alpha}{\alpha!} + \varphi_1.\end{aligned}$$

Here $L^{(0)}(x,\partial_x)$ denotes the leading part of the operator L at the conical point $x^{(1)} = 0$ (see Definition 6.2.1) and

$$\varphi_1 = \varphi + \sum_{|\alpha|=l-s} \Big((Ku_\alpha)(r)\, L^{(0)}(x,\partial_x)\, \frac{x^\alpha}{\alpha!} - L(x,\partial_x)\, (Ku_\alpha)(r)\, \frac{x^\alpha}{\alpha!}\Big)$$

is an element of the space $V^{l+k-2m}_{2,s+k-n/2}(\mathcal{G})$. By our assumptions on L, there is the representation

$$p_{l-s-2m-1}(f) - L\,p_{l-s-1}(u) = F_1 + F_2 + F_3\,,$$

where $F_1 \in V^{l+k-2m}_{2,s+k-n/2}(\mathcal{G})$, F_2 is a finite sum of terms of the form

$$a(\omega)\, r^\sigma\,,\ \sigma < l-s-2m, \quad \text{and} \quad a(\omega)\, (\log r)^j\, r^\sigma,\ \sigma \le l-s,\ j \ge 1,$$

and F_3 is a finite sum of terms of the form $a(\omega)\, r^{l-s-2m}$ with $a \in C^\infty(\overline{\Omega})$.

Furthermore, we have

$$\sum_{|\alpha|=l-s-2m} (Kf_\alpha)(r)\, \frac{x^\alpha}{\alpha!} - \sum_{|\alpha|=l-s} (Ku_\alpha)(r)\, L^{(0)}(x,\partial_x)\, \frac{x^\alpha}{\alpha!} = F_4 + F_5\,,$$

where $F_4 \in V^{l+k-2m}_{2,s+k-n/2}(\mathcal{G})$ and the function F_5 is a finite sum of terms of the form

$$a(\omega)\, (Kg)(r)\, r^{l-s-2m}$$

with $a \in C^\infty(\overline{\Omega})$ and $g \in W_2^{1/2}((0,\varepsilon))$. Since $v \in V^l_{2,s-n/2}(\mathcal{G})$, it follows from (7.3.25) that $F_2 + F_3 + F_5 \in V^{l-2m}_{2,s-n/2}(\mathcal{G})$. This implies $F_2 = 0$ and $F_3 + F_5 \in V^{l+k-2m}_{2,s+k-n/2}(\mathcal{G})$. Hence $Lv \in V^{l+k-2m}_{2,s+k-n/2}(\mathcal{G})$.

Analogously, we can prove that $Bv + C\underline{v} \in V^{l+k-\underline{\mu}-1/2}_{2,s+k-n/2}(\partial\mathcal{G})$. Thus, by Lemma 6.3.1, we obtain $(v, \underline{v}) \in V^{l+k}_{2,s+k-n/2}(\mathcal{K}) \times V^{l+k+\vec{\tau}-1/2}_{2,s+k-n/2}(\partial\mathcal{K})$. This proves the Theorem. ■

Note that the assertion of Theorem 7.2.4 is also true if $\beta + n/2$ is an integer, $1 \le \beta + n/2 \le l + \max(0, \tau_j)$, since every solution $(u, \underline{u}) \in W^l_{2,\beta}(\mathcal{G}) \times W^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ belongs also to $W^l_{2,\beta+\varepsilon}(\mathcal{G}) \times W^{l+\underline{\tau}-1/2}_{2,\beta+\varepsilon}(\partial\mathcal{G})$ for $\varepsilon > 0$.

However, if $\beta_1 + n/2 \in \{1, 2, \dots, l_1 - \min(2m, \mu_k)\}$, then the asymptotics of u contains additional terms of the form

$$(Kh)(r)\, r^{l_1-\beta_1-n/2}\, p(\omega, \log r),$$

where $h \in W^{1/2}_2((0,\varepsilon))$ and $p(\omega, r)$ is a polynomial of $\log r$ with infinitely differentiable coefficients in $\overline{\Omega}$. Analogously, the asymptotics of u_j contains additional terms of the form

$$(Kh)(r)\, r^{l_1+\tau_j-\beta_1-n/2}\, p(\omega, \log r).$$

7.3.4. Examples. Let $\mathcal{G}$ be a plane domain with $d \ge 2$ angular points on the boundary. Outside the set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$ of the angular points the boundary is assumed to be smooth. Furthermore, we suppose that the domain $\mathcal{G}$ coincides with the plane wedge $\mathcal{K} = \{x : r > 0,\ 0 < \theta < \alpha\}$, $0 < \alpha < 2\pi$, in a neighbourhood of the corner $x^{(1)} = 0$. Here r, θ are the polar coordinates of the point x.

The Dirichlet problem for the Laplace operator. We consider the generalized solution $u \in W^1_2(\mathcal{G})$ of the Dirichlet problem

$$-\Delta u = f \quad \text{in } \mathcal{G}, \qquad u = g \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S},$$

where $f \in W^0_{2,\beta}(\mathcal{G})$, $g \in W^{3/2}_{2,\beta}(\partial\mathcal{G})$, $\beta = (\beta_1, \dots, \beta_d)$, $\beta_\tau \le 1$ for $\tau = 1, \dots, d$, and $(1-\beta_1)\alpha/\pi$ is noninteger.

If $v \in W^2_{2,\beta}(\mathcal{G})$ is an extension of the function g to the domain $\mathcal{G}$, then $w = u - v$ is a solution of the Dirichlet problem

$$-\Delta w = f + \Delta v \quad \text{in } \mathcal{G}, \qquad w = 0 \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}.$$

where $f + \Delta v \in V^0_{2,\beta}(\mathcal{G})$. Consequently, by (6.6.8), the function $u - v$ has the asymptotics

$$u - v = \sum_j c_j\, r^{j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha} + w_0$$

in a neighbourhood of $x^{(1)}$, where $w_0 \in V^2_{2,\beta}(\mathcal{G})$ and the summation is extended over all positive j less than $(1-\beta_1)\alpha/\pi$. Decomposing v into a polynomial (or quasipolynomial) and a function from $V^2_{2,\beta}(\mathcal{G})$, we get

$$u = v(0) + \sum_j c_j\, r^{j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha} + u_0 \quad \text{if } 0 < \beta < 1,$$

$$u = v(0) + (Kv_1)(r)\, x_1 + (Kv_2)(r)\, x_2 + \sum_j c_j\, r^{j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha} + u_0 \ \text{ if } \beta = 0,$$

$$u = v(0) + \frac{\partial v}{\partial x_1}(0)\, x_1 + \frac{\partial v}{\partial x_2}(0)\, x_2 + \sum_j c_j\, r^{j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha} + u_0 \ \text{ if } \beta \in (-1, 0),$$

$$u = \sum_j c_j\, r^{j\pi/\alpha} \sin\frac{j\pi\theta}{\alpha} + u_0 \quad \text{if } \beta \le -1,$$

where $u_0 \in V^2_{2,\beta}(\mathcal{G})$, $v_1, v_2 \in W_2^{1/2}((0,\varepsilon))$. Here $v(0) = g(0)$.

The Neumann problem for the Laplace operator. Let $u \in W_2^1(\mathcal{G})$ be the generalized solution of the Neumann problem

$$-\Delta u = f \quad \text{in } \mathcal{G}, \qquad \frac{\partial u}{\partial \nu} = g_\tau \quad \text{on } \Gamma_\tau, \quad \tau = 1, \dots, d,$$

where $f \in W^0_{2,\beta}(\mathcal{G})$, $g_\tau \in W^{1/2}_{2,\beta}(\Gamma_\tau)$, $\beta = (\beta_1, \dots, \beta_d)$, $\beta_\tau < 1$ for $\tau = 1, \dots, d$, and $(1 - \beta_\tau)\alpha/\pi$ is noninteger. Here again Γ_τ denotes the side of the boundary $\partial\mathcal{G}$ which connects the corners $x^{(\tau)}$ and $x^{(\tau+1)}$. In particular, the point $x^{(1)}$ is the intersection of $\overline{\Gamma}_1$ and $\overline{\Gamma}_d$.

Note that the mapping

$$W_2^1(\mathcal{G}) \ni v \to \int_{\mathcal{G}} \overline{f}\, v\, dx + \sum_{\tau=1}^{d} \int_{\Gamma_\tau} \overline{g}_\tau\, v\, d\sigma$$

defines a linear and continuous functional on $W_2^1(\mathcal{G})$ if $\beta_\tau < 1$ for $\tau = 1, \dots, d$.

If $0 < \beta_\tau < 1$ or $\beta_\tau \le -1$, then the space $W^{1/2}_{2,\beta}(\partial\mathcal{G})$ coincides with the space $V^{1/2}_{2,\beta}(\partial\mathcal{G})$. Consequently, by (6.6.14), the solution u has the asymptotics

$$u = c_0 + \sum_j c_j\, r^{j\pi/\alpha} \cos\frac{j\pi\theta}{\alpha} + u_0$$

in a neighbourhood of $x^{(1)}$ if $0 < \beta_1 < 1$ or $\beta_1 \le -1$, where $u_0 \in V^2_{2,\beta_1}(\mathcal{K})$ and the summation is extended over all positive integer j less than $(1 - \beta_1)\alpha/\pi$.

We consider the case $-1 < \beta_1 < 0$. Let ζ_1 be a smooth cut-off function equal to one near $x^{(1)}$ with sufficiently small support. Then

$$g_\tau = g_\tau(0) + h_\tau \quad \text{on } \Gamma_\tau, \quad \text{where } \zeta_1 h_\tau \in V^{1/2}_{2,\beta}(\Gamma_\tau), \ \tau = 1, d.$$

If $\alpha \ne \pi$, then there exists a linear function $v = d_1\, x_1 + d_2\, x_2$ such that $\frac{\partial v}{\partial \nu}\big|_{\Gamma_\tau} = g_\tau(0)$ in a neighbourhood of $x^{(1)}$ for $\tau = 1$ and $\tau = d$. Since $w = u - v$ satisfies the equations

$$-\Delta w = f \quad \text{in } \mathcal{G}, \qquad \frac{\partial u}{\partial \nu} = h_\tau \quad \text{on } \Gamma_\tau$$

in a neighbourhood of $x^{(1)}$, we obtain

$$u = c_0 + d_1\, x_1 + d_2\, x_2 + \sum_j c_j\, r^{j\pi/\alpha} \cos\frac{j\pi\theta}{\alpha} + u_0$$

near $x^{(1)}$, where $u_0 \in V^2_{2,\beta_1}(\mathcal{K})$ and the summation is extended over all noninteger j less than $(1-\beta_1)\alpha/\pi$.

Analogously, in the case $\beta_1 = 0$ the asymptotics

$$u = c_0 + (Kw_1)(r)\, x_1 + (Kw_2)(r)\, x_2 + \sum_j c_j\, r^{j\pi/\alpha} \cos\frac{j\pi\theta}{\alpha} + u_0$$

near $x^{(1)}$ holds, where $u_0 \in V^2_{2,\beta_1}(\mathcal{K})$, $w_1, w_2 \in W_2^{1/2}((0,\varepsilon))$.

CHAPTER 8

Variants and extensions

In the first two sections of this chapter we extend the results of Chapter 6 to elliptic boundary value problems for systems of differential equations. This is done in two steps. In Section 8.1 we consider elliptic boundary value problems for a $2m$ order differential equation without additional conditions on the boundary operators B_k, while in Section 8.2 boundary value problems for systems of differential equations are studied. The proofs are in essential the same as in Chapter 6. Therefore, we give only the formulations of the main theorems without proofs.

Section 8.3 is dedicated to boundary value problems in the variational form. The main results in this section are the Fredholm property of the operator of the variational problem and the asymptotics of the variational solutions near the conical points.

8.1. Elliptic problems with boundary operators of higher order in bounded domains with conical points

As we have seen in Section 6.2, an admissible differential operator L of order $2m$ can be extended to the the space $\tilde{V}_{2,\beta}^{l,2m}$ with integer $l < 2m$. In Chapter 6 it was assumed for simplicity that the orders of the differential operators B_k are less than $2m$. Under this assumption, the restriction of the vector Bu to the boundary is the product of a matrix Q of tangential differential operators and the vector $\mathcal{D}u$. This allows to extend the operator B also to the space $\tilde{V}_{2,\beta}^{l,2m}(\mathcal{G})$ with $l < 2m$. Moreover, by means of the Green formula (6.2.12), we were able to introduce the formally adjoint boundary value problem.

If the condition above on the operators B_k is not satisfied, then we have to extend the operators L and B to weighted Sobolev spaces $\tilde{V}_{2,\beta}^{l,\kappa}(\mathcal{G})$ with $\kappa > 2m$. Furthermore, the introduction of the formally adjoint boundary value problem requires the more general Green formula obtained in Section 4.1.

8.1.1. The boundary value problem and its formally adjoint. As in Chapter 6, let $\mathcal{G}$ be a bounded domain in $\mathbb{R}^n$. We suppose that there exists a finite set $\mathcal{S}$ of points $x^{(\tau)} \in \partial\mathcal{G}$, $\tau = 1, \dots, d$, such that $\partial\mathcal{G}\backslash\mathcal{S}$ is smooth. Furthermore, we assume that for every $\tau = 1, \dots, d$ there exist a neighbourhood $\mathcal{U}_\tau$ of $x^{(\tau)}$ and a cone $\mathcal{K}_\tau$ with vertex $x^{(\tau)}$ such that $\mathcal{G} \cap \mathcal{U}_\tau = \mathcal{K}_\tau \cap \mathcal{U}_\tau$.

We consider the boundary value problem

(8.1.1) $$L(x, \partial_x)u = f \quad \text{in } \mathcal{G},$$

(8.1.2) $$B(x, \partial_x)u + C(x, \partial_x)\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S},$$

where L is an admissible differential operator of order $2m$ (see Definition 6.2.1), B is a vector of admissible differential operators B_k of order μ_k, and C is a matrix of admissible tangential differential operators $C_{k,j}$ on $\partial\mathcal{G}\backslash\mathcal{S}$ of order $\mu_k + \tau_j$. Outside

$\mathcal{S}$ the coefficients of L, B_k, $C_{k,j}$ are assumed to be smooth. Moreover, we suppose that there exists a matrix Q of admissible tangential differential operators $Q_{k,j}$, $k = 1, \dots, m+J$, $j = 1, \dots, \kappa$, $\kappa \geq 2m$, such that

$$B(x, \partial_x)u|_{\partial\mathcal{G}\backslash\mathcal{S}} = Q(x, \partial_x) \cdot \mathcal{D}^{(\kappa)} u|_{\partial\mathcal{G}\backslash\mathcal{S}} \tag{8.1.3}$$

for all $u \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})$. Here $\mathcal{D}^{(\kappa)}$ denotes the column vector with the components $1, D_\nu, \dots, D_\nu^{\kappa-1}$. Then the Green formula (cf. formula (4.1.10))

$$\begin{aligned}(Lu, v)_{\mathcal{G}} + \big(\mathcal{D}^{(\kappa-2m)} Lu\,,\, \underline{w}\big)_{\partial\mathcal{G}} + \big(Bu + C\underline{u}\,,\, \underline{v}\big)_{\partial\mathcal{G}} \\ = \big(u, L^+ v\big)_{\mathcal{G}} + \big(\mathcal{D}^{(\kappa)} u\,,\, P^{(\kappa)} v + Q^+ \underline{v} + R^+ \underline{w}\big)_{\partial\mathcal{G}} + \big(\underline{u}, C^+ \underline{v}\big)_{\partial\mathcal{G}}\end{aligned} \tag{8.1.4}$$

is valid for all $u, v \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})$, $\underline{u} \in C_0^\infty(\partial\mathcal{G}\backslash\mathcal{S})^J$, $\underline{v} \in C_0^\infty(\partial\mathcal{G}\backslash\mathcal{S})^{m+J}$, $\underline{w} \in C_0^\infty(\partial\mathcal{G}\backslash\mathcal{S})^{\kappa-2m}$. Here

$$R = \Big(R_{s,j}(x, \partial_x)\Big)_{1\leq s\leq \kappa-2m,\ 1\leq j\leq\kappa}$$

is a matrix of admissible tangential differential operators on $\partial\mathcal{G}\backslash\mathcal{S}$, $\operatorname{ord} R_{s,j} \leq 2m + s - j$, such that

$$\mathcal{D}^{(\kappa-2m)} Lu\,\big|_{\partial\mathcal{G}\backslash\mathcal{S}} = R \cdot \mathcal{D}^{(\kappa)} u|_{\partial\mathcal{G}\backslash\mathcal{S}} \tag{8.1.5}$$

for $u \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})$, and $P^{(\kappa)}$ is a vector of admissible differential operators P_j, $j = 1, \dots, \kappa$, $\operatorname{ord} P_j = 2m - j$, $P_j = 0$ for $j \geq 2m+1$. The vector $P^{(\kappa)}$ is uniquely determined by the equality

$$\big(Lu, v\big)_{\mathcal{G}} = \big(u, L^+ v\big)_{\mathcal{G}} + \big(\mathcal{D}^{(\kappa)} u\,,\, P^{(\kappa)} v\big)_{\partial\mathcal{G}}, \qquad u, v \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S}).$$

If $\kappa = 2m$, then formula (8.1.4) coincides with the Green formula (6.2.12).

The boundary value problem

$$L^+ v = f \quad \text{in } \mathcal{G}, \tag{8.1.6}$$

$$P^{(\kappa)} v + R^+ \underline{w} + Q^+ \underline{v} = \underline{g} \quad \text{on } \partial\mathcal{G}, \tag{8.1.7}$$

$$C^+ \underline{v} = \underline{h} \quad \text{on } \partial\mathcal{G} \tag{8.1.8}$$

is said to be *formally adjoint* to the boundary value problem (8.1.1), (8.1.2) with respect to the Green formula (8.1.4). Note that problems (8.1.1), (8.1.2) and (8.1.6)–(8.1.8) are simultaneously elliptic.

8.1.2. The operator of the boundary value problem. Let l be an integer, $l \geq \kappa$. Then the operator

$$\begin{aligned}V_{2,\beta}^l(\mathcal{G}) \times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G}) \ni (u, \underline{u}) \\ \to \big(Lu, Bu|_{\partial\mathcal{G}\backslash\mathcal{S}} + C\underline{u}\big) \in V_{2,\beta}^{l-2m}(\mathcal{G}) \times V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})\end{aligned}$$

of the boundary value problem (8.1.1), (8.1.2) can be identified with the operator

$$\begin{aligned}&\tilde{V}_{2,\beta}^{l,\kappa}(\mathcal{G}) \times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G}) \ni \big(u, \mathcal{D}^{(\kappa)} u|_{\partial\mathcal{G}\backslash\mathcal{S}}\,,\, \underline{u}\big) \\ &\to \big(Lu,\, \mathcal{D}^{(\kappa-2m)} Lu|_{\partial\mathcal{G}\backslash\mathcal{S}}\,,\, Bu|_{\partial\mathcal{G}\backslash\mathcal{S}} + C\underline{u}\big) \in \tilde{V}_{2,\beta}^{l-2m,\kappa-2m}(\mathcal{G}) \times V_{2,\beta}^{2m-\underline{\mu}-1/2}(\partial\mathcal{G}).\end{aligned}$$

We denote both operators by $\mathcal{A}_{l,\beta}$. Analogously to Theorems 4.1.2 and 6.2.1, we can extend the operator $\mathcal{A}_{\kappa,\beta+(\kappa-l)\vec{1}}$ to the space $\tilde{V}_{2,\beta}^{l,\kappa}(\mathcal{G}) \times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G})$ with $l < \kappa$.

First we consider the extension

$$L\,:\ \tilde{V}_{2,\beta}^{l,\kappa}(\mathcal{G}) \to \tilde{V}_{2,\beta}^{l-2m,\kappa-2m}(\mathcal{G}), \quad l < \kappa, \tag{8.1.9}$$

of the operator

$$\tilde{V}^{\kappa,\kappa}_{2,\beta+(\kappa-l)\vec{1}}(\mathcal{G}) \ni \big(u,\ \mathcal{D}^{(\kappa)}u|_{\partial\mathcal{G}\backslash\mathcal{S}}\big)$$
$$\to \big(Lu,\ \mathcal{D}^{(\kappa-2m)}Lu|_{\partial\mathcal{G}\backslash\mathcal{S}}\big) \in \tilde{V}^{l-2m,\kappa-2m}_{2,\beta+(\kappa-l)\vec{1}}(\mathcal{G}).$$

By Lemma 6.2.4 and (8.1.5), this extension has the form

$$(u,\underline{\phi}) \to (f, R\underline{\phi}),$$

where $f = L(x,\partial_x)\,u$ if $2m \le l < \kappa$, while in the cases $l \le 0$ and $0 < l < 2m$ the functional $f \in V^{2m-l}_{2,-\beta}(\mathcal{G})^*$ is given by the equalities (6.2.19) and (6.2.20), respectively.

Thus, we obtain the following theorem (cf. Theorem 6.2.1).

THEOREM 8.1.1. *The operator $\mathcal{A}_{\kappa,\beta+(\kappa-l)\vec{1}}$ can be extended in a unique way to a continuous operator*

$$\mathcal{A}_{l,\beta}\ :\ \tilde{V}^{l,\kappa}_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G}) \to \tilde{V}^{l-2m,\kappa-2m}_{2,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$$

with $l < \kappa$. This extension has the form

$$(u,\underline{\phi},\underline{u}) \to \big(L(u,\underline{\phi}),\ Q\underline{\phi} + C\underline{u}\big)$$

where L denotes the operator (8.1.9) and Q is given by (8.1.3).

In particular, in the case $l \le 0$ the triple

$$(f,\underline{\Phi},\underline{g}) = \mathcal{A}_{l,\beta}\,(u,\underline{\phi},\underline{u})$$

satisfies the equality

$$(f,v)_{\mathcal{G}} + (\underline{\Phi},\underline{w})_{\partial\mathcal{G}} + (\underline{g},\underline{v})_{\partial\mathcal{G}}$$
$$= (u,L^+v)_{\mathcal{G}} + \big(\underline{\phi},P^{(\kappa)}v + R^+\underline{w} + Q^+\underline{v}\big)_{\partial\mathcal{G}} + (\underline{u},C^+\underline{v})_{\partial\mathcal{G}}$$

for all $v \in V^{2m-l}_{2,-\beta}(\mathcal{G})$, $\underline{w} \in \prod_{s=1}^{\kappa-2m} V^{2m-l+s-1/2}_{2,-\beta}(\partial\mathcal{G})$, $\underline{v} \in V^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{G})$. This means that $\mathcal{A}_{l,\beta}$ is adjoint to the operator

$$V^{2m-l}_{2,-\beta}(\mathcal{G}) \times \Big(\prod_{s=1}^{\kappa-2m} V^{2m-l+s-1/2}_{2,-\beta}(\partial\mathcal{G})\Big) \times V^{-l+\underline{\mu}+1/2}_{2,-\beta}(\partial\mathcal{G}) \ni (v,\underline{w},\underline{v})$$
$$\to \big(L^+v,\ P^{(\kappa)}v|_{\partial\mathcal{G}\backslash\mathcal{S}} + R^+\underline{w} + Q^+\underline{v},\ C^+\underline{v}\big)$$
$$\in V^{-l}_{2,-\beta}(\mathcal{G}) \times \Big(\prod_{j=1}^{\kappa} V^{-l+j-1/2}_{2,-\beta}(\partial\mathcal{G})\Big) \times V^{-l-\underline{\tau}+1/2}_{2,-\beta}(\partial\mathcal{G})$$

of the formally adjoint problem (8.1.6)–(8.1.8).

8.1.3. Solvability of the boundary value problem. We write the leading parts $L^{(\tau)}$, $B^{(\tau)}_k$, and $C^{(\tau)}_{k,j}$ of the operators L, B_k, and $C_{k,j}$ at the point $x^{(\tau)}$ (see Definition 6.2.1) in the form

$$L^{(\tau)}(x,\partial_x) = r^{-2m}\mathcal{L}^{(\tau)}(\omega,\partial_\omega,r\partial_r), \qquad B^{(\tau)}_k(x,\partial_x) = r^{-\mu_k}\,\mathcal{B}^{(\tau)}_k(\omega,\partial_\omega,r\partial_r)$$
$$C^{(\tau)}_{k,j}(x,\partial_x) = r^{-\mu_k-\tau_j}\,\mathcal{C}^{(\tau)}_{k,j}(\omega,\partial_\omega,r\partial_r),$$

where $r = |x - x^{(\tau)}|$ and ω are coordinates on the unit sphere $|x - x^{(\tau)}| = 1$. Let Ω_τ be the intersection of the cone $\mathcal{K}_\tau$ with the sphere $|x - x^{(\tau)}| = 1$. As in Chapter 6, we denote by $\mathfrak{A}_\tau(\lambda)$ the operator of the boundary value problem

$$\mathcal{L}^{(\tau)}(\omega, \partial_\omega, \lambda)u = f \quad \text{in } \Omega_\tau,$$

$$\mathcal{B}_k^{(\tau)}(\omega, \partial_\omega, \lambda)u + \sum_{j=1}^{J} \mathcal{C}_{k,j}^{(\tau)}(\omega, \partial_\omega, \lambda + \tau_j)u_j = g_k \text{ on } \partial\Omega_\tau, \ k = 1, \dots, m + J.$$

Furthermore, let $\mathcal{N}_\beta^*$ be the set of all solutions

$$(v, \underline{w}, \underline{v}) \in \bigcap_{l \geq 2m} \Big(V_{2,\beta+l\vec{1}}^{l}(\mathcal{G}) \times \prod_{s=1}^{\kappa-2m} V_{2,\beta+l\vec{1}}^{l+s-1/2}(\partial\mathcal{G}) \times \prod_{k=1}^{m+J} V_{2,\beta+l\vec{1}}^{l-2m+\mu_k+1/2}(\partial\mathcal{G}) \Big)$$

of the homogeneous formally adjoint problem (8.1.6)–(8.1.8).

The following result generalizes Theorem 6.3.3.

THEOREM 8.1.2. *Suppose that the boundary value problem (8.1.1), (8.1.2) is elliptic, the operators L, B_k and $C_{k,j}$ are admissible, and the representation (8.1.3) is valid for the vector B. Furthermore, we assume that there are no eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ on the line $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ for $\tau = 1, \dots, d$. Then the operator $\mathcal{A}_{l,\beta}$ is Fredholm. The kernel of $\mathcal{A}_{l,\beta}$ depends only on the difference $\beta - l\vec{1}$. The range of the operator $\mathcal{A}_{l,\beta}$ is the set of all $(f, \underline{\Phi}, \underline{g}) \in \tilde{V}_{2,\beta}^{l-2m,\kappa-2m}(\mathcal{G}) \times V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})$ such that*

$$(f, v)_{\mathcal{G}} + (\underline{\Phi}, \underline{w})_{\partial\mathcal{G}} + (\underline{g}, \underline{v})_{\partial\mathcal{G}} = 0$$

for all $(v, \underline{w}, \underline{v}) \in \mathcal{N}^*_{-\beta+(l-2m)\vec{1}}$.

8.1.4. Asymptotics of the solution. We suppose that the conditions of Theorem 8.1.2 are satisfied for $l = l_1 \geq \kappa$ and for a certain β and consider the solution $(u, \underline{u}) \in V_{2,\beta}^{l_1}(\mathcal{G}) \times V_{2,\beta}^{l_1+\underline{\tau}-1/2}(\partial\mathcal{G})$ of the boundary value problem (8.1.1), (8.1.2), where

$$(f, \underline{g}) \in V_{2,\gamma}^{l_2-2m}(\mathcal{G}) \times V_{2,\gamma}^{l_2-\underline{\mu}-1/2}(\partial\mathcal{G}),$$

$l_2 \geq \kappa$. Then the assertions of Theorems 6.4.1 and 6.4.2 remain valid if we replace the assumption $l_1, l_2 \geq 2m$ by $l_1, l_2 \geq \kappa$. In particular, if the operators L, B_k, $C_{k,j}$ are δ-admissible (see Definition 6.4.1), there are no eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ on the lines $\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$ and $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$, where $0 < (l_2 - \gamma_\tau) - (l_1 - \beta_\tau) < \delta$, then the solution $(u, \underline{u})$ admits the decomposition

$$(u, \underline{u}) = \sum_{\mu=1}^{M} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s} \, (u_{\mu,j,s} \, , \underline{u}_{\mu,j,s}) + (w, \underline{w}) \tag{8.1.10}$$

in a neighbourhood of the conical point $x^{(\tau)}$. Here $(w, \underline{w})$ belongs to the space $V_{2,\gamma_\tau}^{l_2}(\mathcal{K}_\tau) \times V_{2,\gamma_\tau}^{l_2+\underline{\tau}-1/2}(\partial\mathcal{K}_\tau)$, $c_{\mu,j,s}$ are constants, and the functions $u_{\mu,j,s}$ and vector-functions $\underline{u}_{\mu,j,s}$ are given by the equalities

$$\begin{cases} u_{\mu,j,s} = r^{\lambda_\mu} \displaystyle\sum_{\sigma=0}^{s} \frac{(\log r)^\sigma}{\sigma!} \, \varphi_{j,s-\sigma}^{(\mu)}(\omega), \\ \underline{u}_{\mu,j,s} = r^{\lambda_\mu} \displaystyle\sum_{\sigma=0}^{s} \frac{(\log r)^\sigma}{\sigma!} \, \big(r^{\tau_1} \varphi_{1;j,s-\sigma}^{(\mu)}(\omega), \dots, r^{\tau_J} \varphi_{J;j,s-\sigma}^{(\mu)}(\omega) \big), \end{cases} \tag{8.1.11}$$

where $\lambda_1, \dots, \lambda_M$ are the eigenvalues of $\mathfrak{A}_\tau(\lambda)$ in the strip $-\beta_\tau + l_1 - n/2 < \operatorname{Re}\lambda < -\gamma_\tau + l_2 - n/2$ and

$$\{(\varphi^{(\mu)}_{j,s}, \varphi^{(\mu)}_{1;j,s}, \dots, \varphi^{(\mu)}_{J;j,s})\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

are canonical systems of Jordan chains of the operator pencil $\mathfrak{A}_\tau(\lambda)$ corresponding to the eigenvalues λ_μ.

8.1.5. A formula for the coefficients in the asymptotics. The coefficients $c_{\mu,j,s}$ in (8.1.10) can be calculated analogously to the case $\kappa = 2m$ (see Section 6.4).

Let $(L^+)^{(\tau)}(x,\partial_x)$, $P_j^{(\tau)}(x,\partial_x)$, $(Q^+_{k,j})^{(\tau)}(x,\partial_x)$, $(C^+_{k,j})^{(\tau)}(x,\partial_x)$, $(R^+_{s,j})^{(\tau)}(x,\partial_x)$ be the leading parts of the operators L^+, P_j, $Q^+_{k,j}$, $C^+_{k,j}$, and $R^+_{s,j}$ at the point $x^{(\tau)}$ (see Defintion 6.2.1). We write these operators in the form

$$\begin{aligned}
(L^+)^{(\tau)}(x,\partial_x) &= r^{-2m}\,\mathcal{L}^+(\omega,\partial_\omega, r\partial_r),\\
P_j^{(\tau)}(x,\partial_x) &= r^{-2m+j}\,\mathcal{P}_j(\omega,\partial_\omega, r\partial_r),\\
(Q^+_{k,j})^{(\tau)}(x,\partial_x) &= r^{-\mu_k+j-1}\,\mathcal{Q}^+_{k,j}(\omega,\partial_\omega, r\partial_r),\\
(C^+_{k,j})^{(\tau)}(x,\partial_x) &= r^{-\mu_k-\tau_j}\,\mathcal{C}^+_{k,j}(\omega,\partial_\omega, r\partial_r),\\
(R^+_{s,j})^{(\tau)}(x,\partial_x) &= r^{-2m-s+j}\,\mathcal{R}^+_{s,j}(\omega,\partial_\omega, r\partial_r),
\end{aligned}$$

where $r = |x - x^{(\tau)}|$ and ω are coordinates on the sphere $|x - x^{(\tau)}| = 1$, and denote the operator of the boundary value problem

$$\begin{aligned}
&\mathcal{L}^+(\omega,\partial_\omega, -\lambda+2m-n)\,v = f \quad \text{in } \Omega_\tau,\\
&\mathcal{P}_j(\omega,\partial_\omega,-\lambda+2m-n)\,v + \sum_{k=1}^{m+J} \mathcal{Q}^+_{k,j}(\omega,\partial_\omega,-\lambda+\mu_k+1-n)\,v_k\\
&\qquad + \sum_{s=1}^{\kappa-2m} \mathcal{R}^+_{s,j}(\omega,\partial_\omega,-\lambda+2m+s-n)\,w_s = g_j \quad \text{on } \partial\Omega_\tau,\ j=1,\dots,\kappa,\\
&\sum_{k=1}^{m+J} \mathcal{C}^+_{k,j}(\omega,\partial_\omega,-\lambda+\mu_k+1-n)\,v_k = h_j \quad \text{on } \partial\Omega_\tau,\ j=1,\dots,J,
\end{aligned}$$

by $\mathfrak{A}^+_\tau(\lambda)$. The operator $\mathfrak{A}^+_\tau(\lambda)$ is formally adjoint to $\mathfrak{A}_\tau(\overline{\lambda})$ with respect to a Green formula analogous to (6.1.31). Hence λ_μ is an eigenvalue of the pencil $\mathfrak{A}_\tau(\lambda)$ if and only if $\overline{\lambda}_\mu$ is an eigenvalue of $\mathfrak{A}^+_\tau(\lambda)$. The geometric, algebraic and partial multiplicities of these eigenvalues coincide.

For every $\mu = 1,\dots,M$ let $\{(\psi^{(\mu)}_{j,s}, \underline{\chi}^{(\mu)}_{j,s}, \underline{\psi}^{(\mu)}_{j,s})\}$ be a canonical systems of Jordan chains of the operator pencil $\mathfrak{A}^+_\tau(\lambda)$ corresponding to the eigenvalue $\overline{\lambda}_\mu$ satisfying the biorthonormality condition

$$\sum_{p=0}^{\sigma}\sum_{q=p+1}^{p+s+1} \frac{1}{q!}\Big\langle \mathfrak{A}^{(q)}(\lambda_\mu)\,(\varphi^{(\mu)}_{j,p+s+1-q}, \underline{\varphi}^{(\mu)}_{j,p+s+1-q})\,,\,(\psi^{(\mu)}_{k,\sigma-p}, \underline{\chi}^{(\mu)}_{k,\sigma-p}, \underline{\psi}^{(\mu)}_{k,\sigma-p})\Big\rangle$$
$$= \delta_{j,k}\cdot\delta_{s,\kappa_{\mu,k}-1-\sigma}$$

for $j,k=1,\dots,I_\mu$, $s=0,\dots,\kappa_{\mu,j}-1$, $\sigma=0,\dots,\kappa_{\mu,k}-1$. Here $\mathfrak{A}^{(q)}(\lambda)$ denotes the q-th derivative (with respect to λ) of the mapping

$$(\varphi,\underline{\varphi})\to\Big(\mathcal{L}(\lambda)\varphi,\ \mathcal{D}^{(\kappa-2m)}\mathcal{L}(\lambda)\varphi\big|_{\partial\Omega_\tau},\ \big(\mathcal{B}_k(\lambda)\varphi\big|_{\partial\Omega_\tau}+\sum_{j=1}^{J}\mathcal{C}_{k,j}(\lambda+\tau_j)\varphi_j\big)_{1\le k\le m+J}\Big)$$

and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L_2(\Omega_\tau)\times L_2(\partial\Omega_\tau)^{\kappa-2m}\times L_2(\partial\Omega_\tau)^{m+J}$. We set

$$(8.1.12)\quad v_{\mu,j,s}=-r^{-\overline{\lambda}_\mu+2m-n}\sum_{\sigma=0}^{s}\frac{(-\log r)^\sigma}{\sigma!}\,\psi^{(\mu)}_{j,s-\sigma}(\omega),$$

$$(8.1.13)\quad v_{k;\mu,j,s}=-r^{-\overline{\lambda}_\mu+\mu_k+1-n}\sum_{\sigma=0}^{s}\frac{(-\log r)^\sigma}{\sigma!}\,\psi^{(\mu)}_{k;j,s-\sigma}(\omega),\ k=1,\dots,m+J,$$

$$(8.1.14)\quad w_{k;\mu,j,s}=-r^{-\overline{\lambda}_\mu+2m+k-n}\sum_{\sigma=0}^{s}\frac{(-\log r)^\sigma}{\sigma!}\,\chi^{(\mu)}_{k;j,s-\sigma}(\omega),\ k=1,\dots,\kappa-2m,$$

where $\psi^{(\mu)}_{k;j,s}$, $\chi^{(\mu)}_{k;j,s}$ denote the k-th components of the vectors $\underline{\psi}^{(\mu)}_{j,s}$ and $\underline{\chi}^{(\mu)}_{j,s}$, respectively. Let $\underline{v}_{\mu,j,s}$ and $\underline{w}_{\mu,j,s}$ be the vector-functions with the components (8.1.13) and (8.1.14), respectively. Then the triples $(v_{\mu,j,s},\underline{w}_{\mu,j,s},\underline{v}_{\mu,j,s})$ are special solutions of the homogeneous formally adjoint problem (8.1.6)–(8.1.8).

Analogously to Theorem 6.4.3, the following result holds.

THEOREM 8.1.3. *Suppose that the operators L, B_k, $C_{k,j}$ of the elliptic boundary value problem (8.1.1), (8.1.2) are δ-admissible near the conical point $x^{(\tau)}$ and the representation (8.1.3) is valid. Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ on the lines* $\operatorname{Re}\lambda=-\beta_\tau+l_1-n/2$ *and* $\operatorname{Re}\lambda=-\gamma_\tau+l_2-n/2$, *where* $0<(l_2-\gamma_\tau)-(l_1-\beta_\tau)<\delta$, $l_1\ge\kappa$, $l_2\ge\kappa$. *If $(u,\underline{u})$ is a solution of the boundary value problem (8.1.1), (8.1.2) such that*

$$(8.1.15)\qquad \eta\,(u,\underline{u})\in V^{l_1}_{2,\beta_\tau}(\mathcal{K}_\tau)\times V^{l_1+\underline{\tau}-1/2}_{2,\beta_\tau}(\partial\mathcal{K}_\tau)$$

and

$$(8.1.16)\qquad \eta\,(f,\underline{g})\in V^{l_2-2m}_{2,\gamma_\tau}(\mathcal{K}_\tau)\times V^{l_2-\underline{\mu}-1/2}_{2,\gamma_\tau}(\partial\mathcal{K}_\tau),$$

where η is an infinitely differentiable function with support in $\mathcal{U}_\tau$ equal to one in a neighbourhood of $x^{(\tau)}$, then there is the representation (8.1.10) in this neighbourhood.

Let $(v,\underline{w},\underline{v})$ be a solution of the homogeneous formally adjoint problem (8.1.6)–(8.1.8) such that

$$(v,\underline{w},\underline{v})=\big(v_{\mu,j,\kappa_{\mu,j}-1-s}\,,\underline{w}_{\mu,j,\kappa_{\mu,j}-1-s}\,,\underline{v}_{\mu,j,\kappa_{\mu,j}-1-s}\big)+(V,\underline{W},\underline{V}),$$

where

$$\eta(V,\underline{W},\underline{V})\in V^{2m}_{2,-\beta_\tau+l_1}(\mathcal{K}_\tau)\times\prod_{s=1}^{\kappa-2m}V^{2m+s-1/2}_{2,-\beta_\tau+l_1}(\partial\mathcal{K}_\tau)\times V^{\underline{\mu}+1/2}_{2,-\beta_\tau+l_1}(\partial\mathcal{K}_\tau).$$

Then the constants $c_{\mu,j,s}$ *in (8.1.10) are given by the formula*

$$(8.1.17) \qquad c_{\mu,j,s} = \int\limits_{\mathcal{G}} L(\zeta u)\cdot \overline{v}\, dx + \int\limits_{\partial\mathcal{G}} \big(\mathcal{D}^{(\kappa-2m)} L(\zeta u), \underline{w}\big)_{\mathbb{C}^{\kappa-2m}}\, d\sigma + \int\limits_{\partial\mathcal{G}} \big(B(\zeta u) + C(\zeta \underline{u})\, ,\, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma.$$

Here ζ *is an arbitrary smooth cut-off function equal to one in a neighbourhood of* $x^{(\tau)}$ *such that* $\zeta\eta = \zeta$.

Formula (8.1.17) is based on the formula

$$c_{\mu,j,s} = \big(f, v_{\mu,j,\kappa_{\mu,j}-1-s}\big)_{\mathcal{K}_\tau} + \big(\mathcal{D}^{(\kappa-2m)} f, \underline{w}_{\mu,j,\kappa_{\mu,j}-1-s}\big)_{\partial\mathcal{K}_\tau} + \big(\underline{g}, \underline{v}_{\mu,j,\kappa_{\mu,j}-1-s}\big)_{\partial\mathcal{K}_\tau}$$

for the coefficients in the asymptotics of the solutions of the model problem in the cone $\mathcal{K}_\tau$ (cf. Theorem 6.1.6).

Analogously, the assertions of Theorem 6.4.4 can be generalized. Under our conditions on B, the formula for the coefficients d_s in Theorem 6.4.4 contains the additional term $\big(\mathcal{D}^{(\kappa-2m)} f, \underline{w}^{(s)}\big)_{\partial\mathcal{G}}$ on the right-hand side.

8.2. Elliptic problems for systems of differential equations

Now we extend the main results of Chapter 6 to boundary value problems for elliptic systems of differential equations.

8.2.1. The boundary value problem and its formally adjoint. As in the previous section, we suppose that $\mathcal{G}$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{G}$ which is smooth outside the subset $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\} \subset \partial\mathcal{G}$ and that in a neighbourhood $\mathcal{U}_\tau$ of each of the points $x^{(\tau)}$ the domain $\mathcal{G}$ coincides with a cone $\mathcal{K}_\tau$ with vertex $x^{(\tau)}$. Let

$$L(x,\partial_x) = \big(L_{i,j}(x,\partial_x)\big)_{1\le i,j\le N}$$

be a matrix of admissible differential operators of order $s_i + t_j$ (see Definition 6.2.1), where s_i and t_j are integer numbers, $\max(s_1, \dots, s_N) = 0$, $t_j \ge 0$, $i,j = 1,\dots,N$.

Furthermore, let

$$B(x,\partial_x) = \Big(B_{k,j}(x,\partial_x)\Big)_{1\le k\le m+J,\ 1\le j\le N}$$

be a matrix of admissible differential operators of order $\sigma_k + t_j$ and

$$C(x,\partial_x) = \Big(C_{k,\nu}(x,\partial_x)\Big)_{1\le k\le m+J,\ 1\le \nu\le J}$$

a matrix of admissible tangential differential operators $C_{k,\nu}$ of order $\mu_k + \tau_\nu$ on $\partial\mathcal{G}\backslash\mathcal{S}$.

Throughout this section, we suppose that the boundary value problem

$$(8.2.1) \qquad L\underline{\mathfrak{u}} = \underline{\mathfrak{f}} \quad \text{in } \mathcal{G},$$

$$(8.2.2) \qquad B\underline{\mathfrak{u}} + C\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}$$

is elliptic, i.e., L is properly elliptic in $\overline{\mathcal{G}}\backslash\mathcal{S}$ and condition (ii) in Definition 4.2.2 is satisfied for all $x^{(0)} \in \partial\mathcal{G}\backslash\mathcal{S}$. Here $\underline{\mathfrak{u}} = (\mathfrak{u}_1, \dots, \mathfrak{u}_N)$, $\underline{\mathfrak{f}} = (\mathfrak{f}_1, \dots, \mathfrak{f}_N)$ are vector-functions on $\mathcal{G}$ and $\underline{u} = (u_1, \dots, u_J)$, $\underline{g} = (g_1, \dots, g_{m+J})$ are vector-functions on

$\partial\mathcal{G}\backslash\mathcal{S}$. We set

$$\sigma_0 \overset{def}{=} \max(0, \sigma_1 + 1, \dots, \sigma_{m+J} + 1) \tag{8.2.3}$$

and write the vector $B\underline{\mathfrak{u}}$ in the form

$$B\underline{\mathfrak{u}}\big|_{\partial\mathcal{G}\backslash\mathcal{S}} = \sum_{j=1}^{N} Q^{(j)}\, \mathcal{D}^{(\sigma_0+t_j)} \mathfrak{u}_j\big|_{\partial\mathcal{G}\backslash\mathcal{S}}\,, \tag{8.2.4}$$

where $\mathcal{D}^{(\sigma_0+t_j)}$ denotes the column vector with the components $1, D_\nu, \dots, D_\nu^{\sigma_0+t_j-1}$ and

$$Q^{(j)} = \left(Q^{(j)}_{k,q}\right)_{1\le k\le m+J,\ 1\le q\le \sigma_0+t_j}$$

are $(m+J)\times(\sigma_0+t_j)$-matrices of admissible tangential differential operators $Q^{(j)}_{k,q}$ of order $\sigma_k + t_j - q + 1$ on $\partial\mathcal{G}\backslash\mathcal{S}$. Furthermore, let

$$\mathcal{D}^{(\sigma_0-s_i)} L_{i,j}\mathfrak{u}_j\big|_{\partial\mathcal{G}\backslash\mathcal{S}} = R^{i,j}\, \mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\big|_{\partial\mathcal{G}\backslash\mathcal{S}}\,, \tag{8.2.5}$$

where

$$R^{i,j} = \left(R^{i,j}_{s,q}\right)_{1\le s\le \sigma_0-s_i,\ 1\le q\le\sigma_0+t_j}$$

are $(\sigma_0 - s_i)\times(\sigma_0+t_j)$-matrices of admissible tangential differential operators $R^{i,j}_{s,q}$ of order $s_i + t_j + q - s$ on $\partial\mathcal{G}\backslash\mathcal{S}$. By (4.2.10), there exist $(\sigma_0 + t_j)\times N$-matrices $\mathfrak{P}^{(j)}$ such that

$$\int_{\mathcal{G}} (L\underline{\mathfrak{u}}, \underline{\mathfrak{v}})_{\mathbb{C}^N}\, dx = \int_{\mathcal{G}} (\underline{\mathfrak{u}}, L^+\underline{\mathfrak{v}})_{\mathbb{C}^N}\, dx + \sum_{j=1}^{N}\int_{\partial\mathcal{G}} (\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\,, \mathfrak{P}^{(j)}\underline{\mathfrak{v}})_{\mathbb{C}^{\sigma_0+t_j}}\, d\sigma \tag{8.2.6}$$

for $\underline{\mathfrak{u}}, \underline{\mathfrak{v}} \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})^N$. Due to Lemma 6.2.3, the elements $P^{(j)}_{q,i}$ of the matrix $\mathfrak{P}^{(j)}$ are admissible differential operators of order $s_i + t_j - q$.

Formulas (8.2.4)–(8.2.6) yield the following *Green formula* which is valid for all $\underline{\mathfrak{u}}, \underline{\mathfrak{v}} \in C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})^N$, $\underline{u} \in C_0^\infty(\partial\mathcal{G}\backslash\mathcal{S})^J$, $\underline{v} \in C^\infty(\partial\mathcal{G}\backslash\mathcal{S})^{m+J}$, $\underline{w}^{(i)} \in C^\infty(\partial\mathcal{G}\backslash\mathcal{S})^{\sigma_0-s_i}$:

$$\begin{aligned}\int_{\mathcal{G}} (L\underline{\mathfrak{u}}, \underline{\mathfrak{v}})_{\mathbb{C}^N}\, dx &+ \int_{\partial\mathcal{G}} \sum_{i=1}^{N}\sum_{j=1}^{N} (\mathcal{D}^{(\sigma_0-s_i)} L_{i,j}\mathfrak{u}_j\,, \underline{w}^{(i)})_{\mathbb{C}^{\sigma_0-s_i}}\, d\sigma \\ &+ \int_{\partial\mathcal{G}} (B\underline{\mathfrak{u}} + C\underline{u}, \underline{v})_{\mathbb{C}^{m+J}}\, d\sigma = \int_{\mathcal{G}} (\underline{\mathfrak{u}}, L^+\underline{\mathfrak{v}})_{\mathbb{C}^N}\, dx + \int_{\partial\mathcal{G}} (\underline{u}, C^+\underline{v})_{\mathbb{C}^J}\, d\sigma \\ &+ \sum_{j=1}^{N}\int_{\partial\mathcal{G}} \Big(\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j\,, \mathfrak{P}^{(j)}\underline{\mathfrak{v}} + (Q^{(j)})^+\underline{v} + \sum_{i=1}^{N}(R^{i,j})^+\underline{w}^{(i)}\Big)_{\mathbb{C}^{\sigma_0+t_j}}\, d\sigma.\end{aligned} \tag{8.2.7}$$

As in Section 4.2, the boundary value problem

$$L^+\,\underline{\mathfrak{v}} = \underline{\mathfrak{f}} \quad \text{in } \mathcal{G}, \tag{8.2.8}$$

$$\mathfrak{P}^{(j)}\,\underline{\mathfrak{v}} + (Q^{(j)})^+\underline{v} + \sum_{i=1}^{N}(R^{i,j})^+\underline{w}^{(i)} = \underline{g}^{(j)} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S},\ j = 1, \dots, N, \tag{8.2.9}$$

$$C^+\,\underline{v} = \underline{h} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S} \tag{8.2.10}$$

is said to be *formally adjoint* to the boundary value problem (8.2.1), (8.2.2). This problem is elliptic if and only if problem (8.2.1), (8.2.2) is elliptic.

8.2.2. Solvability of the boundary value problem. The boundary value problem (8.2.1), (8.2.2) generates a continuous mapping

$$\mathcal{A}_{l,\beta}\ :\ V_{2,\beta}^{l+\underline{t}}(\mathcal{G})\times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G})\ni(\mathfrak{u},\underline{u})\to(\mathfrak{f},\underline{g})\in V_{2,\beta}^{l-\underline{s}}(\mathcal{G})\times V_{2,\beta}^{l-\underline{\sigma}-1/2}(\partial\mathcal{G})$$

for arbitrary integer $l\geq\sigma_0$ and arbitrary $\beta=(\beta_1,\dots,\beta_d)\in\mathbb{R}^d$. Here we have used the notation

$$V_{2,\beta}^{l+\underline{t}}(\mathcal{G})=\prod_{j=1}^{N}V_{2,\beta}^{l+t_j}(\mathcal{G})\quad\text{and}\quad V_{2,\beta}^{l-\underline{s}}(\mathcal{G})=\prod_{i=1}^{N}V_{2,\beta}^{l-s_i}(\mathcal{G}).$$

Analogous notation will be used for the products of the spaces $\tilde V_{2,\beta}^{l+t_j,\sigma_0+t_j}(\mathcal{G})$ and $\tilde V_{2,\beta}^{l-s_i,\sigma_0-s_i}(\mathcal{G})$. Then we can identify the operator $\mathcal{A}_{l,\beta}$ with the operator

$$\tilde V_{2,\beta}^{l+\underline{t},\sigma_0+\underline{t}}(\mathcal{G})\times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G})\ni\Big(\big\{\big(\mathfrak{u}_j,\mathcal{D}^{(\sigma_0+t_j)}\mathfrak{u}_j|_{\partial\mathcal{G}\setminus\mathcal{S}}\big)\big\}_{1\leq j\leq N}\,,\,\underline{u}\Big)$$
$$\to\Big(\big\{\big(\mathfrak{f}_i,\mathcal{D}^{(\sigma_0-s_i)}\mathfrak{f}_i|_{\partial\mathcal{G}\setminus\mathcal{S}}\big)\big\}_{1\leq i\leq N}\,,\,\underline{g}\Big)\in\tilde V_{2,\beta}^{l-\underline{s},\sigma_0-\underline{s}}(\mathcal{G})\times V_{2,\beta}^{l-\underline{\sigma}-1/2}(\partial\mathcal{G}),$$

where

$$\mathfrak{f}_i=\sum_{j=1}^{N}L_{i,j}\,\mathfrak{u}_j\,,\qquad\underline{g}=B\underline{\mathfrak{u}}|_{\partial\mathcal{G}\setminus\mathcal{S}}+C\underline{u}.$$

We denote the last operator also by $\mathcal{A}_{l,\beta}$. The operator $\mathcal{A}_{\sigma_0,\beta+(\sigma_0-l)\vec{1}}$ can be uniquely extended to a continuous operator

$$\mathcal{A}_{l,\beta}\ :\ \tilde V_{2,\beta}^{l+\underline{t},\sigma_0+\underline{t}}(\mathcal{G})\times V_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{G})\to\tilde V_{2,\beta}^{l-\underline{s},\sigma_0-\underline{s}}(\mathcal{G})\times V_{2,\beta}^{l-\underline{\sigma}-1/2}(\partial\mathcal{G})\,,\quad l<\sigma_0,$$

which has the form

$$\big(\{(\mathfrak{u}_j,\underline{\phi}^{(j)})\}_{1\leq j\leq N}\,,\,\underline{u}\big)\to\big(\{(\mathfrak{f}_i,\underline{\psi}^{(i)})\}_{1\leq i\leq N}\,,\,\underline{g}\big),$$

where

$$(\mathfrak{f}_i,\underline{\psi}^{(i)})=\sum_{j=1}^{N}L_{i,j}\,(\mathfrak{u}_j,\underline{\phi}^{(j)})\quad\text{for }i=1,\dots,N,\tag{8.2.11}$$

$$\underline{g}=\sum_{j=1}^{N}Q^{(j)}\,\underline{\phi}^{(j)}+C\underline{u}\,.\tag{8.2.12}$$

Here the operator $L_{i,j}\,:\,\tilde V_{2,\beta}^{l+t_j,\sigma_0+t_j}(\mathcal{G})\to\tilde V_{2,\beta}^{l-s_i,\sigma_0-s_i}(\mathcal{G})$ is defined analogously to the operator (8.1.9) in the previous section.

In particular, the vector-functions $\underline{\psi}^{(i)}$ in (8.2.11) are given by

$$\underline{\psi}^{(i)}=\sum_{j=1}^{N}R^{i,j}\,\underline{\phi}^{(j)}$$

where $R^{i,j}$ is the matrix in (8.2.5). Moreover, in the case $l\leq-\max(t_1,\dots,t_N)$ the vector-functional $\underline{\mathfrak{f}}\in V_{2,-\beta}^{-l+\underline{s}}(\mathcal{G})^*$ satisfies the equality

$$\big(\underline{\mathfrak{f}},\underline{\mathfrak{v}}\big)_{\mathcal{G}}=\big(\underline{\mathfrak{u}},L^+\underline{\mathfrak{v}}\big)_{\mathcal{G}}+\sum_{j=1}^{N}\big(\underline{\phi}^{(j)},\mathfrak{P}^{(j)}\underline{\mathfrak{v}}\big)_{\partial\mathcal{G}}\quad\text{for all }\underline{\mathfrak{v}}\in V_{2,-\beta}^{-l+\underline{s}}(\mathcal{G}).$$

for all $\underline{\mathfrak{v}}\in V_{2,-\beta}^{-l+\underline{s}}(\mathcal{G})$.

We write the leading parts of the operators $L^{(\tau)}_{i,j}$, $B^{(\tau)}_{k,j}$ and $C^{(\tau)}_{k,\nu}$ at the conical point $x^{(\tau)}$ in the form

$$L^{(\tau)}_{i,j}(x,\partial_x) = r^{-(s_i+t_j)}\,\mathcal{L}^{(\tau)}_{i,j}(\omega,\partial_\omega,r\partial_r),$$
$$B^{(\tau)}_{k,j}(x,\partial_x) = r^{-(\sigma_k+t_j)}\,\mathcal{B}^{(\tau)}_{k,j}(\omega,\partial_\omega,r\partial_r),$$
$$C^{(\tau)}_{k,\nu}(x,\partial_x) = r^{-(\sigma_k+\tau_\nu)}\,\mathcal{C}^{(\tau)}_{k,\nu}(\omega,\partial_\omega,r\partial_r),$$

where $r = |x-x^{(\tau)}|$, ω are coordinates on the domain $\Omega_\tau = \{x \in \mathcal{K}_\tau : |x-x^{(\tau)}| = 1\}$. Let $\mathfrak{A}_\tau(\lambda)$ be the operator of the boundary value problem

$$\text{(8.2.13)}\quad \sum_{j=1}^{N} \mathcal{L}^{(\tau)}_{i,j}(\omega,\partial_\omega,\lambda+t_j)\,\Phi_j = \mathfrak{f}_i \quad \text{in } \Omega_\tau,\ i=1,\dots,N,$$

$$\text{(8.2.14)}\quad \sum_{j=1}^{N} \mathcal{B}^{(\tau)}_{k,j}(\omega,\partial_\omega,\lambda+t_j)\,\Phi_j + \sum_{\nu=1}^{J} \mathcal{C}^{(\tau)}_{k,\nu}(\omega,\partial_\omega,\lambda+\tau_\nu)\varphi_\nu = g_k \quad \text{on } \partial\Omega_\tau,$$
$$k=1,\dots,m+J.$$

Then the following statement holds.

THEOREM 8.2.1. *Suppose that the boundary value problem (8.2.1), (8.2.2) is elliptic and the line* $\mathrm{Re}\,\lambda = -\beta_\tau + l - n/2$ *does not contain eigenvalues of the pencil* $\mathfrak{A}_\tau(\lambda)$ *for* $\tau = 1,\dots,d$. *Then the operator* $\mathcal{A}_{l,\beta}$ *is Fredholm. The kernel of* $\mathcal{A}_{l,\beta}$ *depends only on* $\beta - l\vec{1}$.

Let $\mathcal{N}^*_\beta$ *be the set of all solutions* $\big(\underline{\mathfrak{v}}, (\underline{w}^{(i)})_{1\le i\le N}, \underline{v}\big)$ *of the homogeneous formally adjoint problem (8.2.8)–(8.2.10) which belong to the space*

$$\bigcap_{k\ge \max t_j} \Big(V^{k+\underline{s}}_{2,\beta+k\vec{1}}(\mathcal{G}) \times \prod_{i=1}^{N}\prod_{\nu=1}^{\sigma_0-s_i} V^{k+s_i+\nu-1/2}_{2,\beta+k\vec{1}}(\partial\mathcal{G}) \times V^{k+\underline{\sigma}+1/2}_{2,\beta+k\vec{1}}(\partial\mathcal{G})\Big).$$

Then the range of the operator $\mathcal{A}_{l,\beta}$ *is the set of all*

$$(\underline{\mathfrak{f}}, (\underline{\psi}^{(i)})_{1\le i\le N}, \underline{g}) \in \tilde{V}^{l-\underline{s},\sigma_0-\underline{s}}_{2,\beta}(\mathcal{G}) \times V^{l-\underline{\sigma}-1/2}_{2,\beta}(\partial\mathcal{G})$$

such that

$$\text{(8.2.15)}\qquad \big(\underline{\mathfrak{f}},\underline{\mathfrak{v}}\big)_{\mathcal{G}} + \sum_{i=1}^{N}\big(\underline{\psi}^{(i)},\underline{w}^{(i)}\big)_{\partial\mathcal{G}} + (\underline{g},\underline{v})_{\partial\mathcal{G}} = 0$$

for all $\big(\underline{\mathfrak{v}}, (\underline{w}^{(i)})_{1\le i\le N}, \underline{v}\big) \in \mathcal{N}^*_{-\beta+l\vec{1}}$.

Furthermore, an a priori estimate analogous to that given in Theorem 6.3.1 holds for the solutions of the boundary value problem (8.2.1), (8.2.2).

8.2.3. Asymptotics of the solution. Now we assume that the operators $L_{i,j}$, $B_{k,j}$, and $C_{k,\nu}$ are δ-admissible near the conical point $x^{(\tau)}$ (see Definition 6.4.1). More precisely, $L_{i,j}$ are δ-admissible operators of order s_i+t_j, $B_{k,j}$ are δ-admissible operators of order σ_k+t_j, and $C_{k,\nu}$ are δ-admissible operators of order $\mu_k+\tau_\nu$. Moreover, we assume that the boundary value problem (8.2.1), (8.2.2) is elliptic. Let l_1 and l_2 be integer numbers, $l_1, l_2 \ge \sigma_0$, and let β_τ, γ_τ be real numbers satisfying the inequalities

$$0 < (l_2-\gamma_\tau) - (l_1-\beta_\tau) < \delta.$$

We suppose that the lines $\mathrm{Re}\,\lambda = -\beta_\tau + l_1 - n/2$, $\mathrm{Re}\,\lambda = -\gamma_\tau + l_2 - n/2$ do not contain eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ and denote the eigenvalues in the strip $-\beta_\tau + l_1 - n/2 < \mathrm{Re}\,\lambda = -\gamma_\tau + l_2 - n/2$ by $\lambda_1, \dots, \lambda_M$. For every $\mu = 1, \dots, M$ let

$$\{(\underline{\Phi}^{(\mu)}_{j,s}, \underline{\varphi}^{(\mu)}_{j,s})\}_{j=1,\dots,I_\mu,\ s=0,1,\dots,\kappa_{\mu,j}-1}$$

be a canonical system of Jordan chains of the operator pencil $\mathfrak{A}_\tau(\lambda)$ corresponding to the eigenvalue λ_μ. We introduce the vector-functions

$$\begin{aligned} \underline{\mathfrak{u}}_{\mu,j,s} &= r^{\lambda_\mu} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma \big(r^{t_1} \Phi^{(\mu)}_{1;j,s-\sigma}(\omega), \dots, r^{t_N} \Phi^{(\mu)}_{N;j,s-\sigma}(\omega)\big), \\ \underline{u}_{\mu,j,s} &= r^{\lambda_\mu} \sum_{\sigma=0}^{s} \frac{1}{\sigma!} (\log r)^\sigma \big(r^{\tau_1} \varphi^{(\mu)}_{1;j,s-\sigma}(\omega), \dots, r^{\tau_N} \varphi^{(\mu)}_{N;j,s-\sigma}(\omega)\big), \end{aligned}$$

where $r = |x - x^{(\tau)}|$, ω are coordinates on the sphere $|x - x^{(\tau)}| = 1$, and $\Phi^{(\mu)}_{k;j,s}$, $\varphi^{(\mu)}_{k;j,s}$ denote the components of the vector-functions $\underline{\Phi}^{(\mu)}_{j,s}$ and $\underline{\varphi}^{(\mu)}_{j,s}$, respectively.

Analogously to Theorem 6.4.1, the following statement holds.

THEOREM 8.2.2. *Let* $(\underline{\mathfrak{u}}, \underline{u}) \in V^{l_1+\underline{t}}_{2,\beta}(\mathcal{G}) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ *be a solution of the boundary value problem (8.2.1), (8.2.2), where* $\underline{\mathfrak{f}} \in V^{l_2-\underline{s}}_{2,\gamma}(\mathcal{G})$, $\underline{g} \in V^{l_2-\underline{\sigma}-1/2}_{2,\gamma}(\partial\mathcal{G})$. *If all assumptions given before the formulation of this theorem are valid, then the solution* $(\underline{\mathfrak{u}}, \underline{u})$ *admits the decomposition*

$$(8.2.16) \qquad (\underline{\mathfrak{u}}, \underline{u}) = \sum_{\mu=1}^{M} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s} \big(\underline{\mathfrak{u}}_{\mu,j,s}, \underline{u}_{\mu,j,s}\big) + (\underline{\mathfrak{w}}, \underline{w})$$

in a neighbourhood of $x^{(\tau)}$, *where* $(\underline{\mathfrak{w}}, \underline{w}) \in V^{l_2+\underline{t}}_{2,\gamma}(\mathcal{G}) \times V^{l_2+\underline{\tau}-1/2}_{2,\gamma}(\partial\mathcal{G})$.

An analogous result holds for the generalized solution $\big(\mathfrak{u}, (\underline{\phi}^{(j)})_{1\le j\le N}, \underline{u}\big) \in \tilde{V}^{l_1+\underline{t},\sigma_0+\underline{t}}_{2,\beta}(\mathcal{G}) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta}(\partial\mathcal{G})$ of problem (8.2.1), (8.2.2) in the case $l_1 < \sigma_0$.

A formula for the coefficients in the asymptotics. We introduce the formally adjoint operator pencil $\mathfrak{A}^+_\tau(\lambda)$ to $\mathfrak{A}_\tau(\lambda)$. To this end, we write the leading parts of the differential operators $L^+_{i,j}(x,\partial_x)$, $P^{(j)}_{q,i}(x,\partial_x)$, $(Q^{(j)}_{k,q})^+(x,\partial_x)$, $(R^{i,j}_{s,q})^+(x,\partial_x)$ and $C^+_{k,\nu}(x,\partial_x)$ at the point $x^{(\tau)}$ (see Definition 6.2.1) in the form

$$\begin{aligned} (L^{(\tau)}_{i,j})^+(x,\partial_x) &= r^{-(s_i+t_j)}\, \mathcal{L}^+_{i,j}(\omega,\partial_\omega, r\partial_r), \\ P^{(j;\tau)}_{q,i}(x,\partial_x) &= r^{-(s_i+t_j-q)}\, \mathcal{P}^{(j)}_{q,i}(\omega,\partial_\omega, r\partial_r), \\ (Q^{(j;\tau)}_{k,q})^+(x,\partial_x)^+ &= r^{-(\sigma_k+t_j+1-q)}\, (\mathcal{Q}^{(j)}_{k,q})^+(\omega,\partial_\omega, r\partial_r), \\ (R^{(i,j;\tau)}_{s,q})^+(x,\partial_x) &= r^{-(s_i+t_j+q-s)}\, (\mathcal{R}^{i,j}_{s,q})^+(\omega,\partial_\omega, r\partial_r), \\ (C^{(\tau)}_{k,\nu})^+(x,\partial_x) &= r^{-(\sigma_k+\tau_\nu)}\, \mathcal{C}^+_{k,\nu}(\omega,\partial_\omega, r\partial_r)\,, \end{aligned}$$

$i,j=1,\dots,N$, $k=1,\dots,m+J$, $\nu=1,\dots,J$, $q=1,\dots,\sigma_0+t_j$, $s=1,\dots,\sigma_0-s_i$. Let $\mathfrak{A}_\tau^+(\lambda)$ be the operator of the boundary value problem

$$\sum_{i=1}^N \mathcal{L}_{i,j}^+(\omega,\partial_\omega,-\lambda+s_i-n)\,\Psi_i=\mathfrak{f}_j \quad \text{in } \Omega,\ j=1,\dots,N,$$

$$\sum_{i=1}^n \mathcal{P}_{q,i}^{(j)}(\omega,\partial_\omega,-\lambda+s_i-n)\Psi_i+\sum_{k=1}^{m+J}(\mathcal{Q}_{k,q}^{(j)})^+(\omega,\partial_\omega,-\lambda+\sigma_k+1-n)\,\psi_k$$
$$+\sum_{i=1}^N\sum_{s=1}^{\sigma_0-s_i}(\mathcal{R}_{s,q}^{i,j})^+(\omega,\partial_\omega,-\lambda+s_i+s-n)\,\chi_s^{(i)}=g_q^{(j)} \quad \text{on } \partial\Omega,$$
$$j=1,\dots,N,\ q=1,\dots,\sigma_0+t_j,$$

$$\sum_{k=1}^{m+J}\mathcal{C}_{k,\nu}^+(\omega,\partial_\omega,-\lambda+\sigma_k+1-n)\,\psi_k=h_\nu \quad \text{on } \partial\Omega,\ \nu=1,\dots,J.$$

If we replace the parameter λ in this problem by $\overline{\lambda}$, then we obtain the formally adjoint problem to (8.2.13), (8.2.14). Hence λ_μ is an eigenvalue of the pencil $\mathfrak{A}_\tau(\lambda)$ if and only if $\overline{\lambda}_\mu$ is an eigenvalue of the pencil $\mathfrak{A}_\tau^+(\lambda)$. Both eigenvalues have the same geometric and algebraic multiplicities. Let

$$\Big\{\big(\underline{\Psi}_{j,s}^{(\mu)},(\underline{\chi}_{j,s}^{(i,\mu)})_{1\le i\le N},\underline{\psi}_{j,s}^{(\mu)}\big)\Big\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

be a canonical system of Jordan chains of the operator pencil $\mathfrak{A}_\tau^+(\lambda)$ satisfying the biorthonormality condition

$$\sum_{p=0}^{\sigma}\sum_{q=p+1}^{p+s+1}\frac{1}{q!}\Big\langle\mathfrak{A}^{(q)}(\lambda_\mu)\,\big(\underline{\Phi}_{j,p+s+1-q}^{(\mu)},\underline{\varphi}_{j,p+s+1-q}^{(\mu)}\big),\big(\underline{\Psi}_{k,\sigma-p}^{(\mu)},(\underline{\chi}_{k,\sigma-p}^{(i,\mu)})_{1\le i\le N},\underline{\psi}_{k,\sigma-p}^{(\mu)}\big)\Big\rangle$$
$$=\delta_{j,k}\cdot\delta_{s,\kappa_{\mu,k}-1-\sigma}$$

for $j=1,\dots,I_\mu$, $s=0,\dots,\kappa_{\mu,j}-1$, $\sigma=0,\dots,\kappa_{\mu,k}-1$. Here $\mathfrak{A}^{(q)}(\lambda)$ denotes the q-th derivative (with respect to λ) of the mapping

$$(\underline{\Phi},\underline{\varphi})\to\big((\mathfrak{f}_i)_{1\le i\le N},(\mathcal{D}^{(\sigma_0-s_i)}\mathfrak{f}_i)_{1\le i\le N},(g_k)_{1\le k\le m+J}\big),$$

where $\mathfrak{f}_i$, g_k are given by (8.2.13), (8.2.14), and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L_2(\Omega_\tau)^N\times\prod_{i=1}^N L_2(\partial\Omega_\tau)^{\sigma_0-s_i}\times L_2(\partial\Omega_\tau)^{m+J}$. We set

$$\mathfrak{v}_{i;\mu,j,s} = -r^{-\overline{\lambda}_\mu+s_i-n}\sum_{\sigma=0}^{s}\frac{(-\log r)^\sigma}{\sigma!}\Psi_{i;j,s-\sigma}^{(\mu)}(\omega),$$

$$v_{k;\mu,j,s} = -r^{-\overline{\lambda}_\mu+\sigma_k+1-n}\sum_{\sigma=0}^{s}\frac{(-\log r)^\sigma}{\sigma!}\psi_{k;j,s-\sigma}^{(\mu)}(\omega),$$

$$w_{p;\mu,j,s}^{(i)} = -r^{-\overline{\lambda}_\mu+s_i+p-n}\sum_{\sigma=0}^{s}\frac{(-\log r)^\sigma}{\sigma!}\chi_{p;j,s}^{(i;\mu)}(\omega)$$

($i=1,\dots,N$, $k=1,\dots,m+J$, $p=1,\dots,\sigma_0-s_i$). These functions form special solutions of the homogeneous formally adjoint model problem in the cone $\mathcal{K}_\tau$.

THEOREM 8.2.3. *Suppose that the operators $L_{i,j}$, $B_{k,j}$, $C_{k,\nu}$ of the elliptic boundary value problem (8.2.1), (8.2.2) are $\delta-$admissible near the conical point $x^{(\tau)}$. Furthermore, we assume that there are no eigenvalues of $\mathfrak{A}_\tau(\lambda)$ on the lines*

$\operatorname{Re}\lambda = -\beta_\tau + l_1 - n/2$ *and* $\operatorname{Re}\lambda = -\gamma_\tau + l_2 - n/2$, *where* $0 < (\beta_\tau - \gamma_\tau) - (l_1 - l_2) < \delta$, $l_1 \geq \sigma_0$, $l_2 \geq \sigma_0$. *If* $(u, \underline{u})$ *is a solution of the boundary value problem (8.2.1), (8.2.2) such that*

$$\eta\,(u, \underline{u}) \in V^{l_1+\underline{t}}_{2,\beta_\tau}(\mathcal{K}_\tau) \times V^{l_1+\underline{\tau}-1/2}_{2,\beta_\tau}(\partial\mathcal{K}_\tau) \tag{8.2.17}$$

and

$$\eta\,(f, \underline{g}) \in V^{l_2-\underline{s}}_{2,\gamma_\tau}(\mathcal{K}_\tau) \times V^{l_2-\underline{\sigma}-1/2}_{2,\gamma_\tau}(\partial\mathcal{K}_\tau), \tag{8.2.18}$$

where η *is an infinitely differentiable function with support in* $\mathcal{U}_\tau$ *equal to one in a neighbourhood of* $x^{(\tau)}$, *then there is the representation (8.2.16) in this neighbourhood.*

Let $\big(\underline{\mathfrak{v}}, (\underline{w}^{(i)})_{1\leq i\leq N}, \underline{v}\big)$ *be a solution of the homogeneous formally adjoint problem (8.2.8)–(8.2.10) such that*

$$\begin{aligned}\big(\underline{\mathfrak{v}}, (\underline{w}^{(i)})_{1\leq i\leq N}, \underline{v}\big) \;&=\; \big(\underline{\mathfrak{v}}_{\mu,j,\kappa_{\mu,j}-1-s}\,, (\underline{w}^{(i)}_{\mu,j,\kappa_{\mu,j}-1-s})_{1\leq i\leq N}\,, \underline{v}_{\mu,j,\kappa_{\mu,j}-1-s}\big) \\ &\quad + \big(\underline{\mathfrak{V}}, (\underline{W}^{(i)})_{1\leq i\leq N}, \underline{V}\big),\end{aligned}$$

where

$$\begin{aligned}\eta\big(\underline{\mathfrak{V}}, (\underline{W}^{(i)})_{1\leq i\leq N}, \underline{V}\big) \in V^{t_0+\underline{s}}_{2,-\beta_\tau+l_1+t_0}(\mathcal{K}_\tau) \times \prod_{i=1}^{N} \prod_{\nu=1}^{\sigma_0-s_i} V^{t_0+s_i+\nu-1/2}_{2,-\beta_\tau+l_1+t_0}(\partial\mathcal{K}_\tau), \\ \times V^{t_0+\underline{\sigma}+1/2}_{2,-\beta_\tau+l_1+t_0}(\partial\mathcal{K}_\tau)\end{aligned}$$

and $t_0 = \max t_j$. *Then the constants* $c_{\mu,j,s}$ *in (8.2.16) are determined by the formula*

$$\begin{aligned}c_{\mu,j,s} = \int\limits_{\mathcal{G}} \big(L(\zeta\underline{u}), \underline{\mathfrak{v}}\big)_{\mathbb{C}^N}\, dx + \int\limits_{\partial\mathcal{G}} \sum_{i,j=1}^{N} \big(\mathcal{D}^{(\sigma_0-s_i)} L_{i,j}(\zeta u_j), \underline{w}^{(i)}\big)_{\mathbb{C}^{\sigma_0-s_i}}\, d\sigma \\ + \int\limits_{\mathcal{G}} \big(B(\zeta\underline{u}) + C(\zeta\underline{u})\,,\, \underline{v}\big)_{\mathbb{C}^{m+J}}\, d\sigma.\end{aligned} \tag{8.2.19}$$

Here ζ *is an arbitrary smooth cut-off function equal to unity in a neighbourhood of* $x^{(\tau)}$ *such that* $\zeta\eta = \zeta$.

8.3. Boundary value problems in the variational form

This section is concerned with elliptic boundary value problems in domains with conical points which are given in the variational form. First we study the relations between the solutions of variational problems and the generalized solutions of corresponding boundary value problems. Using these relations and the results of Chapter 6, we prove the Fredholm property of the operator of the variational problem and describe the asymptotics of the solutions near the conical points.

8.3.1. Formulation of the problem. As in the previous sections, $\mathcal{G}$ is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{G}$ which contains a finite set $\mathcal{S}$ of conical points. Outside the set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$ the boundary $\partial\mathcal{G}$ is assumed to be smooth. Let L be an admissible elliptic differential operator of order $2m$ on $\mathcal{G}$ given in the form

$$Lu = \sum_{|\alpha|\leq l} \sum_{|\beta|\leq 2m-l} D^\beta_x\, \big(a_{\alpha,\beta}(x)\, D^\alpha_x\, u\big)\ ,$$

where l is an arbitrary nonnegative integer not greater than $2m$. Furthermore, let

$$a(u,v) = \int\limits_{\Omega} \sum_{|\alpha|\le l} \sum_{|\beta|\le 2m-l} a_{\alpha,\beta}(x)\, D_x^\alpha u\, \overline{D_x^\beta v}\, dx \tag{8.3.1}$$

be the corresponding sesquilinear form.

Moreover, let B be a vector of admissible operators B_j, $j=1,\dots,J$ $(J \ge m-l)$, $\operatorname{ord} B_j = \sigma_j \le 2m-l-1$, M a vector of admissible operators M_s, $s=1,\dots,J+l-m$, $\operatorname{ord} M_s = \mu_s \le l-1$, P a vector of admissible operators P_k, and G a vector of admissible operators G_k, $k=1,\dots,N$, $\operatorname{ord} P_k \le l-1$, $\operatorname{ord} G_k \le 2m-l-1$.

We introduce the sesquilinear form

$$b(u,v) = a(u,v) + \int\limits_{\partial\mathcal{G}} \big(Pu, Gv\big)_{\mathbb{C}^N}\, d\sigma$$

on $V_{2,\beta}^l(\mathcal{G}) \times V_{2,-\beta}^{2m-l}(\mathcal{G})$ and define the space $\mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})$ as the set of all functions $v \in V_{2,-\beta}^{2m-l}(\mathcal{G})$ such that $Bv = 0$ on $\partial\mathcal{G}\backslash\mathcal{S}$. Analogously to Section 4.3, we consider the following problem:

Find a function $u \in V_{2,\beta}^l(\mathcal{G})$ which satisfies the equation

$$b(u,v) = (f,v)_{\mathcal{G}} \qquad \textit{for each } v \in \mathcal{V}_{-\beta}^{2m-l}(\mathcal{G}) \tag{8.3.2}$$

and the boundary condition $Mu = \underline{g}$ on $\partial\Omega$, i.e.,

$$M_s u = g_s \qquad \textit{on } \partial\Omega \quad \textit{for } s = 1,\dots,J+l-m. \tag{8.3.3}$$

Here f is a given linear and continuous functional on $\mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})$ and g_s are given functions from $V_{2,\beta}^{l-\mu_s-1/2}(\partial\mathcal{G})$. The variational problem (8.3.2), (8.3.3) generates a linear operator $\mathcal{B} : u \to (f, \underline{g})$ continuously mapping $V_{2,\beta}^l(\mathcal{G})$ into $\mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})^* \times V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})$.

An equivalent formulation of the variational problem. We assume that the following condition is satisfied:

(B) For every vector-function $\underline{g} \in V_{2,-\beta}^{2m-l-\underline{\sigma}-1/2}(\partial\mathcal{G})$ there exists a function $v \in V_{2,-\beta}^{2m-l}(\mathcal{G})$ such that $Bv = \underline{g}$ on $\partial\mathcal{G}\backslash\mathcal{S}$.

Then there exists a continuous right inverse $\Lambda : V_{2,-\beta}^{2m-l-\underline{\sigma}-1/2}(\partial\mathcal{G}) \to V_{2,-\beta}^{2m-l}(\mathcal{G})$ to the mapping $v \to Bv|_{\partial\mathcal{G}\backslash\mathcal{S}}$. Condition (B) is satisfied, e.g., if the operators B_j form a normal system on $\partial\mathcal{G}\backslash\mathcal{S}$ (see Definition 3.1.4).

Let Λ be the mentioned right inverse. Then every functional $f \in \mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})^*$ can be continuously extended to a functional $F \in V_{2,-\beta}^{2m-l}(\mathcal{G})^*$ as follows:

$$(F,v)_{\mathcal{G}} = \big(f\,,\, v - \Lambda(Bv|_{\partial\mathcal{G}\backslash\mathcal{S}})\big)_{\mathcal{G}}\,, \qquad v \in V_{2,-\beta}^{2m-l}(\mathcal{G}). \tag{8.3.4}$$

The disadvantage of the above given formulation of the variational problem consists in the use of functionals over the space $\mathcal{V}_B^{2m-l}(\mathcal{G})$. The following lemma enables us to give another (equivalent) formulation of this problem with functionals over the space $V_{2,-\beta}^{2m-l}(\mathcal{G})$ on the right-hand side.

LEMMA 8.3.1. *Let F be a linear and continuous functional on $V_{2,-\beta}^{2m-l}(\mathcal{G})$ satisfying the condition*

$$(F,v)_{\mathcal{G}} = 0 \qquad \textit{for each } v \in \mathcal{V}_{-\beta}^{2m-l}(\mathcal{G}).$$

Then there exists a vector-function $\underline{u} \in V_{2,\beta}^{l-2m+\underline{\sigma}+1/2}(\partial\Omega)$ *such that*

$$(F,v)_{\mathcal{G}} = \big(\underline{u}, Bv|_{\partial\mathcal{G}\setminus\mathcal{S}}\big)_{\partial\mathcal{G}} \qquad \text{for each } v \in V_{2,-\beta}^{2m-l}(\Omega). \tag{8.3.5}$$

The vector-function $\underline{u}$ *is uniquely determined by* F.

The proof of this lemma is the same as for Lemma 4.3.1.

THEOREM 8.3.1. *Suppose that Condition (B) is satisfied. Then the function* $u \in V_{2,\beta}^l(\mathcal{G})$ *is a solution of the variational problem (8.3.2), (8.3.3) if and only if there exists a vector-function* $\underline{u} \in V_{2,\beta}^{l-2m+\underline{\sigma}+1/2}(\partial\mathcal{G})$ *such that*

$$b(u,v) + \big(\underline{u}, Bv|_{\partial\mathcal{G}\setminus\mathcal{S}}\big)_{\partial\mathcal{G}} = (F,v)_{\mathcal{G}} \qquad \text{for each } v \in V_{2,-\beta}^{2m-l}(\mathcal{G}), \tag{8.3.6}$$

$$Mu = \underline{g} \qquad \text{on } \partial\mathcal{G}, \tag{8.3.7}$$

where $F \in V_{2,-\beta}^{2m-l}(\Omega)^*$ *is an extension of the functional* $f \in \mathcal{V}_{-\beta}^{2m-l}(\Omega)^*$ *to the whole space* $V_{2,-\beta}^{2m-l}(\mathcal{G})$. *The vector-function* $\underline{u}$ *depends on the choice of the extension* F *and satisfies the estimate*

$$\sum_{j=1}^{J} \|u_j\|_{V_{2,\beta}^{l-2m+\sigma_j+1/2}(\partial\mathcal{G})} \le c\Big(\|u\|_{V_{2,\beta}^l(\mathcal{G})} + \|F\|_{V_{2,-\beta}^{2m-l}(\mathcal{G})^*}\Big). \tag{8.3.8}$$

Proof: It is evident that (8.3.6), (8.3.7) imply (8.3.2), (8.3.3). Suppose u is a solution of problem (8.3.2), (8.3.3) and $F \in V_{2,-\beta}^{2m-l}(\mathcal{G})^*$ is a functional which coincides with f on $\mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})$. Then

$$(F,v)_{\mathcal{G}} - b(u,v) = 0 \quad \text{for all } v \in \mathcal{V}_{-\beta}^{2m-l}(\mathcal{G}).$$

Hence by Lemma 8.3.1, there exists a vector $\underline{u}$ such that

$$(F,v)_{\mathcal{G}} - b(u,v) = \big(\underline{u},\ Bv|_{\partial\mathcal{G}\setminus\mathcal{S}}\big)_{\partial\mathcal{G}}$$

for each $v \in V_{2,-\beta}^{2m-l}(\mathcal{G})$. Therefore, $(u,\underline{u})$ is a solution of the problem (8.3.6), (8.3.7).

We prove (8.3.8). Inserting $v = \Lambda\underline{w}$ into (8.3.6), where $\underline{w}$ is an arbitrary vector-function in $V_{2,-\beta}^{2m-l-\underline{\sigma}-1/2}(\partial\mathcal{G})$ and Λ is a continuous right inverse to the operator $v \to Bv|_{\partial\mathcal{G}\setminus\mathcal{S}}$, we get

$$(\underline{u},\underline{w})_{\partial\mathcal{G}} = (F,\Lambda\underline{w})_{\mathcal{G}} - b(u,\Lambda\underline{w}).$$

This implies (8.3.8). ■

8.3.2. The boundary value problem corresponding to the variational problem. We suppose that the solution $(u,\underline{u})$ of the variational problem (8.3.6), (8.3.7) belongs to the subspace $V_{2,\beta+(2m-l)\vec{1}}^{2m}(\mathcal{G}) \times V_{2,\beta+(2m-l)\vec{1}}^{\underline{\sigma}+1/2}(\partial\mathcal{G})$ of $V_{2,\beta}^l(\mathcal{G}) \times V_{2,\beta}^{l-2m+\underline{\sigma}+1/2}(\partial\mathcal{G})$. Then we have

$$a(u,v) = \int_{\mathcal{G}} Lu\cdot\overline{v}\,dx + \int_{\partial\mathcal{G}} \big(Su, \mathcal{D}^{(2m-l)}v\big)_{\mathbb{C}^{2m-l}}\,d\sigma \tag{8.3.9}$$

for arbitrary $v \in V_{2,-\beta}^{2m-l}(\mathcal{G})$, where $\mathcal{D}^{(2m-l)}$ is the vector with the components $1, D_\nu, \dots, D_\nu^{2m-l-1}$ and S is a vector of admissible differential operators S_k, $k = 1,\dots,2m-l$, $\operatorname{ord} S_k \le 2m-k$. The vector Su admits the representation

$$Su|_{\partial\mathcal{G}\setminus\mathcal{S}} = Q\cdot\mathcal{D}^{(2m)}u|_{\partial\mathcal{G}\setminus\mathcal{S}} \tag{8.3.10}$$

(cf. Lemma 4.3.2), where

$$Q = \begin{pmatrix} Q_{1,1} & \cdots & Q_{1,l+1} & Q_{1,l+2} & \cdots & Q_{1,2m-1} & Q_{1,2m} \\ Q_{2,1} & \cdots & Q_{2,l+1} & Q_{2,l+2} & \cdots & Q_{2,2m-1} & 0 \\ \\ Q_{2m-l,1} & \cdots & Q_{2m-l,l+1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is a trapezium matrix of admissible tangential differential operators $Q_{k,j}$ of order $2m+1-k-j$ on $\partial\mathcal{G}\backslash\mathcal{S}$, $Q_{k,j} \equiv 0$ for $k+j > 2m+1$. If L is admissible and elliptic in $\overline{\mathcal{G}}\backslash\mathcal{S}$, then the matrix elements $Q_{1,2m}, Q_{2,2m-1}, \dots , Q_{2m-l,l+1}$ are functions which do not vanish on $\partial\mathcal{G}$.

Furthermore, there exist matrices

$$C = (C_{j,k})_{1\le j\le J,\ 1\le k\le 2m-l} \qquad \text{and} \qquad H = (H_{j,k})_{1\le j\le N,\ 1\le k\le 2m-l}$$

of admissible tangential differential operators $C_{j,k}$, $\operatorname{ord} C_{j,k} \le \sigma_j - k + 1$ ($C_{j,k} \equiv 0$ if $k > \sigma_j + 1$), and admissible tangential differential operators $H_{j,k}$, $\operatorname{ord} H_{j,k} \le 2m - l - k$, on $\partial\mathcal{G}\backslash\mathcal{S}$, such that

$$Bv|_{\partial\mathcal{G}\backslash\mathcal{S}} = C \cdot \mathcal{D}^{(2m-l)}v\,|_{\partial\mathcal{G}\backslash\mathcal{S}} \tag{8.3.11}$$

and

$$Gv|_{\partial\mathcal{G}\backslash\mathcal{S}} = H \cdot \mathcal{D}^{(2m-l)}v\,|_{\partial\mathcal{G}\backslash\mathcal{S}}\,. \tag{8.3.12}$$

Hence by means of (8.3.9), we get

$$\begin{aligned} & b(u,v) + \int\limits_{\partial\mathcal{G}} (\underline{u}, Bv)_{\mathbb{C}^J}\, d\sigma \\ & = \int\limits_{\mathcal{G}} Lu \cdot \overline{v}\, dx + \int\limits_{\partial\mathcal{G}} \big((S + H^+P)u + C^+\underline{u}\,,\, \mathcal{D}^{(2m-l)}v\big)_{\mathbb{C}^{2m-l}}\, d\sigma \end{aligned} \tag{8.3.13}$$

for all $v \in V^{2m-l}_{2,-\beta}(\mathcal{G})$. Consequently, the functional F on the right side of (8.3.6) is an element of the space $D^{0,2m-l}_{2,\beta+(2m-l)\vec{1}}(\mathcal{G})$ which was introduced Section 6.3, i.e., F has the form

$$(F,v)_{\mathcal{G}} = (f,v)_{\mathcal{G}} + \big(\underline{h}\,,\, \mathcal{D}^{(2m-l)}v|_{\partial\mathcal{G}\backslash\mathcal{S}}\big)_{\partial\mathcal{G}}\,, \qquad v \in V^{2m-l}_{2,-\beta}(\mathcal{G}), \tag{8.3.14}$$

where $f \in V^0_{2,\beta+(2m-l)\vec{1}}(\mathcal{G})$ and $\underline{h} = (h_1, \dots , h_{2m-l})$ is a vector of functions $h_k \in V^{k-1/2}_{2,\beta+(2m-l)\vec{1}}(\partial\mathcal{G})$. Moreover, the following assertion holds.

LEMMA 8.3.2. *Suppose that the differential operators B_k satisfy Condition (B) and $(u, \underline{u}) \in V^{2m}_{2,\beta+(2m-l)\vec{1}}(\mathcal{G}) \times V^{\underline{\sigma}+1/2}_{2,\beta+(2m-l)\vec{1}}(\partial\mathcal{G})$ is a solution of the variational problem (8.3.6), (8.3.7), where $g_s \in V^{2m-\mu_s-1/2}_{2,\beta+(2m-l)\vec{1}}(\partial\mathcal{G})$ for $s = 1, \dots , J+l-m$ and F is a functional of the form (8.3.14) with $f \in V^0_{2,\beta+(2m-l)\vec{1}}(\mathcal{G})$, $h_k \in V^{k-1/2}_{2,\beta+(2m-l)\vec{1}}(\partial\mathcal{G})$. Then $(u, \underline{u})$ is a solution of the boundary value problem*

$$Lu = f \quad \text{in } \mathcal{G}, \tag{8.3.15}$$

$$Mu = \underline{g}\,, \qquad (S + H^+P)u + C^+\underline{u} = \underline{h} \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}. \tag{8.3.16}$$

The operator $\mathcal{A}$ of the boundary value problem (8.3.15), (8.3.16) continuously maps the space $\tilde{V}^{2m,2m}_{2,\beta+(2m-l)\vec{1}}(\mathcal{G}) \times V^{\underline{\sigma}+1/2}_{2,\beta+(2m-l)\vec{1}}(\partial\mathcal{G})$ into the space

$$V^{0}_{2,\beta+(2m-l)\vec{1}}(\mathcal{G}) \times \prod_{s=1}^{J+l-m} V^{2m-\mu_s-1/2}_{2,\beta+(2m-l)\vec{1}}(\partial\mathcal{G}) \times \prod_{k=1}^{2m-l} V^{k-1/2}_{2,\beta+(2m-l)\vec{1}}(\partial\mathcal{G})$$

and can be extended to a continuously mapping from

$$\tilde{V}^{l,2m}_{2,\beta}(\mathcal{G}) \times V^{l-2m+\underline{\sigma}+1/2}_{2,\beta}(\partial\mathcal{G}) \tag{8.3.17}$$

into the space

$$V^{2m-l}_{2,-\beta}(\mathcal{G})^* \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}) \times \prod_{k=1}^{2m-l} V^{l-2m+k-1/2}_{2,\beta}(\partial\mathcal{G}). \tag{8.3.18}$$

Let $(u, \underline{\phi}, \underline{u})$, $(f, \underline{g}, \underline{h})$ be arbitrary elements of the spaces (8.3.17) and (8.3.18), respectively. Then the equation $\mathcal{A}\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g}, \underline{h})$ means (cf. (4.3.25), (4.3.26)):

$$\begin{aligned} (f, v)_{\mathcal{G}} &= a(u, v) - \big(Q\underline{\phi}, \mathcal{D}^{(2m-l)}v\big)_{\partial\mathcal{G}} \quad \text{for all } v \in V^{2m-l}_{2,-\beta}(\mathcal{G}), \\ \underline{g} &= Mu|_{\partial\mathcal{G}\setminus\mathcal{S}}\,, \\ \underline{h} &= Q\underline{\phi} + H^+ Pu|_{\partial\mathcal{G}\setminus\mathcal{S}} + C^+\underline{u}. \end{aligned}$$

Thus, we obtain the following relations between the solution

$$(u, \underline{u}) \in V^{l}_{2,\beta}(\mathcal{G}) \times V^{l-2m+\underline{\sigma}+1/2}_{2,\beta}(\partial\mathcal{G})$$

of the variational problem (8.3.6), (8.3.7) and the generalized solution

$$(u, \underline{\phi}, \underline{u}) \in \tilde{V}^{l}_{2,\beta}(\mathcal{G}) \times V^{l-2m+\underline{\sigma}+1/2}_{2,\beta}(\partial\mathcal{G})$$

of the boundary value problem (8.3.15), (8.3.16) (cf. Lemma 4.3.4).

LEMMA 8.3.3. *Suppose that Condition (B) is satisfied and $(u, \underline{\phi}, \underline{u})$, $(f, \underline{g}, \underline{h})$ are elements of the spaces (8.3.17) and (8.3.18), respectively. Furthermore, let the functional $F \in V^{2m-l}_{2,-\beta}(\mathcal{G})^*$ be defined as*

$$(F, v)_{\mathcal{G}} = (f, v)_{\mathcal{G}} + \big(\underline{h}, \mathcal{D}^{(2m-l)}v|_{\partial\mathcal{G}\setminus\mathcal{S}}\big)_{\partial\mathcal{G}}\,, \qquad v \in V^{2m-l}_{2,-\beta}(\mathcal{G})^*. \tag{8.3.19}$$

Then $(u, \underline{\phi}, \underline{u})$ is a solution of the equation

$$\mathcal{A}\,(u, \underline{\phi}, \underline{u}) = (f, \underline{g}, \underline{h})$$

if and only if $(u, \underline{u})$ is a solution of the variational problem (8.3.6), (8.3.7) and $\underline{\phi} = (\phi_1, \dots, \phi_{2m})$ satisfies the equations

$$\phi_j = D_\nu^{j-1}u|_{\partial\mathcal{G}\setminus\mathcal{S}} \ \ \text{for } j = 1, \dots, l, \quad Q\underline{\phi} + H^+ Pu|_{\partial\mathcal{G}\setminus\mathcal{S}} + C^+\underline{u} = \underline{h}. \tag{8.3.20}$$

Note that the system of the equations (8.3.20) has a unique solution $\underline{\phi}$ for arbitrary given u, $\underline{u}$, and $\underline{h}$ if the operator L is elliptic.

8.3.3. Solvability of the variational problem. Let $\mathfrak{A}_\tau(\lambda)$ denote the operator pencils of the parameter-depending boundary value problems in Ω_τ which are generated by the boundary value (8.3.15), (8.3.16), $\tau = 1, \dots, d$ (see Section 6.3). Using the relations between the variational problem (8.3.2), (8.3.3) and the corresponding boundary value problem (see Theorem 8.3.1 and Lemma 8.3.3), we can show that the operator

$$\mathcal{B} \,:\, V^l_{2,\beta}(\mathcal{G}) \to \mathcal{V}^{2m-l}_{-\beta}(\mathcal{G})^* \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{8.3.21}$$

of problem (8.3.2), (8.3.3), the operator

$$\mathcal{B}' \,:\, V^l_{2,\beta}(\mathcal{G}) \times V^{l-2m+\underline{\sigma}+1/2}_{2,\beta}(\partial\mathcal{G}) \to V^{2m-l}_{2,-\beta}(\mathcal{G})^* \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}) \tag{8.3.22}$$

of problem (8.3.6), (8.3.7), and the operator

$$\begin{aligned}\mathcal{A} : \tilde{V}^l_{2,\beta}(\mathcal{G}) &\times V^{l-2m+\underline{\sigma}+1/2}_{2,\beta}(\partial\mathcal{G}) \\ &\to \mathcal{V}^{2m-l}_{-\beta}(\mathcal{G})^* \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\mathcal{G}) \times \prod_{k=1}^{2m-l} V^{l-2m+k-1/2}_{2,\beta}(\partial\mathcal{G})\end{aligned}$$

of the boundary value problem (8.3.15), (8.3.16) are simultaneously Fredholm. Furthermore, analogously to Lemma 4.3.6, it can be shown that

$$(u, \underline{v}, \underline{w}) \in V^{2m}_{2,-\beta+l\vec{1}}(\mathcal{G}) \times V^{\underline{\mu}+1/2}_{2,-\beta+l\vec{1}}(\partial\mathcal{G}) \times \prod_{k=1}^{2m-l} V^{2m-k+1/2}_{2,-\beta+l\vec{1}}(\partial\mathcal{G})$$

is a solution of the homogeneous formally adjoint boundary value problem to (8.3.15), (8.3.16) if and only if $(v, \underline{v})$ is a solution of the problem

$$b(u,v) + \big(Mu|_{\partial\mathcal{G}\setminus\mathcal{S}}, \underline{v}\big)_{\partial\mathcal{G}} = 0 \quad \text{for all } u \in V^l_{2,\beta}(\mathcal{G}), \tag{8.3.23}$$

$$Bv = 0 \quad \text{on } \partial\mathcal{G}\setminus\mathcal{S} \tag{8.3.24}$$

and the vector-function $\underline{w}$ coincides with $\mathcal{D}^{(2m-l)}v|_{\partial\mathcal{G}\setminus\mathcal{S}}$.

Together with Theorem 6.3.3 this leads to the following statement (cf. Theorem 4.3.2).

THEOREM 8.3.2. *Suppose that the boundary value (8.3.15), (8.3.16) is elliptic, Condition (B) is satisfied, and the line* $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ *does not contain eigenvalues of the pencil* $\mathfrak{A}_\tau(\lambda)$ *for* $\tau = 1, \dots, d$. *Then the operators (8.3.21) and (8.3.22) are Fredholm. The kernel of the operator* $\mathcal{B}'$ *is the set of all solutions*

$$(u, \underline{u}) \in \bigcap_j \Big(V^{l+j}_{2,\beta+j\vec{1}}(\mathcal{G}) \times V^{l-2m+j+\underline{\sigma}+1/2}_{2,\beta+j\vec{1}}(\partial\mathcal{G})\Big)$$

of the homogeneous boundary value problem (8.3.15), (8.3.16), while the kernel of the operator $\mathcal{B}$ *consists of all*

$$u \in \bigcap_j V^{l+j}_{2,\beta+j\vec{1}}(\mathcal{G})$$

for which there exists a vector-function

$$\underline{u} \in \bigcap_j V^{l-2m+j+\underline{\sigma}+1/2}_{2,\beta+j\vec{1}}(\partial\mathcal{G})$$

such that $(u, \underline{u})$ *is a solution of the homogeneous boundary value problem (8.3.15), (8.3.16). The range of the operator* $\mathcal{B}'$ *is the set of all* $(F, \underline{g}) \in V_{2,-\beta}^{2m-l}(\mathcal{G})^* \times V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})$ *satisfying the condition*

$$(F, v)_{\mathcal{G}} + (\underline{g}, \underline{v})_{\partial\mathcal{G}} = 0$$

for all solutions $(v, \underline{v}) \in V_{2,-\beta+l\vec{1}}^{2m}(\mathcal{G}) \times V_{2,-\beta+l\vec{1}}^{\underline{\mu}+1/2}(\partial\mathcal{G})$ *of problem (8.3.23), (8.3.24), and the range of the operator* $\mathcal{B}$ *is the set of all* $(f, \underline{g}) \in \mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})^* \times V_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{G})$ *satisfying the condition*

$$(f, v)_{\mathcal{G}} + (\underline{g}, \underline{v})_{\partial\mathcal{G}} = 0$$

for all solutions $(v, \underline{v}) \in V_{2,-\beta+l\vec{1}}^{2m}(\mathcal{G}) \times V_{2,-\beta+l\vec{1}}^{\underline{\mu}+1/2}(\partial\mathcal{G})$ *of problem (8.3.23), (8.3.24).*

8.3.4. Asymptotics of the solution. Now let $u \in V_{2,\beta}^{l}(\mathcal{G})$ be a solution of the variational problem (8.3.2), (8.3.3), where f coincides with a functional $F \in D_{2,\gamma}^{l_1-2m,2m-l}(\mathcal{G})$ on $\mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})$, $l_1 \ge l$, $\gamma = (\gamma_1, \dots, \gamma_d)$, $l_1 - \gamma_\tau \ge l - \beta_\tau$ for $\tau = 1, \dots, d$. Then there exist a function $\Phi \in \tilde{V}_{2,\gamma}^{l_1-2m,0}(\mathcal{G})$ and a vector-function

$$\underline{h} \in \prod_{k=1}^{2m-l} V_{2,\gamma}^{l_1-2m+k-1/2}(\partial\mathcal{G})$$

such that

$$(f, v)_{\mathcal{G}} = (\Phi, v)_{\mathcal{G}} + \big(\underline{h}, \mathcal{D}^{(2m-l)} v\big)_{\partial\mathcal{G}} \quad \text{for } v \in \mathcal{V}_{-\beta}^{2m-l}(\mathcal{G}). \tag{8.3.25}$$

We denote the eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ in the strip $-\beta_\tau + l - n/2 < \operatorname{Re}\lambda < -\gamma_\tau + l_1 - n/2$ by $\lambda_1, \dots, \lambda_M$. Furthermore, we assume that

$$\Big\{(\varphi_{j,s}^{(\mu)}, \underline{\varphi}_{j,s}^{(\mu)})\Big\}_{j=1,\dots,I_\mu,\ s=0,\dots,\kappa_{\mu,j}-1}$$

is a canonical system of Jordan chains of the operator pencil $\mathfrak{A}_\tau(\lambda)$ corresponding to the eigenvalue λ_μ, $\mu = 1, \dots, M$.

THEOREM 8.3.3. *Suppose that the boundary value problem (8.3.15), (8.3.16) is elliptic and Condition (B) is satisfied. Furthermore, we assume that f coincides with a functional* $F \in D_{2,\gamma}^{l_1-2m,2m-l}(\mathcal{G})$ *on* $\mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})$ *and the vector-function* $\underline{g}$ *belongs to* $V_{2,\gamma}^{l_1-\underline{\sigma}-1/2}(\partial\mathcal{G})$, *where* $l_1 \ge l$, $\gamma = (\gamma_1, \dots, \gamma_d)$, $0 < (l_1 - \gamma_\tau) - (l - \beta_\tau) < \delta$. *If the lines* $\operatorname{Re}\lambda = -\beta_\tau + l - n/2$ *and* $\operatorname{Re}\lambda = -\gamma_\tau + l_1 - n/2$ *do not contain eigenvalues of the pencil* $\mathfrak{A}_\tau(\lambda)$, *then every solution* $u \in V_{2,\beta}^{l}(\mathcal{G})$ *of problem (8.3.2), (8.3.3) admits the decomposition*

$$u = \sum_{\mu=1}^{M} \sum_{j=1}^{I_\mu} \sum_{s=0}^{\kappa_{\mu,j}-1} c_{\mu,j,s}\, r^{\lambda_\mu} \sum_{k=0}^{s} \frac{1}{k!} (\log r)^k \varphi_{j,s-k}^{(\mu)}(\omega) + w \tag{8.3.26}$$

near the conical point $x^{(\tau)}$, *where* $w \in V_{2,\gamma}^{l_1}(\mathcal{G})$.

Proof: Let the functional f have the representation (8.3.25), where Φ belongs to the space $\tilde{V}_{2,\gamma}^{l_1-2m,0}(\mathcal{G})$ and $\underline{h} \in \prod V_{2,\gamma}^{l_1-2m+k-1/2}(\partial\mathcal{G})$. By Theorem 8.3.1, there exists a unique vector-function $\underline{u} \in V_{2,\beta}^{l-2m+\underline{\sigma}+1/2}(\partial\mathcal{G})$ such that $(u, \underline{u})$ is a solution of the problem

$$b(u,v) + (\underline{u}, Bv)_{\partial\mathcal{G}} = (\Phi, v)_{\mathcal{G}} + \big(\underline{h}, \mathcal{D}^{(2m-l)}v\big)_{\partial\mathcal{G}} \quad \text{for } v \in V_{2,-\beta}^{2m-l}(\mathcal{G}),$$
$$Mu = 0 \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S}.$$

Furthermore, according to Lemma 8.3.3, there exists a uniquely determined vector $\underline{\phi} = (\phi_1, \dots, \phi_{2m})$, $\phi_j \in V_{2,\beta}^{l-j+1/2}(\partial\mathcal{G})$, such that $(u, \underline{\phi}) \in \tilde{V}_{2,\beta}^{l,2m}(\mathcal{G})$ and $(u, \underline{\phi}, \underline{u})$ is a solution of the equation $\mathcal{A}\,(u, \underline{\phi}, \underline{u}) = (\Phi, \underline{g}, \underline{h})$, where

$$(\Phi, \underline{g}, \underline{h}) \in \tilde{V}_{2,\gamma}^{l_1-2m,0}(\mathcal{G}) \times V_{2,\gamma}^{l_1-\underline{\sigma}-1/2}(\partial\mathcal{G}) \times \prod_{k=1}^{m+J} V_{2,\gamma}^{l_1-2m+k-1/2}(\partial\mathcal{G}).$$

Applying Theorem 6.4.1, we obtain (8.3.26). ∎

As a consequence of the previous theorem, the following regularity assertion for the solutions of problem (8.3.2), (8.3.3) holds.

COROLLARY 8.3.1. *Let the assumptions of Theorem 8.3.3 be valid. Moreover, we assume that the strip* $-\beta_\tau + l - n/2 \le \operatorname{Re}\lambda \le -\gamma_\tau + l_1 - n/2$ *does not contain eigenvalues of the pencil* $\mathfrak{A}_\tau(\lambda)$ *for* $\tau = 1, \dots, d$. *Then every solution* $u \in V_{2,\beta}^l(\mathcal{G})$ *of problem (8.3.2), (8.3.3) belongs to the space* $V_{2,\gamma}^{l_1}(\mathcal{G})$.

Using Lemma 6.3.1, Theorem 8.3.1 and Lemma 8.3.3, we obtain the following regularity assertion which requires only the ellipticity of the boundary value problem (8.3.15), (8.3.16).

THEOREM 8.3.4. *Suppose that the operators* L, B_j, M_s, P_k, *and* G_k *are admissible, the boundary value problem (8.3.15), (8.3.16) is elliptic, and Condition (B) is satisfied. If* $\underline{g} \in V_{2,\beta+(l_1-l)\vec{1}}^{l_1-\underline{\sigma}-1/2}(\partial\mathcal{G})$ *and* f *coincides with a functional* $F \in D_{2,\beta+(l_1-l)\vec{1}}^{l_1-2m,2m-l}(\mathcal{G})$ *on* $\mathcal{V}_{-\beta}^{2m-l}(\mathcal{G})$, *then every solution* $u \in V_{2,\beta}^l(\mathcal{G})$ *of problem (8.3.2), (8.3.3) belongs to the space* $V_{2,\beta+(l_1-l)\vec{1}}^{l_1}(\mathcal{G})$.

Moreover, if $l_1 \ge 2m$, *then there exists a vector-function* $\underline{u} \in V_{2,\beta+(l_1-l)\vec{1}}^{l_1-2m+\underline{\sigma}+1/2}(\partial\mathcal{G})$ *such that* $(u, \underline{u})$ *is a solution of the boundary value problem (8.3.15), (8.3.16).*

8.4. Further results

8.4.1. Estimates in L_p and Hölder spaces. The main results of Chapter 6 can be extended to weighted L_p Sobolev spaces $V_{p,\beta}^l$ and weighted Hölder classes $N_\beta^{l,\alpha}$.

As in Chapter 6, we suppose that $\mathcal{G}$ is a bounded domain with boundary $\partial\mathcal{G}$ which is smooth outside the set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$ of conical points. By ζ_τ, $\tau = 1, \dots, d$, we denote smooth cut-off functions equal to one near $x^{(\tau)}$ with sufficiently small supports. Furthermore, we set $\zeta_0 = 1 - \zeta_1 - \dots - \zeta_d$. Let l be a nonnegative integer, p a real number greater than one, and $\beta = (\beta_1, \dots, \beta_d)$ a real d-tuple. Then $V_{p,\beta}^l(\mathcal{G})$ is the closure of the set $C_0^\infty(\overline{\mathcal{G}}\backslash\mathcal{S})$ with respect to the norm

$$\|u\|_{V_{p,\beta}^l(\mathcal{G})} = \|\zeta_0 u\|_{W_p^l(\mathcal{G})} + \sum_{\tau=1}^{d} \sum_{|\alpha|\le l} \|\zeta_\tau r^{\beta_\tau - l + |\alpha|}\, D_x^\alpha u\|_{L_p(\mathcal{G})}\,.$$

Here r denotes the distance to the set $\mathcal{S}$. The weighted Hölder space $N_\beta^{l,\alpha}(\mathcal{G})$, where $0 < \alpha < 1$, is equipped with the norm

$$\begin{aligned} \|u\|_{N_\beta^{l,\alpha}(\mathcal{G})} &= \|\zeta_0 u\|_{C^{l,\alpha}(\mathcal{G})} + \sum_{\tau=1}^{d} \sum_{|\gamma|\le l} \sup_{x\in\mathcal{G}\cap supp\,\zeta_\tau} r(x)^{\beta_\tau - l-\alpha+|\gamma|} \, |D_x^\gamma u(x)| \\ &+ \sum_{\tau=1}^{d} \sum_{|\gamma|=l} \sup_{x,y\in\mathcal{G}\cap supp\,\zeta_\tau} |x-y|^{-\alpha} \left| r(x)^{\beta_\tau} D_x^\gamma u(x) - r(y)^{\beta_\tau} D_y^\gamma u(y) \right|, \end{aligned}$$

where

$$\|u\|_{C^{l,\alpha}(\mathcal{G})} = \sum_{|\gamma|\le l} \sup_{x\in\mathcal{G}} |D_x^\gamma u(x)| + \sum_{|\gamma|=l} \sup_{x,y\in\mathcal{G}} |x-y|^{-\alpha} \left| D_x^\gamma u(x) - D_y^\gamma u(y) \right|.$$

Moreover, $V_{p,\beta}^{l-1/p}(\partial\mathcal{G})$, $l \ge 1$, and $N_\beta^{l,\alpha}(\partial\mathcal{G})$ denote the spaces of traces on $\partial\mathcal{G}$ of functions from $V_{p,\beta}^{l}(\mathcal{G})$ and $N_\beta^{l,\alpha}(\mathcal{G})$, respectively.

We consider the boundary value problem

$$(8.4.1) \qquad Lu = f \quad \text{in } \mathcal{G}, \qquad Bu + C\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}$$

with the same assumptions on L, B, and C as in Section 6.2. The operator of problem (8.4.1) realizes continuous mappings

$$(8.4.2) \quad \mathcal{A}_{p,l,\beta} \,:\, V_{p,\beta}^{l}(\mathcal{G}) \times \prod_{j=1}^{J} V_{p,\beta}^{l+\tau_j-1/p}(\partial\mathcal{G}) \to V_{p,\beta}^{l-2m}(\mathcal{G}) \times \prod_{k=1}^{m+J} V_{p,\beta}^{l-\mu_k-1/p}(\partial\mathcal{G})$$

and

(8.4.3)

$$\mathcal{A}_{l,\beta}^{\alpha} \,:\, N_\beta^{l,\alpha}(\mathcal{G}) \times \prod_{j=1}^{J} N_\beta^{l+\tau_j-1/p,\alpha}(\partial\mathcal{G}) \to N_{p,\beta}^{l-2m,\alpha}(\mathcal{G}) \times \prod_{k=1}^{m+J} N_\beta^{l-\mu_k-1/p,\alpha}(\partial\mathcal{G}),$$

where $l \ge 2m$, $1 < p < \infty$, $0 < \alpha < 1$, $\beta \in \mathbb{R}^d$. The following theorems were proved (for boundary value problems without unknowns on the boundary) by V. G. Maz'ya and B. A. Plamenevskiĭ [**147**] (see also [**148**], [**153**]).

THEOREM 8.4.1. *Suppose that the boundary value problem (8.4.1) is elliptic and the operators L, B_k, $C_{k,j}$ are admissible. If there are no eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ on the line $\mathrm{Re}\,\lambda = -\beta_\tau + l - n/p$ for $\tau = 1, \dots, d$, then the operator (8.4.2) is Fredholm. If the line $\mathrm{Re}\,\lambda = -\beta_\tau + l + \alpha$ is free of eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ for $\tau = 1, \dots, d$, then the operator (8.4.3) is Fredholm. Furthermore, a priori estimates analogous to (6.3.13) are valid for the solutions $(u, \underline{u}) \in V_{p,\beta}^{l}(\mathcal{G}) \times V_{p,\beta}^{l+\underline{\tau}-1/p}(\partial\mathcal{G})$ and $(u, \underline{u}) \in N_\beta^{l,\alpha}(\mathcal{G}) \times N_\beta^{l+\underline{\tau}-1/p,\alpha}(\partial\mathcal{G})$.*

THEOREM 8.4.2. *Suppose that the boundary value problem (8.4.1) is elliptic and the operators L, B_k, $C_{k,j}$ are admissible. Furthermore, we assume that $l \ge 2m$, $\beta_\tau > \gamma_\tau$, and there are no eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ on the lines $\mathrm{Re}\,\lambda = -\beta_\tau + l - n/p$ and $\mathrm{Re}\,\lambda = -\gamma_\tau + l - n/p$ for a certain $\tau = 1, \dots, d$. Then every solution $(u, \underline{u}) \in V_{p,\beta}^{l}(\mathcal{G}) \times V_{p,\beta}^{l+\underline{\tau}-1/p}(\partial\mathcal{G})$ of problem (8.4.1) with*

$$f \in V_{p,\beta}^{l-2m}(\mathcal{G}) \cap V_{p,\gamma}^{l-2m}(\mathcal{G}), \quad \underline{g} \in V_{p,\beta}^{l-\underline{\mu}-1/p}(\partial\mathcal{G}) \cap V_{p,\gamma}^{l-\underline{\mu}-1/p}(\partial\mathcal{G})$$

admits the representation

$$(u,\underline{u}) = \sum_{j=1}^{\kappa} c_j\, U_j + (w,\underline{w})$$

in a neighbourhood of the conical point $x^{(\tau)}$*, where* $(w,\underline{w})$ *is an element of the space*

$$V^l_{p,\gamma}(\mathcal{G}) \times V^{l'+\underline{\tau}-1/2}_{p,\beta'}(\partial\mathcal{G}), \tag{8.4.4}$$

c_j *are constants,* κ *is the sum of the algebraic multiplicities of all eigenvectors of* $\mathfrak{A}_\tau(\lambda)$ *lying in the strip* $-\beta_\tau + l - n/p < \mathrm{Re}\lambda < -\gamma_\tau + l - n/p$*, and* $U_j \in V^l_{p,\beta}(\mathcal{G}) \times V^{l+\underline{\tau}-1/2}_{p,\beta}(\partial\mathcal{G})$ *satisfy the homogeneous equations (8.4.1) in a neighbourhood of* $x^{(\tau)}$*, are zero outside a neighbourhood of* $x^{(\tau)}$ *and linearly independent modulo (8.4.4).*

An analogous statement holds in the class of the spaces $N^{l,\alpha}_\beta$*.*

Moreover, under additional assumptions on the operators L, B_k, and $C_{k,j}$ and on the functions f, g_k (see Section 6.4), the same asymptotic representations as in Theorems 6.4.1, 6.4.2 hold for the solution $(u,\underline{u}) \in V^l_{p,\beta}(\mathcal{G}) \times V^{l-2m}_{p,\beta}(\partial\mathcal{G})$. The formulas (6.4.20), (6.4.28) for the coefficients in the asymptotics are also valid for such solutions. For this we refer to the papers [**142**], [**144**] of V. G. Maz'ya and B. A. Plamenevskiĭ.

Note that in the same way as was done in Chapter 6 for the case $p = 2$, the operator of the boundary value problem (8.4.1) can be extended to a continuous mapping from $\tilde{V}^{l,2m}_{p,\beta}(\partial\mathcal{G}) \times V^{l+\underline{\tau}-1/p}_{p,\beta}(\partial\mathcal{G})$ into $\tilde{V}^{l-2m,0}_{p,\beta}(\partial\mathcal{G}) \times V^{l-\underline{\mu}-1/p}_{p,\beta}(\partial\mathcal{G})$. Here $\tilde{V}^{l,k}_{p,\beta}(\mathcal{G})$ is the set of all pairs $(u,\underline{\phi})$, where

$$u \in \begin{cases} V^l_{p,\beta}(\mathcal{G}) & \text{if } l \ge 0,\\ V^{-l}_{p',-\beta}(\mathcal{G})^* & \text{if } l < 0,\end{cases}$$

$p' = p/(p-1)$, and $\underline{\phi} \in \prod_{j=1}^k V^{l-j+1-1/p}_{p,\beta}(\partial\mathcal{G})$. Then the conditions of Theorem 8.4.1 ensure the Fredholm property of the operator of problem (8.4.1) for arbitrary integer l. Furthermore, an assertion analogous to Theorem 8.4.2 holds for the generalized solutions of problem (8.4.1).

8.4.2. The comparison principle for solutions of elliptic boundary value problems in a cone. In Chapters 6, 7 we have considered solutions of elliptic boundary value problems in weighted L_2 Sobolev spaces. As we have mentioned above, it is possible to extend the results to weighted L_p Sobolev spaces.

There exists another approach proposed by V. A. Kozlov and V. G. Maz'ya [**109, 110, 114**] and based upon comparison of solutions of a boundary value problem and an ordinary differential equation. We present a sketch of this theory.

Let $\mathcal{K}$ be an open cone in $\mathbb{R}^n$ with vertex at the origin and with boundary $\partial\mathcal{K}$ cutting out a region Ω with smooth boundary on the unit sphere. We consider the boundary value problem with normal boundary conditions (see Definition 3.1.4)

$$L^{(0)}(x,\partial_x)\, u = f \ \text{ in } \mathcal{K}, \quad B^{(0)}_k(x,\partial_x)\, u = g_k \ \text{ on } \partial\mathcal{K}\backslash\{0\},\ k = 1,\dots,m, \tag{8.4.5}$$

where $L^{(0)}$, $B^{(0)}_k$ are model operators of order $2m$, μ_k, respectively, which can be represented in the form

$$L^{(0)}(x,\partial_x) = r^{-2m}\, \mathcal{L}(\omega,\partial_\omega, r\partial_r), \qquad B^{(0)}_k(x,\partial_x) = r^{-\mu_k}\, \mathcal{B}_k(\omega,\partial_\omega, r\partial_r).$$

We characterize the behaviour of the solution u by the function

$$M^l_{p;\mathcal{K}}(u;\rho) = \Big(\sum_{|\alpha|\le l} \rho^{p|\alpha|} \int_\rho^{2\rho} \int_\Omega |D^\alpha_x u|^p \, d\omega \, \frac{dr}{r} \Big)^{1/p},$$

where $p \in (1,\infty)$ and l is a nonnegative integer.

The space $W^l_{p,loc}(\overline{\mathcal{K}}\backslash\{0\})$ is defined as the set of all functions u such that $M^l_{p;\mathcal{K}}(u;\rho) < \infty$ for every $\rho > 0$. Let $W^{l-1/p}_{p,loc}(\partial\mathcal{K}\backslash\{0\})$ be the trace space on $\partial\mathcal{K}\backslash\{0\}$ for the space $W^l_{p,loc}(\overline{\mathcal{K}}\backslash\{0\})$, $l = 1, 2, \dots$. We introduce the family of seminorms

$$M^{l-1/p}_{p;\partial\mathcal{K}}(v;\rho) = \inf M^l_{p;\mathcal{K}}(u;\rho), \quad \rho > 0,$$

where the infimum is taken over all $u \in W^l_{p;loc}(\overline{\mathcal{K}}\backslash\{0\})$ such that $u = v$ on $\partial\Omega \times (\rho, 2\rho)$.

For arbitrary functions $f \in W^{l-2m}_{p,loc}(\overline{\mathcal{K}}\backslash\{0\})$ and $g_k \in W^{l-1/p}_{p,loc}(\partial\mathcal{K}\backslash\{0\})$ we set

$$F^l_p(\rho) = M^{l-2m}_{p;\mathcal{K}}(f;\rho) + \sum_{k=1}^m M^{l-\mu_k-1/p}_{p;\partial\mathcal{K}}(g_k;\rho).$$

As in Section 6.1, we denote by

$$\mathfrak{A}(\lambda) \,:\, W^l_2(\Omega) \to W^{l-2m}_2(\Omega) \times \prod_{k=1}^m W^{l-\mu_k-1/2}_2(\partial\Omega)$$

the operator of the parameter-depending problem

$$\mathcal{L}(\omega, \partial_\omega, \lambda)\, u = f \quad \text{in } \Omega,$$
$$\mathcal{B}_k(\omega, \partial_\omega, \lambda)\, u = g_k \quad \text{on } \partial\Omega, \ k = 1, \dots, m.$$

We fix real numbers λ_-, λ_+, $\lambda_- < \lambda_+$, such that the strip

$$\lambda_- < \operatorname{Re}\lambda < \lambda_+$$

does not contain eigenvalues of the pencil $\mathfrak{A}(\lambda)$. By $m_\pm$ we denote an arbitrary integer upper bound for the maximal lengths of the Jordan chains corresponding to the eigenvalues of $\mathfrak{A}(\lambda)$ on the lines $\operatorname{Re}\lambda = \lambda_\pm$ (if there are no eigenvalues on the line $\operatorname{Re}\lambda = \lambda_\pm$, we set $m_\pm = 1$).

The following result was obtained in [**109**].

THEOREM 8.4.3. *Let l be an integer not less than $2m$ and $1 < p < \infty$. Furthermore, let $f \in W^{l-2m}_{p,loc}(\overline{\mathcal{K}}\backslash\{0\})$, $g_k \in W^{l-\mu_k-1/p}_{p,loc}(\partial\mathcal{K}\backslash\{0\})$, $k = 1, \dots, m$, be functions satisfying the condition*

(8.4.6)
$$\int_0^1 \rho^{-\lambda_-} \big(1 + |\log\rho|\big)^{m_- - 1} F^l_p(\rho)\, \frac{d\rho}{\rho} + \int_1^\infty \rho^{-\lambda_+} \big(1 + |\log\rho|\big)^{m_+ - 1} F^l_p(\rho)\, \frac{d\rho}{\rho} < \infty.$$

Then the model problem (8.4.5) has a solution $u \in W^l_{p,loc}(\overline{\mathcal{K}}\backslash\{0\})$ *satisfying the inequality*

$$(8.4.7)\qquad M^l_{p;\mathcal{K}}(u;r) \;\le\; c\Big(\int_0^r \Big(\frac{\rho}{r}\Big)^{-\lambda_-}\Big(1+\log\frac{r}{\rho}\Big)^{m_--1} F^l_p(\rho)\,\frac{d\rho}{\rho} + \int_r^\infty \Big(\frac{\rho}{r}\Big)^{-\lambda_+}\Big(1+\log\frac{\rho}{r}\Big)^{m_+-1} F^l_p(\rho)\,\frac{d\rho}{\rho}\Big)$$

with a constant c *independent of* $f, g_1, \dots, g_m$.

It follows directly from (8.4.7) that

$$(8.4.8)\qquad M^l_{p;\mathcal{K}}(u;r) = \begin{cases} o(r^{\lambda_-}) & \text{as } r \to 0, \\ o(r^{\lambda_+}) & \text{as } r \to \infty. \end{cases}$$

According to [**109**], the solution $u \in W^l_{p,loc}(\overline{\mathcal{K}}\backslash\{0\})$ of problem (8.4.5) subject to (8.4.8) is unique.

We introduce new variables t and τ by

$$r = e^t \quad \text{and} \quad \rho = e^\tau.$$

Then (8.4.7) takes the form

$$M^l_{p;\mathcal{K}}(u;e^t) \;\le\; c\Big(\int_{-\infty}^t e^{\lambda_-(t-\tau)}\,(1+t-\tau)^{m_--1}\,F^l_p(e^\tau)\,d\tau + \int_t^{+\infty} e^{\lambda_+(t-\tau)}\,(1+\tau-t)^{m_+-1}\,F^l_p(e^\tau)\,d\tau\Big).$$

The last estimate proves to be equivalent to

$$M^l_{p;\mathcal{K}}(u;e^t) \le c_0 \int_{-\infty}^{+\infty} G(t-\tau)\,F^l_p(e^\tau)\,d\tau\,,$$

where c_0 is a positive constant independent of $u, f, g_1, \dots, g_m$, and G is the Green function of the operator

$$M(\partial_t) = (-\partial_t - \lambda_-)^{m_-}\,(\partial_t + \lambda_+)^{m_+}.$$

In other words, the function $\mathbb{R}_+ \ni r \to M^l_{p;\mathcal{K}}(u;r)$ satisfies the comparison principle

$$(8.4.9)\qquad M^l_{p;\mathcal{K}}(u;r) \le c_0\, w(\log r),$$

where w is the solution of the equation $M(\partial_t)\,w(t) = F^l_p(e^t)$ on $\mathbb{R}$ satisfying the condition $w(t) = o(e^{\lambda_\pm t})$ as $t \to \pm\infty$. Condition (8.4.6) which ensures this comparison principle can not be relaxed, because it is necessary and sufficient for the existence of w.

It is possible to generalize the comparison principle to the boundary value problem

$$(8.4.10)\qquad L(x,\partial_x)\,u = f \;\text{ in } \mathcal{K}, \quad B_k(x,\partial_x)\,u = g_k \;\text{ on } \partial\mathcal{K}\backslash\{0\},\; k = 1,\dots,m,$$

which is understood as a perturbation of the model problem (8.4.5). We set

$$\mathcal{K}_r = \{x \in \mathcal{K} : r < |x| < 2r\}, \quad \Gamma_r = \{x \in \partial\mathcal{K} : r < |x| < 2r\}$$

and introduce the function

$$\gamma(\log r) = c_0 \Big(\|L - L^{(0)}\|_{W_p^l(\mathcal{K}_r)\to W_p^{l-2m}(\mathcal{K}_r)} + \sum_{k=1}^{m} \|B_k - B_k^{(0)}\|_{W_p^l(\mathcal{K}_r)\to W_p^{l-\mu_k-1/p}(\Gamma_r)} \Big),$$

where c_0 is the constant in (8.4.9). One can easily give upper estimates for γ in terms of the coefficients of the operators L and B_k. We assume that the function γ satisfies the inequality

$$\sup \gamma(t) < m_+^{m_+}\, m_-^{m_-} \Big(\frac{\lambda_+ - \lambda_-}{m_+ + m_-}\Big)^{m_+ + m_-}$$

which guarantees the existence of the Green function G_γ of the equation

$$\big(M(\partial_t) - \gamma(t)\big)\, w = h \quad \text{on } \mathbb{R}.$$

THEOREM 8.4.4. *Let*

$$\int_{-\infty}^{+\infty} G_\gamma(0,\tau)\, F_p^l(e^\tau)\, d\tau < \infty.$$

Then there exists a solution $u \in W_{p,loc}^l(\overline{\mathcal{K}}\backslash\{0\})$ *of the boundary value problem (8.4.10) such that (8.4.9) is valid, where* w *is the solution of the equation*

$$\big(M(\partial_t) - \gamma(t)\big)\, w(t) = F_p^l(e^t) \quad \text{on } \mathbb{R}. \tag{8.4.11}$$

This comparison principle enables one to obtain various pieces of information on solutions of the boundary value problem (8.4.10) by using estimates for the solution of the ordinary differential equation (8.4.11). Since the comparison principle (8.4.9) gives precise "pointwise" information on the solutions, it leads, in particular, to estimates in the weighted Sobolev spaces $V_{p,\beta}^l(\mathcal{K})$.

8.4.3. The Miranda-Agmon maximum principle. The Miranda-Agmon maximum principle for solutions of strongly elliptic equations in smooth bounded domains (see Section 4.4) is one of the fundamental results in the theory of elliptic boundary value problems. Using this result, V. G. Maz'ya and B. A. Plamenevskiĭ [**147**, Th. 10.3] obtained the following "weighted" maximum principle for solutions of a strongly elliptic equation in a bounded domain $\mathcal{G}$ which is smooth outside the set $\mathcal{S} = \{x^{(1)}, \dots, x^{(d)}\}$ of conical points.

THEOREM 8.4.5. *Let* $L(x,\partial_x)$ *be a strongly elliptic admissible (see Definition 6.2.1) differential operator of order* $2m$ *in* $\mathcal{G}$. *Furthermore, let* $r = r(x)$ *be a smooth function in* $\overline{\mathcal{G}}$ *which coincides with the distance of* x *to the conical points* $x^{(\tau)}$ *near* $x^{(\tau)}$, $\tau = 1, \dots, d$. *We suppose that the operator pencils* $\mathfrak{A}_\tau(\lambda)$, $\tau = 1, \dots, d$, *generated by the Dirichlet problem*

$$Lu = 0 \quad \text{in } \mathcal{G}, \qquad \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = g_k \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S},\ k = 1, \dots, m, \tag{8.4.12}$$

(see Section 6.3) do not have eigenvalues on the line $\operatorname{Re}\lambda = -\beta + m - 1$. *Then every solution* u *of (8.4.12) with finite norm* $\|r^\beta u\|_{L_2(\mathcal{G})}$ *satisfies the inequality*

$$\|r^\beta u\|_{C^{m-1}(\mathcal{G})} + \|r^{\beta-m+1} u\|_{C(\mathcal{G})} \le c_1 \sum_{k=1}^{m} \Big(\|r^\beta g_k\|_{C^{k-m}(\partial\mathcal{G})} + \|r^{\beta-k+m} g_k\|_{C(\partial\mathcal{G})} \Big) + c_2\, \|u\|_{L(\mathcal{G}_1)}\,, \tag{8.4.13}$$

where $\mathcal{G}_1$ is a compact subset of $\mathcal{G}$. If the Dirichlet problem has at most one solution of the class indicated, then the last term on the right of (8.4.13) can be omitted.

Both the validity of the weighted maximum principle (8.4.13) and the validity of the Miranda-Agmon maximum principle

$$\|u\|_{C^{m-1}(\mathcal{G})} \le c\Big(\sum_{k=1}^{m} \|D_\nu^{k-1}u\|_{C^{m-k}(\partial\mathcal{G}\setminus\mathcal{S})} + \|u\|_{L(\mathcal{G}_1)}\Big). \tag{8.4.14}$$

depend on the spectral properties of the pencils $\mathfrak{A}_\tau(\lambda)$. By means of a result of V. A. Kozlov and V. G. Maz'ya [**111**], V. G. Maz'ya and J. Roßmann [**154**] proved the validity of inequality (8.4.14) for solutions of elliptic equations with real coefficients in plane polygonal and threedimensional polyhedral domains. For special differential equations (in particular, the biharmonic equation) we refer also to the papers of V. G. Maz'ya and B. A. Plamenevskiĭ [**150, 152**]. It was first shown by J. Pipher and G. Verchota [**197**] that the Miranda-Agmon maximum principle for solutions of the biharmonic equation fails, in general, if $\mathcal{G}$ is a Lipschitz domains in $\mathbb{R}^n$, $n \ge 4$. V. G. Maz'ya and J. Roßmann found independently an explicit counterexample [**155**].

Note that for nonsmooth domains two versions of the Miranda-Agmon maximum principle exist. The first one consists in the validity of the inequality (8.4.14) for the generalized solutions $u \in W_2^m(\mathcal{G})$, while in the second one the validity of (8.4.14) is required only for smooth solutions of problem (8.4.12). The following theorem was proved by V. G. Maz'ya and J. Roßmann [**155**].

THEOREM 8.4.6. *Let L be a strongly elliptic differential operator of order $2m$ with smooth coefficients.*

1) *The generalized solution $u \in W_2^m(\mathcal{G})$ of problem (8.4.12) satisfies the inequality (8.4.14) for all $g_k \in C^{m-k}(\partial\mathcal{G}\setminus\mathcal{S})$ if and only if the strip $m-n/2 \le \operatorname{Re}\lambda \le m-1$ does not contain eigenvalues of the pencils $\mathfrak{A}_\tau(\lambda)$, $\tau = 1, \dots, d$. (This condition is satisfied, e.g., in the case $n = 2$ and in the case $n = 3$ if the coefficients of L are real.)*

2) *The estimate (8.4.14) is satisfied for all solutions $u \in C^\infty(\overline{\mathcal{G}})$ of the equation $Lu = 0$ if and only if there are no eigenvalues of the pencils $\mathfrak{A}_\tau(\lambda)$ on the line* $\operatorname{Re}\lambda = m - 1$.

Moreover, in [**155**] it was shown that for every $n = 4, 5, \dots$ and for every strongly elliptic differential operator L there exist domains $\mathcal{G} \subset \mathbb{R}^n$ such that (8.4.14) is not satisfied for all smooth solutions of the equation $Lu = f$ in $\mathcal{G}$.

8.4.4. Estimates of the Green function. In the same way as it was done for boundary value problems in smooth domains, Green functions (or generalized Green functions) can be introduced for domains with conical points. In contrast to the smooth case, then the Green functions have singularities both at the diagonal and at the singular boundary points. In the following, we present pointwise estimates of the Green functions which were proved by V. G. Maz'ya and B. A. Plameneviskiĭ [**149**]

First we consider the Green function for the model problem

$$L(x,\partial_x)\,u = f \ \text{ in } \mathcal{K}, \quad B_k(x,\partial_x)\,u = g_k \ \text{ on } \partial\mathcal{K}\setminus\{0\},\ k = 1,\dots,m, \tag{8.4.15}$$

in a cone $\mathcal{K} \subset \mathbb{R}^n$ with vertex at the origin which cuts out a smooth domain Ω on the unit sphere. Here L is an elliptic model operator of order $2m$ and B_k are model operators of order $\mu_k < 2m$ which form a normal system (see Definition 3.1.4) and cover the operator L on $\partial\mathcal{K}\backslash\mathcal{S}$. We complete the system $\{B_1, \dots, B_m\}$ by model operators $B_{m+1}, \dots, B_{2m}$, $\operatorname{ord} B_{m+k} = \mu_{m+k} < 2m$, to a Dirichlet system of order $2m$ on $\partial\mathcal{K}\backslash\mathcal{S}$. Then there exist model operators B'_k such that the following Green formula is satisfied for all $u, v \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$:

$$\int_{\mathcal{K}} Lu \cdot \overline{v}\, dx + \sum_{k=1}^{m} \int_{\partial\mathcal{K}} B_k u \cdot \overline{B'_{k+m} v}\, d\sigma = \int_{\mathcal{K}} u \cdot \overline{L^+ v}\, dx + \sum_{k=1}^{m} \int_{\partial\mathcal{K}} B_{k+m} u \cdot \overline{B'_k v}\, d\sigma.$$

The model operators L, L^+, B_k, B'_k can be represented in the form

$$L = r^{-2m}\, \mathcal{L}(\omega, \partial_\omega, r\partial_r), \quad L^+ = r^{-2m}\, \mathcal{L}^+(\omega, \partial_\omega, r\partial_r),$$
$$B_k = r^{-\mu_k}\, \mathcal{B}_k(\omega, \partial_\omega, r\partial_r), \quad B'_k = r^{\mu_{m+k}-2m+1}\, \mathcal{B}'_k(\omega, \partial_\omega, r\partial_r), \quad k = 1, \dots, m.$$

As in Section 6.1, we denote by $\mathfrak{A}(\lambda)$, $\mathfrak{A}_c^+(\lambda)$ the operators of the parameter-depending problems

$$\mathcal{L}(\omega, \partial_\omega, \lambda)\, u = f \ \text{ in } \Omega, \quad \mathcal{B}'_k(\omega, \partial_\omega, \lambda)\, u = g_k \ \text{ on } \partial\Omega,\ k = 1, \dots, m,$$

and

$$\mathcal{L}^+(\omega, \partial_\omega, -\lambda + 2m - n)\, v = f \ \text{ in } \Omega,$$
$$\mathcal{B}'_k(\omega, \partial_\omega, -\lambda + 2m - n)\, u = g_k \ \text{ on } \partial\Omega,\ k = 1, \dots, m,$$

respectively.

THEOREM 8.4.7. *Suppose that the line* $\operatorname{Re}\lambda = -\beta + 2m - n/2$ *does not contain eigenvalues of the operator pencil* $\mathfrak{A}(\lambda)$. *Then the following assertions hold.*

1) *There exists a unique solution* $G(\cdot, y)$ *of the boundary value problem*

$$L(x, \partial_x)\, G(x, y) = \delta(x - y), \quad x, y \in \mathcal{K},$$
$$B_k(x, \partial_x)\, G(x, y) = 0, \quad x \in \partial\mathcal{K}\backslash\{0\},\ y \in \mathcal{K},\ k = 1, \dots, m,$$

such that the function $x \to \eta(|y|^{-1}|x|)\, G(x, y)$ *lies in the space* $V_{2,\beta+s}^{2m+s}(\mathcal{K})$, $s = 0, 1, \dots$, *for each fixed* $y \in \mathcal{K}$ *and for each function* $\eta \in C^\infty((0, \infty))$, $\eta(t) = 0$ *for* $\frac{3}{4} < t < \frac{3}{2}$, $\eta(t) = 1$ *for* $t < \frac{1}{2}$, $t > 2$.

2) *The function* G *is the unique solution of the boundary value problem*

$$L^+(y, \partial_y)\, \overline{G(x, y)} = \delta(x - y), \quad x, y \in \mathcal{K},$$
$$B'_k(y, \partial_y)\, \overline{G(x, y)} = 0, \quad y \in \partial\mathcal{K}\backslash\{0\},\ x \in \mathcal{K},\ k = 1, \dots, m,$$

such that the function $y \to \eta(|y|^{-1}|x|)\, G(x, y)$ *lies in the space* $V_{2,-\beta+2m+s}^{2m+s}(\mathcal{K})$, $s = 0, 1, \dots$, *for each fixed* $x \in \mathcal{K}$.

Let β be a real number such that no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ lie on the line $\operatorname{Re}\lambda = -\beta + 2m - n/2$. We denote by $\lambda_1^+, \lambda_2^+, \dots$ all the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ which lie above that line, enumerated in the order of increase of the real part, and by $\lambda_1^-, \lambda_2^-, \dots$ all the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ which lie above that line, enumerated in the order of decreasing the real parts. The algebraic multiplicities of the eigenvalues $\lambda_\mu^\pm$ are denoted by $\kappa_\mu^\pm$, while $\kappa_{\mu,1}^\pm, \dots, \kappa_{\mu,I_\mu^\pm}^\pm$ denote the partial multiplicities of the eigenvalues $\lambda_\mu^\pm$, $\kappa_{\mu,1}^\pm, \dots, \kappa_{\mu,I_\mu^\pm}^\pm = \kappa_\mu^\pm$. Let

$$\{\varphi_{j,s}^{\mu,\pm}\}_{j=1,\dots,I_\mu^\pm,\ s=0,\dots,\kappa_{\mu,j}^\pm} \quad \text{and} \quad \{\psi_{j,s}^{\mu,\pm}\}_{j=1,\dots,I_\mu^\pm,\ s=0,\dots,\kappa_{\mu,j}^\pm}$$

be canonical systems of Jordan chains of the pencil $\mathfrak{A}(\lambda)$, $\mathfrak{A}_c^+(\lambda)$ to the eigenvalues $\lambda_\mu^\pm$ and $\overline{\lambda_\mu^\pm}$, respectively, satisfying the biorthonormality condition (6.1.61). Using Theorem 6.1.7, we obtain the following statement.

THEOREM 8.4.8. *Let $G(x,y)$ be the Green function of the previous theorem. Then:*

1. *For* $2|x| < |y|$,

$$G(x,y) = -\sum_{\mu=1}^{N} \frac{|x|^{\lambda_\mu^+}}{|y|^{\lambda_\mu^+ + n - 2m}} \sum_{j=0}^{I_\mu^+} \sum_{s=0}^{\kappa_{\mu,j}^+ - 1} \frac{1}{(\kappa_{\mu,j}^+ - s - 1)!} \Big(\log \frac{|x|}{|y|}\Big)^{\kappa_{\mu,j}^+ - s - 1} \times \sum_{t=0}^{s} \varphi_{j,t}^{\mu,+}\Big(\frac{x}{|x|}\Big) \overline{\psi_{j,s-t}^{\mu,+}\Big(\frac{y}{|y|}\Big)} + R_+(x,y),$$

where N is a natural number such that $\operatorname{Re}\lambda_{N+1}^+ > \operatorname{Re}\lambda_N^+$ *and $R_+(x,y)$ satisfies the inequality*

$$|D_x^\alpha D_y^\gamma R_+(x,y)| \le c \frac{|x|^{Re\,\lambda_{N+1}^+ - |\alpha|}}{|y|^{Re\,\lambda_{N+1}^+ - 2m + n + |\gamma|}} \Big|\log \frac{|y|}{|x|}\Big|^{\kappa^+}$$

in which α and γ are arbitrary multi-indices and κ^+ is the maximal partial multiplicity of eigenvalues of $\mathfrak{A}(\lambda)$ *on the line* $\operatorname{Re}\lambda = \operatorname{Re}\lambda_{N+1}^+$.

2. *For* $2|y| < |x|$,

$$G(x,y) = \sum_{\mu=1}^{N} \frac{|x|^{\lambda_\mu^-}}{|y|^{\lambda_\mu^- + n - 2m}} \sum_{j=0}^{I_\mu^-} \sum_{s=0}^{\kappa_{\mu,j}^- - 1} \frac{1}{(\kappa_{\mu,j}^- - s - 1)!} \Big(\log \frac{|x|}{|y|}\Big)^{\kappa_{\mu,j}^- - s - 1} \times \sum_{t=0}^{s} \varphi_{j,t}^{\mu,-}\Big(\frac{x}{|x|}\Big) \overline{\psi_{j,s-t}^{\mu,-}\Big(\frac{y}{|y|}\Big)} + R_-(x,y),$$

where N is a natural number such that $\operatorname{Re}\lambda_{N+1}^- < \operatorname{Re}\lambda_N^-$ *and $R_-(x,y)$ satisfies the inequality*

$$|D_x^\alpha D_y^\gamma R_-(x,y)| \le c \frac{|x|^{Re\,\lambda_{N+1}^- - |\alpha|}}{|y|^{Re\,\lambda_{N+1}^- - 2m + n + |\gamma|}} \Big|\log \frac{|x|}{|y|}\Big|^{\kappa^-}$$

in which α and γ are arbitrary multi-indices and κ^- is the maximal partial multiplicity of eigenvalues of $\mathfrak{A}(\lambda)$ *on the line* $\operatorname{Re}\lambda = \operatorname{Re}\lambda_{N+1}^-$.

3. *For* $|x|/2 < |y| < 2|x|$,

$$|D_x^\alpha D_y^\gamma G(x,y)| \le c\Big(|x-y|^{2m-n-|\alpha|-|\gamma|} + |y|^{2m-n-|\alpha|-|\gamma|}\Big) \quad \text{if } |\alpha| + |\gamma| \ne 2m - n,$$

$$|D_x^\alpha D_y^\gamma G(x,y)| \le c\Big(\Big|\log \frac{|x-y|}{|y|}\Big| + 1\Big) \quad \text{in the contrary case.}$$

Similar asymptotic representations hold for the Green functions of elliptic boundary value problems in bounded domains with conical points. Let $\mathcal{G}$ be a bounded domain in $\mathbb{R}^n$ with conical points $x^{(\tau)}$, $\tau = 1, \dots, d$, on the boundary. Outside the set $\mathcal{S} = \{x^{(1)}, \dots, x^{(\tau)}\}$ the boundary $\partial\mathcal{G}$ is assumed to be smooth. We consider the boundary value problem

$$Lu = f \quad \text{in } \mathcal{G}, \qquad B_k u = g_k \quad \text{on } \partial\mathcal{G}\backslash\mathcal{S},\ k = 1, \dots, m, \tag{8.4.16}$$

where L is an elliptic differential operator of order $2m$, and B_k are differential operators of order $\mu_k < 2m$ which form a normal system and cover the operator L on $\partial\mathcal{G}\backslash\mathcal{S}$. Furthermore, we suppose that the operators L and B_k are admissible and that problem (8.4.15) is uniquely solvable in $V_{2,\beta}^{2m}(\mathcal{G})$ for arbitrary $f \in V_{2,\beta}^{0}(\mathcal{G})$, $g_k \in V_{2,\beta}^{2m-\mu_k-1/2}(\partial\mathcal{G})$. Here $\beta = (\beta_1, \dots, \beta_d)$ is a given real d-tuple. We denote by $\mathfrak{A}_\tau(\lambda)$ the operator pencil generated by the model problem in the cone with vertex in $x^{(\tau)}$ (see Section 6.3). From the above assumption on the unique solvability of problem (8.4.16) it follows that the line $\operatorname{Re}\lambda = -\beta_\tau + 2m - n/2$ does not contain eigenvalues of the pencil $\mathfrak{A}_\tau(\lambda)$ for $\tau = 1, \dots, d$. Let $\Lambda_\tau^- < \operatorname{Re}\lambda < \Lambda_\tau^+$ be the largest strip in the complex plane which contains the line $\operatorname{Re}\lambda = -\beta_\tau + 2m - n/2$ and is free of eigenvalues of $\mathfrak{A}_\tau(\lambda)$. As a consequence of Theorems 3.5, 3.6 in [**149**] the following statement holds.

THEOREM 8.4.9. 1) *There exists a unique solution $G(\cdot, y)$ of the boundary value problem*

$$L(x,\partial_x)\, G(x,y) = \delta(x-y), \quad x,y \in \mathcal{G},$$
$$B_k(x,\partial_x)\, G(x,y) = 0, \quad x \in \partial\mathcal{G}\backslash\mathcal{S},\ y \in \mathcal{G},\ k = 1, \dots, m,$$

such that the function $x \to \eta(x,y)\, G(x,y)$ lies in the space $V_{2,\beta+s\vec{1}}^{2m+s}(\mathcal{G})$, $s = 0, 1, \dots$, for each fixed $y \in \mathcal{G}$ and for each smooth function η on $\overline{\mathcal{G}} \times \overline{\mathcal{G}}$ equal to zero in a neighbourhood of the diagonal.

2) *The function $G(x,y)$ is infinitely differentiable in all its arguments for $x, y \in \overline{\mathcal{G}}\backslash\mathcal{S}$, $x \neq y$.*

3) *The function G is the unique solution of the boundary value problem*

$$L^+(y,\partial_y)\, \overline{G(x,y)} = \delta(x-y), \quad x,y \in \mathcal{G},$$
$$B_k'(y,\partial_y)\, \overline{G(x,y)} = 0, \quad y \in \partial\mathcal{G}\backslash\mathcal{S},\ x \in \mathcal{G},\ k = 1, \dots, m,$$

such that the function $y \to \eta(x,y)\, G(x,y)$ lies in the space $V_{2,-\beta+2m+s\vec{1}}^{2m+s}(\mathcal{G})$, $s = 0, 1, \dots$, for each fixed $x \in \mathcal{G}$.

4) *Let $r_\tau(x)$ denote the distance of x to the point $x^{(\tau)}$. If x and y lie in a neighbourhood of the same conical point $x^{(\tau)}$, then the following estimates are valid:*

(i) *For $2r_\tau(x) < r_\tau(y)$,*

$$|D_x^\alpha\, D_y^\gamma\, G(x,y)| \le c\, r_\tau(x)^{\Lambda_\tau^+ - |\alpha| - \varepsilon}\, r_\tau(y)^{-\Lambda_\tau^+ - n + 2m - |\gamma| + \varepsilon}$$

(here ε is an arbitrary small real number).

(ii) *For $2r_\tau(y) < r_\tau(x)$,*

$$|D_x^\alpha\, D_y^\gamma\, G(x,y)| \le c\, r_\tau(x)^{\Lambda_\tau^- - |\alpha| + \varepsilon}\, r_\tau(y)^{-\Lambda_\tau^- - n + 2m - |\gamma| - \varepsilon}\,.$$

(iii) *For $\frac{1}{2}r_\tau(x) < r_\tau(y) < 2r_\tau(x)$,*

$$|D_x^\alpha\, D_y^\gamma\, G(x,y)| \le c\left(|x-y|^{2m-n-|\alpha|-|\gamma|} + r_\tau(y)^{2m-n-|\alpha|-|\gamma|}\right) \quad \textit{if } |\alpha| + |\gamma| \neq 2m-n,$$

$$|D_x^\alpha\, D_y^\gamma\, G(x,y)| \le c\left(\left|\log\frac{|x-y|}{|y|}\right| + 1\right) \quad \textit{in the contrary case.}$$

5) *If x lies in a neighbourhood of $x^{(\tau)}$ and y lies in a neighbourhood of $x^{(\sigma)}$, $\sigma \neq \tau$, then*

$$|D_x^\alpha\, D_y^\gamma\, G(x,y)| \le c\, r_\tau(x)^{\Lambda_\tau^+ - |\alpha| - \varepsilon}\, r_\sigma(y)^{-\Lambda_\sigma^- - n + 2m - |\gamma| - \varepsilon}\,.$$

8.5. Notes

Harmonic functions in domains with singular points on the boundary were already considered in classical works of H. Poincaré, S. Zaremba, H. Lebesgue, T. Carleman, J. Radon at the end of the 19th and beginning of the 20th centuries. Furthermore, already in the first half of the 20th century the question on the behaviour of solutions of boundary value problems near singular boundary points was discussed for applications in fracture mechanics and in aero- and hydrodynamics. The systematic study of elliptic boundary value in domains with angular and conical points began in the 60s. We restrict ourselves here to those works which are dedicated to *general* elliptic boundary value problems. For special boundary value problems we refer to the surveys and bibliographies in the papers of V. G. Maz'ya [**135**], V. A. Kondrat'ev, O. A. Oleĭnik [**100**] and in the book of P. Grisvard [**80**]. ,

1. The results of Chapter 5 are essentially contained in the works of S. Agmon, L. Nirenberg [**9**], V. A. Kondrat'ev [**98, 99**], and V. G. Maz'ya, B. A. Plamenevskiĭ [**144, 147**]. In [**9**] asymptotic formulas for the solutions of elliptic model problems in a cylinder (i.e., boundary value problems with coefficients depending only on the variables in the base of the cylinder) were obtained. In particular, S. Agmon and L. Nirenberg showed that the solutions of the homogeneous boundary value problem can be written as formal infinite series of power-exponential solutions of this problem. The influence of exponentially decaying (for $t \to \infty$) perturbations of the coefficients was studied by A. Pazy [**190**].

V. A. Kondrat'ev [**99**] proved that the inhomogeneous model problem is uniquely solvable in the weighted Sobolev space $\mathcal{W}^l_{2,\beta}$ if and only if the resolvent of the corresponding parameter-depending problem does not contain eigenvalues on the line $\operatorname{Re}\lambda = -\beta$. For this he used an estimate for solutions of the parameter-depending problem which was obtained by M. S. Agranovich and M. I. Vishik [**13**]. Furthermore, he derived an asymptotic representation for the solution. In Theorems 5.2.2, 5.3.2, 5.4.1, 5.4.2 we generalized results of V. A. Kondrat'ev to weighted Sobolev spaces of arbitrary integer order and to boundary value problems with additional unknowns on the boundary. The Fredholm property of the operator of a boundary value problem with t-dependent coefficients stabilizing at infinity (see Theorem 5.5.2) was proved in [**147**].

2. The proof of the above mentioned results for model problems in a cylinder given in V. A. Kondrat'ev's papers [**98, 99**] was the decisive step to obtain the unique solvability of model problems in a cone. Using the invertibility of the operators of model problems in suitable weighted Sobolev spaces, V. A. Kondrat'ev constructed regularizers for the operators of elliptic problems in bounded domains with conical points. In this way, he proved the Fredholm property of the operators of such problems. Moreover, he derived asymptotic formulas for the solutions near the conical points. Note that Theorems 6.1.1–6.1.4, 6.3.3, 6.4.2 and Corollary **??** are generalizations of V. A. Kondrat'ev's results to boundary value problems with additional unknowns on the boundary and to Sobolev spaces of arbitrary integer order. V. G. Maz'ya and B. A. Plamenevskiĭ [**144, 147**] extended the results of V. A. Kondrat'ev to boundary value problems for systems of differential equations and to other function spaces.

Another approach to the theory of elliptic boundary value problems in plane domains with angular points is given in the papers of G. I. Eskin [**65, 66**] and Ya. B. Lopatinskiĭ [**128**]. In contrast to V. A. Kondrat'ev who applied the Mellin transformation directly to the differential operators of the boundary value problem, these two authors applied the Mellin transformation to an integral equation on the boundary of the domain.

3. Theorem 6.3.6 on the structure of solutions of boundary value problems for admissible differential operators and the index formula in Theorem 6.3.7 were proved by V. G. Maz'ya and B. A. Plamenevskiĭ [**144**].

4. The formulas (5.4.24), (6.1.58), (6.4.20), and (6.4.28) for the coefficients in the asymptotics of solutions of boundary value problems both in a cylinder and in domains with conical points are due to V. G. Maz'ya and B. A. Plamenevskiĭ [**142, 144**]. We refer also to the papers of M. Bourlard, M. Dauge, M. S. Lubuma, S. Nicaise [**33**] and S. Nicaise [**185**].

5. Generalized solutions (i.e., solutions of the class $\tilde{V}^l_{2,\beta}$ with $l < 2m$) of elliptic boundary value problems in domains with singularities on the boundary were considered in the papers of J. Roßmann [**211**] and A. K. Aziz, R. B. Kellogg [**22**]. The asymptotics of variational solutions near conical points was studied in papers of V. G. Maz'ya, B. A.Plamenevskiĭ [**147**], H. Blum, R. Rannacher [**31**], J. Roßmann [**209**], and in the monograph of M. Dauge [**53**].

6. The results of V. A. Kondrat'ev have been extended in several papers to other function spaces. V. A. Kondrat'ev [**99**] studied also the solvability and the asymptotics of the solutions in L_2 Sobolev spaces without weight. As we have noted in the previous section, V. G. Maz'ya and B. A. Plamenevskiĭ [**144, 147**] used weighted L_p Sobolev spaces and weighted Hölder classes.

A. Avantagiatti and M. Troisi [**20, 21**] considered solutions of boundary value problems in a plane wedge in weighted Sobolev spaces H^{l,β_1,β_2}, where the exponent β_1 describes the growth of the function near the corner and the exponent β_2 describes the growth at infinity. Analogous spaces in n-dimensional cones ($n > 2$) were used by G. Trombetti [**244**].

The results of Sections 7.1 and 7.2 are essentially contained in the papers [**99**] of V. A. Kondrat'ev and [**143**] of V. G. Maz'ya and B. A. Plamenevskiĭ. In the last paper solutions of elliptic problems in weighted L_p Sobolev spaces and weighted Hölder classes with inhomogeneous norms were considered. However, certain critical values of the weight parameter were excluded there. Relations between weighted Sobolev spaces with homogeneous and inhomogeneous norms for the critical weight parameters were studied by J. Roßmann [**210**] and J. Lang, A. Nekvinda [**123**]. Theorems 7.3.1-7.3.3, 7.3.5 were proved in [**210**] for the two-dimensional case.

Part 3

Elliptic problems in domains with cuspidal points

CHAPTER 9

Elliptic boundary value problems in domains with exterior cusps

In this chapter we study boundary value problems in domains which can be transformed (in a neighbourhood of the singular points) onto a half-cylinder by means of suitable diffeomorphisms. Among the singular points may be, e.g., the points at infinity and vertices of cusps.

We study the solvability of boundary value problems in special weighted Sobolev spaces which depend on the form of the boundary near the singular points. Furthermore, we are interested in regularity assertions for the solutions.

9.1. Elliptic boundary value problems in quasicylindrical domains

This section is dedicated to elliptic boundary value problems in an infinite domain which coincides outside a bounded set with a "quasicylinder". If the coefficients of the differential operators satisfy certain stabilization conditions at infinity, then by means of a special mapping, we obtain a boundary value problem in a cylinder for which the conditions of Chapter 5 are satisfied.

9.1.1. Description of the domain. Let φ be an infinitely differentiable positive function on the interval $[1,+\infty)$ satisfying the conditions

(i) $\lim\limits_{t\to+\infty} \varphi(t)^{k-1}\,\varphi^{(k)}(t) < \infty \qquad$ for $k = 1,2,\dots$,

(ii) $\displaystyle\int\limits_1^\infty \frac{dt}{\varphi(t)} = +\infty$

For example, the function $\varphi(t) = t^\alpha$ satisfies these conditions if $\alpha \le 1$.

Furthermore, we suppose that $\mathcal{G}$ is a domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{G}$ of class C^∞, the set $\{x \in \mathcal{G} : x_n < 1\}$ is bounded, and there exist functions $y_j = h_j(x)$, $j = 1,\dots,n-1$, such that

(iii) the mapping

$$y_j = \frac{h_j(x)}{\varphi(x_n)}, \quad j = 1,\dots,n-1, \qquad y_n = \int\limits_1^{x_n} \frac{dt}{\varphi(t)}$$

transforms the set $\{x \in \mathcal{G} : x_n > 1\}$ onto the half-cylinder

$$\mathcal{C}_+ = \{y = (y', y_n) : y' = (y_1,\dots,y_{n-1}) \in \Omega,\, y_n > 0\}.$$

(iv) $\det\Big(\dfrac{\partial h_j}{\partial x_k}\Big)_{j,k=1,\dots,n-1} \ge \text{const} > 0,$

(v) for all multi-indices α there exist the finite limits (uniform with respect to $y' \in \Omega$)

$$\lim_{x_n \to \infty} \varphi(x_n)^{|\alpha|-1} \, \partial_x^\alpha h_j(x) \,, \qquad j = 1, \dots , n-1.$$

(For $\alpha = 0$ this is a consequence of condition (iii).)

From the equalities

$$\frac{\partial y_n}{\partial x_k} = 0 \quad \text{for } j = 1, \dots , n-1, \qquad \frac{\partial y_n}{\partial x_n} = \varphi(x_n)^{-1} \,,$$

and

$$\frac{\partial y_j}{\partial x_k} = \varphi(x_n)^{-1} \, \frac{\partial h_j}{\partial x_k} \quad \text{for } j,k = 1, \dots , n-1$$

it follows that

$$\text{(9.1.1)} \qquad \det \Big(\frac{\partial y_j}{\partial x_k} \Big)_{j,k=1,\dots,n} = \varphi(x_n)^{-n} \, \det \Big(\frac{\partial h_j}{\partial x_k} \Big)_{j,k=1,\dots,n-1}$$

Moreover, the following assertions hold.

LEMMA 9.1.1. 1) *From conditions (i) and (ii) it follows that*

$$\lim_{t \to \infty} \varphi(t)^{k-1} \, D_t^k \varphi(t) = 0 \quad \textit{for } k > 1.$$

2) *Furthermore, conditions (i), (ii) and (v) imply*

$$\lim_{x_n \to \infty} \varphi(x_n)^{|\alpha|-1} \, D_x^\alpha h_i(x) = 0, \qquad i = 1, \dots , n-1$$

for all multi-indices $\alpha = (\alpha_1, \dots , \alpha_n)$ *such that* $|\alpha| > 1$, $\alpha_n \geq 1$.

Proof: 1) Obviously,

$$\text{(9.1.2)} \qquad \varphi^{k-1} \, \varphi^{(k)} = \varphi \left(\varphi^{(k-2)} \, \varphi^{(k-1)} \right)' - (k-2) \, \varphi' \, \varphi^{k-2} \, \varphi^{(k-1)}.$$

for $k \geq 2$. We denote the limit of the first term on the right by l_k. Then we obtain

$$\varphi^{(k-2)} \, \varphi^{t-1)} \Big|_1^{t_0} = \int_1^{t_0} \frac{l_k + c(t)}{\varphi(t)} \, dt,$$

where $c(t)$ tends to zero as $t \to +\infty$. If $l_k \neq 0$, then by condition (ii), the right side tends to infinity as $t_0 \to +\infty$, while the left side has a finite limit. Consequently, $l_k = 0$, and the limit of $\varphi^{k-1} \, \varphi^{(k)}$ coincides with the limit of the second term on the right of (9.1.2). In particular, for $k = 2$ we conclude that $\varphi \, \varphi'' \to 0$ as $t \to +\infty$. Moreover, using (9.1.2), we obtain by induction that $\varphi^{k-1} \, \varphi^{(k)} \to 0$ for $k \geq 2$.

2) Inserting $\psi = h_j$ into the equality

$$\text{(9.1.3)} \qquad \varphi^{k-1} \, \psi^{(k)} = \varphi \left(\varphi^{(k-2)} \, \psi^{(k-1)} \right)' - (k-2) \, \varphi' \, \varphi^{k-2} \, \psi^{(k-1)},$$

we get

$$\begin{aligned} \varphi(x_n)^{k-1} \, \partial_{x_n}^k h_j(x) &= \varphi(x_n) \, \partial_{x_n} \left(\varphi(x_n)^{k-2} \, \partial_{x_n}^{k-1} h_j \right) \\ &\quad -(k-2) \, \varphi'(x_n) \, \varphi(x_n)^{k-2} \, \partial_{x_n}^{k-1} \, h_j \, . \end{aligned}$$

Consequently, it follows from conditions (i) and (v) that the limit of the function $\varphi(x_n) \, \partial_{x_n} \left(\varphi(x_n)^{k-2} \, \partial_{x_n}^{k-1} h_j \right)$ for $x_n \to +\infty$ exists uniformly with respect to $y' \in$

$\partial\Omega$. Analogously to the first part of the proof, we conclude from condition (ii) that this limit is equal to zero, and therefore,

$$\lim_{x_n\to+\infty} \varphi(x_n)^{k-1}\,\partial_{x_n}^k h_j(x) = -(k-2) \lim_{x_n\to+\infty} \varphi'(x_n)\,\varphi(x_n)^{k-2}\,\partial_{x_n}^{k-1} h_j\,.$$

This implies assertion 2) for the multi-indices $\alpha = (0,\dots,0,\alpha_n)$ with $\alpha_n > 2$.

In the same way, we can prove this assertion for other multi-indices α, $|\alpha| \ge 2$, $\alpha_n \neq 0$, if we insert $\psi = \varphi(x_n)^{|\alpha'|-1}\partial_{x_n}^{\alpha_n} h_j$ into (9.1.3). ■

Example: Let φ be a smooth positive function on $[1,\infty)$ satisfying conditions (i) and (ii). We assume that

$$\{x \in \mathcal{G} :\ x_n > 1\} = \{x \in \mathbb{R}^n :\ x_n > 1, |x'| < \varphi(x_n)\},$$

where $x' = (x_1,\dots,x_{n-1})$ (see Figure 1).

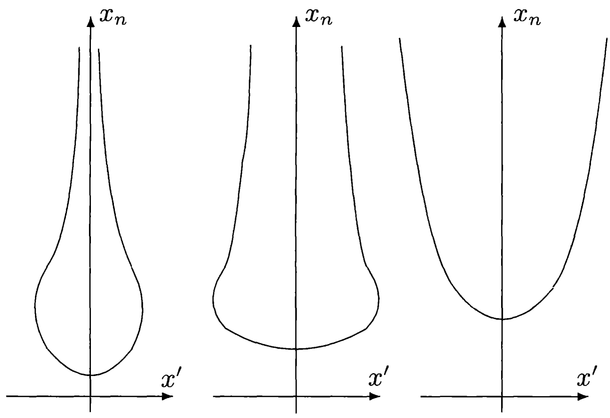


Figure 1: Quasicylinders

Then the mapping

$$y_j = \frac{x_j}{\varphi(x_n)} \quad \text{for } j = 1,\dots,n-1, \quad y_n = \int_1^{x_n} \frac{dt}{\varphi(t)} \tag{9.1.4}$$

takes the set $\{x \in \mathcal{G} : x_n > 1\}$ onto the half-cylinder $\{y \in \mathbb{R}^n : |y'| < 1, y_n > 0\}$. Obviously, the conditions (iv) and (v) are satisfied for the functions $h_j(x) = x_j$.

9.1.2. Weighted Sobolev spaces in $\mathcal{G}$. Let conditions (i)-(v) be satisfied. We extend the function φ to a smooth positive function on $(-\infty,+\infty)$ and define the space $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ as the closure of $C_0^\infty(\overline{\mathcal{G}})$ with respect to the norm

$$\|u\|_{\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})} = \Big(\int_{\mathcal{G}} \sum_{|\alpha|\le l} e^{2\beta y_n(x_n)}\,\varphi(x_n)^{2(\gamma-l+|\alpha|)}\,|D_x^\alpha u(x)|^2\,dx\Big)^{1/2}, \tag{9.1.5}$$

where $y_n(x_n) \overset{def}{=} \int_1^{x_n} \frac{dt}{\varphi(t)}$. Here l is an arbitrary nonnegative integer and β, γ are arbitrary real numbers. For integer $l \geq 1$ we denote the space of traces of functions from $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ on $\partial\mathcal{G}$ by $\mathcal{W}^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$. If l is an integer ≤ 0, then the space $\mathcal{W}^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$ is defined as the dual space of $\mathcal{W}^{-l+1/2}_{2,-\beta,-\gamma}(\partial\mathcal{G})$. Furthermore, for arbitrary integer l and nonnegative integer k we define the space $\tilde{\mathcal{W}}^{l,k}_{2,\beta,\gamma}(\mathcal{G})$ as the set of all pairs $(u, \underline{\phi})$ such that

$$u \in \begin{cases} \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G}) & \text{if } l \geq 0, \\ \mathcal{W}^{-l}_{2,-\beta,-\gamma}(\mathcal{G})^* & \text{if } l < 0, \end{cases}$$

$$\underline{\phi} = (\phi_1, \dots, \phi_k) \in \prod_{j=1}^{k} \mathcal{W}^{l-j+1/2}_{2,\beta,\gamma}(\partial\mathcal{G}), \qquad \phi_j = D_\nu^{j-1} u \ \text{ for } j \leq \min(l,k).$$

The coordinate change $x \to y$ introduced above transforms an arbitrary function $u \in \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ vanishing for $x_n < 1$ into a function on the cylinder $\mathcal{C}$. We will show that this function belongs to the weighted Sobolev space $\mathcal{W}^l_{2,\beta}(\mathcal{C})$ if $\gamma = l - n/2$. For this we need the following lemma.

LEMMA 9.1.2. *Let condition (i)-(v) be satisfied. Then there is the following relation between the x- and y-derivatives:*

$$\partial_x^\alpha = \varphi(x_n)^{-|\alpha|} \sum_{|\gamma| \leq |\alpha|} A_{\alpha,\gamma}(x)\, \partial_y^\gamma \,, \tag{9.1.6}$$

where $A_{\alpha,\gamma}$ are infinitely differentiable functions on $\overline{\mathcal{G}}$ such that there exists the limit of $A_{\alpha,\gamma}$ as $x_n \to +\infty$ (uniformly with respect to $y' \in \Omega$).

Proof: We prove by induction that there is the representation (9.1.6), where $A_{\alpha,\gamma}$ is a linear combination of functions of the form

$$\prod_\delta \left(\varphi^{|\delta|-1}\,(\partial_x^\delta h_j)\right) \quad \text{and} \quad \prod_\delta \left(\varphi^{|\delta|-1}\,(\partial_x^\delta h_j)\right) \cdot \prod_k \left(\varphi^{k-1}\,\varphi^{(k)}\right) \tag{9.1.7}$$

Obviously,

$$\partial_{x_j} = \varphi^{-1} \sum_{i=1}^{n-1} \frac{\partial h_i}{\partial x_j} \cdot D_{y_i} \quad \text{for } j \leq n-1, \ \text{ and}$$

$$\partial_{x_n} = \varphi^{-1} \sum_{i=1}^{n-1} \big(\varphi'\,\varphi^{-1}\, h_i + \frac{\partial h_i}{\partial x_n}\big)\, \partial_{y_i} + \varphi^{-1}\, \partial_{y_n} \,.$$

Hence our assertion is true for $|\alpha| = 1$.

Suppose that (9.1.6) is valid for a given multi-index α. Then for $j = 1, \dots, n-1$ we have

$$\partial_{x_j} \partial_x^\alpha = \varphi^{-|\alpha|} \sum_{|\gamma| \leq |\alpha|} \big(\partial_{x_j} A_{\alpha,\gamma}(x)\big)\, \partial_y^\gamma + \varphi^{-|\alpha|} \sum_{|\gamma| \leq |\alpha|} A_{\alpha,\gamma}(x)\, \partial_{x_j} \partial_y^\gamma$$

Here $\varphi \partial_{x_j} A_{\alpha,\gamma}$ is a linear combination of functions of the form (9.1.7). Thus, we obtain the desired representation for the derivatives $\partial_{x_j} \partial_x^\alpha$, $j = 1, \dots, n-1$. An analogous representation holds for $\partial_{x_n} \partial_x^\alpha$. The lemma is proved. ■

As a consequence of (9.1.1) and Lemma 9.1.2, the following assertion holds.

LEMMA 9.1.3. *Let u be an arbitrary function from $\mathcal{W}^l_{2,\beta,l-n/2}(\mathcal{G})$ and let v be the function which arises from u via the coordinate change $x \to y$. Furthermore, let ζ be a smooth function equal to zero for $y_n < 0$ and to one for $y_n > 1$. Then $\zeta v \in \mathcal{W}^l_{2,\beta}(\mathcal{C})$, where $\mathcal{C} = \Omega \times \mathbb{R}$, and*

$$\|\zeta v\|_{\mathcal{W}^l_{2,\beta}(\mathcal{C})} \le c\, \|u\|_{\mathcal{W}^l_{2,\beta,l-n/2}(\mathcal{G})}$$

with a constant c independent of u. An analogous assertion is true for the spaces $\mathcal{W}^{l-1/2}_{2,\beta,l-n/2}(\partial\mathcal{G})$ and $\tilde{\mathcal{W}}^{l,k}_{2,\beta,l-n/2}(\mathcal{G})$.

Using the fact that $\varphi(x_n)^{k-\alpha} D^k_{x_n} \varphi(x_n)^\alpha$ is bounded on $[1, +\infty)$ for nonnegative integer k and real α, we obtain the following lemma.

LEMMA 9.1.4. *If $u \in \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$, then $\varphi(x_n)^\alpha u \in \mathcal{W}^l_{2,\beta,\gamma-\alpha}(\mathcal{G})$. Analogously, if $u \in \mathcal{W}^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$, then $\varphi(x_n)^\alpha u \in \mathcal{W}^{l-1/2}_{2,\beta,\gamma-\alpha}(\partial\mathcal{G})$.*

9.1.3. Solvability of elliptic boundary value problems in a quasicylinder. We consider the boundary value problem

$$L(x, \partial_x)\, u = f \quad \text{in } \mathcal{G}, \tag{9.1.8}$$

$$B(x, \partial_x)\, u + C(x, \partial_x)\, \underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}, \tag{9.1.9}$$

where again L is an elliptic differential operator of order $2m$, B is a vector of differential operators B_k of order not greater than μ_k, and C is a matrix of tangential differential operators $C_{k,j}$ of order not greater than $\mu_k + \tau_j$, $k = 1, \dots, m + J$, $j = 1, \dots, J$. Suppose that the orders of the differential operators B_k are less than $2m$. Then there exists a $(m + J) \times J$-matrix Q of tangential operators on $\partial\mathcal{G}$ such that

$$Bu|_{\partial\mathcal{G}} = Q(x, \partial_x) \cdot \mathcal{D}u|_{\partial\mathcal{G}} \tag{9.1.10}$$

for arbitrary $u \in C^\infty(\overline{\mathcal{G}})$.

We denote the coefficients of the differential operators L, B_k, $C_{k,j}$ by a_α, $b_{k;\alpha}$ and $c_{k,j;\alpha}$, respectively, and assume that the functions

$$\begin{cases} \hat{a}_\alpha(y) = \varphi\big(x_n(y_n)\big)^{2m-|\alpha|}\, a_\alpha\big(x(y)\big) \\ \hat{b}_{k;\alpha}(y) = \varphi\big(x_n(y_n)\big)^{\mu_k-|\alpha|}\, b_{k;\alpha}\big(x(y)\big) \\ \hat{c}_{k,j;\alpha}(y) = \varphi\big(x_n(y_n)\big)^{\mu_k+\tau_j-|\alpha|}\, c_{k,j;\alpha}\big(x(y)\big) \end{cases} \tag{9.1.11}$$

satisfy the condition of stabilization for $y_n \to +\infty$ in Section 5.5.

Using the properties of the function φ, we obtain the following assertions.

LEMMA 9.1.5. *Let $L(x, \partial_x)$ be a differential operator of order $2m$ on $\mathcal{G}$ with coefficients a_α infinitely differentiable in $\overline{\mathcal{G}}$ such that the functions*

$$\hat{a}_\alpha(y) = \varphi\big(x_n(y_n)\big)^{2m-|\alpha|}\, a_\alpha\big(x(y)\big)$$

stabilize for $y_n \to +\infty$ and let δ be an arbitrary real number. Then the coefficients of the differential operator $\hat{L}(y, \partial_y)$ which arises from the differential operator $\varphi(x_n)^{2m-\delta} L(x, \partial_x)\, \varphi(x_n)^\delta$ via the coordinate change $x \to y$ stabilize for $y_n \to \infty$.

Analogous assertions are valid for the operators $\hat{B}_k(y, \partial_y)$, $\hat{C}_{k,j}(y, \partial_y)$ arising from the operators $\varphi^{\mu_k-\delta} B_k(x, \partial_x)\, \varphi^\delta$ and $\varphi^{\mu_k-\delta} C_{k,j}(x, \partial_x)\, \varphi^{\tau_j+\delta}$, respectively.

Proof: Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an arbitrary multi-index. Then

$$\varphi^{2m-\delta}\, a_\alpha(x)\, \partial_x^\alpha\, (\varphi^\delta u) = \varphi^{2m-\delta}\, a_\alpha(x) \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} (\partial_x^{\alpha-\gamma}\varphi^\delta)\, \partial_x^\gamma u\,.$$

Here

$$\partial_x^\gamma = \varphi(x_n)^{-|\gamma|} \sum_{|\gamma'|\le|\gamma|} A_{\gamma,\gamma'}(x)\, \partial_y^{\gamma'}$$

with coefficients $A_{\gamma,\gamma'}$ which are linear combinations of functions of the form (9.1.7) (see Lemma 9.1.2), and $\partial_x^{\alpha-\gamma}\varphi^\delta$ has the form

$$\partial_x^{\alpha-\gamma}\varphi(x_n)^\delta = \varphi(x_n)^{\delta-|\alpha|+|\gamma|}\, \Phi(x_n),$$

where Φ is a linear combination of functions $\prod \varphi^{k-1}\varphi^{(k)}$ for $\alpha-\gamma \ne 0$. Hence we obtain the representation

$$\varphi^{2m-\delta} L(x,\partial_x)\,(\varphi^\delta u) = \sum_{|\alpha|\le 2m} \varphi(x_n)^{2m-|\alpha|}\, a_\alpha(x) \sum_{|\gamma|\le|\alpha|} a_{\alpha,\gamma}(x)\, \partial_y^\gamma\,,$$

where $a_{\alpha,\gamma}$ are linear combinations of functions of the form (9.1.7). According to conditions (i) and (v), the functions $a_{\alpha,\gamma}$ have finite limits for $x_n \to +\infty$ (uniform with respect to $y' \in \Omega$. Consequently, the coefficients of the operator $\hat L(y,\partial_y)$ stabilize for $y_n \to +\infty$. Similarly this can be proved for the operators $\hat B_k$ and $\hat C_{k,j}$. ■

We assume that the coefficients (9.1.11) stabilize for $y_n \to +\infty$. If we replace the coefficients of the differential operators $\hat L(y,\partial_y)$, $\hat B_k(y,\partial_y)$, $\hat C_{k,j}(y,\partial_y)$ introduced in the previous lemma for $\delta = l-\gamma-n/2$ by their limits for $y_n \to +\infty$, we obtain differential operators $\hat L^{(l-\gamma)}(y',\partial_{y'},\partial_{y_n})$, $\hat B_k^{(l-\gamma)}(y',\partial_{y'},\partial_{y_n})$, and $\hat C_{k,j}^{(l-\gamma)}(y',\partial_{y'},\partial_{y_n})$ with coefficients depending only on $y' = (y_1,\dots,y_{n-1})$. We denote the operator of the parameter-depending boundary value problem

$$\hat L^{(l-\gamma)}(y',\partial_{y'},\lambda)\, u = f, \quad y' \in \Omega,$$

$$\hat B_k^{(l-\gamma)}(y',\partial_{y'},\lambda)\, u + \sum_{j=1}^{J} \hat C_{k,j}^{(l-\gamma)}(y',\partial_{y'},\lambda)\, u_j = g_k \quad y' \in \partial\Omega,\ k = 1,\dots,m+J,$$

by $\mathfrak{A}_{l-\gamma}(\lambda)$. If the boundary value problem (9.1.8), (9.1.9) is elliptic, then the pencil $\mathfrak{A}_{l-\gamma}$ is Fredholm, and its spectrum consists of a countable number of isolated eigenvalues.

REMARK 9.1.1. In general, the operator $\mathfrak{A}_{l-\gamma}(\lambda)$ depends on the difference of l and γ. However, if $\varphi'(t) \to 0$ as $t \to +\infty$, then $\mathfrak{A}_{l-\gamma}$ is independent of $l-\gamma$.

If, for example,

$$\lim_{t\to+\infty} \varphi'(t) = 0 \quad \text{and} \quad h_j(x) = x_j \ \text{ for } j = 1,\dots,n-1,$$

then the coefficients of the operators $\hat L^{(l-\gamma)}$, $\hat B^{(l-\gamma)}$, and $\hat C_{k,j}^{(l-\gamma)}$ are the limits of the functions (9.1.11) for $y_n \to +\infty$.

Indeed, if $\varphi'(t) \to 0$ as $t \to \infty$, then for every multi-index $\alpha \ne 0$ the function $\partial_x^\alpha \varphi(x_n)^\delta$ tends to zero as $x_n \to +\infty$. Hence the limits of the coefficients of the operators $\varphi^{2m-\delta} L(x,\partial_x)\varphi^\delta$ and $\varphi^{2m} L(x,\partial_x)$ coincide.

We identify the operator $(u, \underline{u}) \to (Lu, Bu|_{\partial\mathcal{G}} + C\underline{u})$ of the boundary value problem (9.1.8), (9.1.9) with the operator

$$\mathcal{A} : \big(u, \mathcal{D}u|_{\partial\mathcal{G}}, \underline{u}\big) \to \big(Lu, Bu|_{\partial\mathcal{G}} + C\underline{u}\big)$$

THEOREM 9.1.1. 1) *Suppose the coefficients (9.1.11) stabilize for $y_n \to +\infty$. Then the operator $\mathcal{A}$ of the boundary value problem continuously maps the space*

$$\tilde{\mathcal{W}}^{l,2m}_{2,\beta,\gamma}(\mathcal{G}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G}) \tag{9.1.12}$$

into

$$\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta,\gamma}(\mathcal{G}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G}) \tag{9.1.13}$$

for arbitrary integer l. Here by $\mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$, $\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$ we have denoted the products of the spaces $\mathcal{W}^{l+\tau_j-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$ and $\mathcal{W}^{l-\mu_k-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$, respectively.

2) *If, moreover, the boundary value problem (9.1.8), (9.1.9) is elliptic and the line $\operatorname{Re}\lambda = -\beta$ is free of eigenvalues of the pencil $\mathfrak{A}_{l-\gamma}$, then every solution $(u, \underline{\phi}, \underline{u}) \in \tilde{\mathcal{W}}^{l,2m}_{2,\beta,\gamma}(\mathcal{G}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$ of the equation $\mathcal{A}(u, \underline{\phi}, \underline{u}) = (f, \underline{g})$ satisfies the estimate*

$$\begin{aligned}\|(u, \underline{\phi}, \underline{u})\|_{l,\beta,\gamma} \ \le\ & c\Big(\|f\|_{\mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})} + \|\underline{g}\|_{\mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})} \\ & + \|\zeta(u, \underline{\phi}, \underline{u})\|_{l-1,\beta,\gamma}\Big),\end{aligned} \tag{9.1.14}$$

where $\|\cdot\|_{l,\beta,\gamma}$ denotes the norm in (9.1.12) and ζ is a smooth function on $\overline{\mathcal{G}}$ with compact support.

Proof: 1) Let $(u, \underline{\phi}, \underline{u})$ be a given element of the space (9.1.12), where $l \ge 2m$. Furthermore, let $f = Lu$ and $\underline{g} = Bu|_{\partial\mathcal{G}} + C\underline{u}$. We show that $(f, \underline{g})$ is an element of the space (9.1.13) and the norm of $(f, \underline{g})$ in the space (9.1.13) can be estimated by the norm of $(u, \underline{\phi}, \underline{u})$ in (9.1.12). For functions with compact support this is obvious. Therefore, we may restrict ourselves to the case, when the support of $(u, \underline{\phi}, \underline{u})$ is contained in $\{x \in \overline{\mathcal{G}} : x_n \ge 1\}$.

We set $\delta = l - \gamma - n/2$. Passing in (9.1.8), (9.1.9) to the coordinates y, we obtain (with the notation of Lemma 9.1.5)

$$\hat{L}(y, \partial_y)\, \varphi^{-\delta} u = \varphi^{2m-\delta} f \quad \text{in } \mathcal{C} \tag{9.1.15}$$

$$\hat{B}_k(y, \partial_y)\, \varphi^{-\delta} u + \sum_{j=1}^{J} \hat{C}_{k,j}(y, \partial_y)\, \varphi^{-\delta-\tau_j} u_j = \varphi^{\mu_k-\delta}\, g_k \quad \text{for } y \in \partial\mathcal{C}, \tag{9.1.16}$$

$k = 1, \dots, m + J$. According to Lemma 9.1.4, we have $\varphi^{-\delta} u \in \mathcal{W}^{l}_{2,\beta,l-n/2}(\mathcal{G})$ and $\varphi^{-\delta-\tau_j} u_j \in \mathcal{W}^{l+\tau_j-1/2}_{2,\beta,l+\tau_j-n/2}(\partial\mathcal{G})$. Passing to the y-coordinates, we get $\varphi^{-\delta} u \in \mathcal{W}^{l}_{2,\beta}(\mathcal{C})$ and $\varphi^{-\delta-\tau_j} u_j \in \mathcal{W}^{l+\tau_j-1/2}_{2,\beta}(\partial\mathcal{C})$ (see Lemma 9.1.3). Hence by the continuity of the operators $\hat{L}$, $\hat{B}_k$ and $\hat{C}_{k,j}$ (see Section 5.5), it holds $\varphi^{2m-\delta} f \in \mathcal{W}^{l-2m}_{2,\beta}(\mathcal{C})$ and $\varphi^{\mu_k-\delta} g_k \in \mathcal{W}^{l-\mu_k-1/2}_{2,\beta}(\partial\mathcal{C})$. In the x-coordinates this means that $f \in \mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})$ and $g_k \in \mathcal{W}^{l-\mu_k-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$. Furthermore, the norm of $(f, \underline{g})$ in (9.1.13) can be estimated by the norm of $(u, \underline{\phi}, \underline{u})$ in the space (9.1.12). This proves the continuity of the operator $\mathcal{A}$ from (9.1.12) into (9.1.13) for $l \ge 2m$.

Under our assumptions on the coefficients, the operator of the boundary value problem (9.1.15), (9.1.16) can be extended to a continuous mapping

$$\tilde{\mathcal{W}}_{2,\beta}^{l,2m}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l+\underline{\tau}-1/2}(\partial\mathcal{C}) \to \tilde{\mathcal{W}}_{2,\beta}^{l-2m,0}(\mathcal{C}) \times \mathcal{W}_{2,\beta}^{l-\underline{\mu}-1/2}(\partial\mathcal{C}),$$

with $l < 2m$ (see Theorem 5.5.1). Hence the operator

$$\mathcal{A} : \tilde{\mathcal{W}}_{2,\beta,\gamma-l+2m}^{2m,2m}(\mathcal{G}) \times \mathcal{W}_{2,\beta,\gamma-l+2m}^{l+\underline{\tau}-1/2}(\partial\mathcal{G}) \to \mathcal{W}_{2,\beta,\gamma-l+2m}^{0}(\mathcal{G}) \times \mathcal{W}_{2,\beta,\gamma-l+2m}^{2m-\underline{\mu}-1/2}(\partial\mathcal{G})$$

of the boundary value problem (9.1.8), (9.1.9) can be extended to a continuous mapping from the space (9.1.12) with $l < 2m$ into (9.1.13).

2) For functions with compact support inequality (9.1.14) follows from Theorem 3.2.3, while for functions vanishing for $x_n < 1$ this inequality is a consequence of Lemma 5.5.4. The theorem is proved. ■

It follows from Theorem 9.1.1 that the kernel of the operator $\mathcal{A}$ is finite-dimensional and the range is closed. With the same technique as in Chapter 6 it can be shown that $\dim \operatorname{coker} \mathcal{A} < \infty$. Thus, the following theorem holds.

THEOREM 9.1.2. *Suppose that the boundary value problem (9.1.8), (9.1.9) is elliptic, the coefficients (9.1.11) stabilize for* $y_n \to +\infty$, *and the line* $\operatorname{Re}\lambda = -\beta$ *does not contain eigenvalues of the pencil* $\mathfrak{A}_{l-\gamma}$. *Then the operator* $\mathcal{A}$ *is Fredholm from (9.1.12) into (9.1.13).*

9.1.4. The structure of the solutions. In the proof of Theorem 9.1.1 we have used the fact that the coordinate change $x \to y$ transforms the boundary value problem (9.1.8), (9.1.9) into problem (9.1.15), (9.1.16). If we apply Theorem 5.5.3 to the last problem, we obtain the following result.

THEOREM 9.1.3. *Let* $(u, \underline{u}) \in \mathcal{W}_{2,\beta_1,\gamma_1}^{l_1}(\mathcal{G}) \times \mathcal{W}_{2,\beta_1,\gamma_1}^{l_1+\underline{\tau}-1/2}(\partial\mathcal{G})$ *be a solution of the boundary value problem (9.1.8), (9.1.9), where* $(f, \underline{g})$ *belongs to the intersection of the spaces*

$$\mathcal{W}_{2,\beta_i,\gamma_i}^{l_i-2m}(\mathcal{G}) \times \mathcal{W}_{2,\beta_i,\gamma_i}^{l_i-\underline{\mu}-1/2}(\partial\mathcal{G}) \tag{9.1.17}$$

$(i = 1, 2)$, $\beta_1 < \beta_2$, $l_i \geq 2m$, $l_1 - \gamma_1 = l_2 - \gamma_2$. *We assume that the boundary value problem (9.1.8), (9.1.9) is elliptic, the coefficients (9.1.11) stabilize for* $y_n \to +\infty$, *the lines* $\operatorname{Re}\lambda = -\beta_i$ *are free of eigenvalues of the pencil* $\mathfrak{A}_{l_1-\gamma_1}$, *and the strip* $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$ *contains exactly s eigenvalues counting the algebraic multiplicity. Then the solution* $(u, \underline{u})$ *admits the representation*

$$(u, \underline{u}) = \sum_{k=1}^{s} c_j\, U_j + (w, \underline{w}),$$

for sufficiently large $x_n > T$, *where* $(w, \underline{w}) \in \mathcal{W}_{2,\beta_2,\gamma_2}^{l_2}(\mathcal{G}) \times \mathcal{W}_{2,\beta_2,\gamma_2}^{l_2+\underline{\tau}-1/2}(\partial\mathcal{G})$ *and* U_j *are elements of the space* $\mathcal{W}_{2,\beta_1,\gamma_1}^{l_1}(\mathcal{G}) \times \mathcal{W}_{2,\beta_1,\gamma_1}^{l_1+\underline{\tau}-1/2}(\partial\mathcal{G})$ *which are linearly independent modulo* $\mathcal{W}_{2,\beta_2,\gamma_2}^{l_2}(\mathcal{G}) \times \mathcal{W}_{2,\beta_2,\gamma_2}^{l_2+\underline{\tau}-1/2}(\partial\mathcal{G})$ *and satisfy the homogeneous equations (9.1.8), (9.1.9) for* $x_n > T$.

As a consequence of Theorem 9.1.3, the following regularity assertion for the solutions holds.

COROLLARY 9.1.1. *Let* $(u,\underline{u}) \in \mathcal{W}^{l_1}_{2,\beta_1,\gamma_1}(\mathcal{G}) \times \mathcal{W}^{l_1+\tau-1/2}_{2,\beta_1,\gamma_1}(\partial\mathcal{G})$ *be a solution of the boundary value problem (9.1.8), (9.1.9), where* $(f,\underline{g})$ *belongs to the intersection of the spaces (9.1.17),* $i=1,\,2$, $\beta_1<\beta_2$, $l_i \ge 2m$, $l_1-\gamma_1 = l_2-\gamma_2$. *We assume that the boundary value problem (9.1.8), (9.1.9) is elliptic, the coefficients (9.1.11) stabilize for* $y_n \to +\infty$, *and the strip* $-\beta_2 \le \operatorname{Re}\lambda \le -\beta_1$ *does not contain eigenvalues of the pencil* $\mathfrak{A}_{l_1-\gamma_1}$. *Then* $(u,\underline{u}) \in \mathcal{W}^{l_2}_{2,\beta_2,\gamma_2}(\mathcal{G}) \times \mathcal{W}^{l_2+\tau-1/2}_{2,\beta_2,\gamma_2}(\partial\mathcal{G})$.

Example. We consider the Dirichlet problem

$$(9.1.18) \qquad L(\partial_x)u = f \ \text{ in } \mathcal{G}, \qquad D_\nu^{k-1}u = 0 \ \text{ on } \partial\mathcal{G},\ k=1,\dots,m,$$

for a $2m$ order differential operator $L(\partial_x) = L(\partial_{x'},\partial_{x_n})$ with constant coefficients. Suppose that there exists a positive constant c such that

$$|L(\xi)| \ge c\,|\xi|^{2m} \qquad \text{for all } \xi \in \mathbb{R}^n\,.$$

This assumption guarantees the existence of a uniquely determined weak solution $u \in \overset{\circ}{L}{}_2^m(\mathcal{G})$ for every given functional f on $\overset{\circ}{L}{}_2^m(\mathcal{G})$. Here $\overset{\circ}{L}{}_2^m(\mathcal{G})$ is the closure of $C_0^\infty(\mathcal{G})$ with respect to the norm

$$\|u\|_{L_2^m(\mathcal{G})} = \Big(\sum_{|\alpha|=m} \|D^\alpha u\|^2_{L_2(\mathcal{G})}\Big)^{1/2}.$$

Furthermore, we assume that $\mathcal{G}$ is a domain in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{G}$ such that the set $\{x\in\mathcal{G} : x_n<1\}$ is bounded and the set $\{x\in\mathcal{G} : x_n>1\}$ can be written as $\{x=(x',x_n)\in\mathbb{R}^n : x_n>1,\, x'/\varphi(x_n)\in\Omega\}$, where Ω is a smooth bounded domain in $\mathbb{R}^{n-1}$ and φ is a smooth positive function on $[1,\infty)$ satisfying conditions (i),(ii) in the beginning of this section. Additionally to these assumptions, we suppose that $\varphi'(t)$ tends to zero as $t\to+\infty$.

The coordinate change (9.1.4) takes the set $\{x\in\mathcal{G} : x_n>1\}$ onto the half-cylinder $\mathcal{C}_+ = \Omega\times\mathbb{R}_+$. We consider the operator pencil $\mathfrak{A}_{l-\gamma}$ which appears in the assumptions of Theorems 9.1.1 -9.1.3. Under our conditions on the domain $\mathcal{G}$ and on the function φ, the operator $\mathfrak{A}_{l-\gamma}(\lambda)$ corresponds to the Dirichlet problem

$$(9.1.19) \qquad L(\partial_{x'},\lambda)\,u = f \ \text{ in } \Omega, \qquad D_\nu^{k-1}u = 0 \ \text{ on } \partial\Omega, \quad k=1,\dots,m,$$

and does not depend on $l-\gamma$ (see Remark 9.1.1).

LEMMA 9.1.6. *The space* $\overset{\circ}{L}{}_2^m(\mathcal{G})$ *is continuously imbedded into* $\mathcal{W}^m_{2,0,0}(\mathcal{G})$.

Proof: Let $u \in \overset{\circ}{L}{}_2^m(\mathcal{G})$. We denote by Ω_t the set $\{y'\in\mathbb{R}^{n-1} : x'/\varphi(t)\in\Omega\}$. Then by the Friedrichs inequality, we have

$$\int_{\Omega_t} |u|^2\,dx' \le c\,\varphi(t)^{2k} \sum_{|\alpha'|=k}\int_{\Omega_t} |D_{x'}^{\alpha'}u|^2\,dx',$$

and, therefore,

$$\varphi(x_n)^{2(|\gamma|-m)} \int_{\Omega_{x_n}} |D_x^\gamma u|^2\,dx' \le c \sum_{|\alpha|=m}\int_{\Omega_{x_n}} |D_x^\alpha u|^2\,dx'$$

for $|\gamma|\le m$. Consequently,

$$(9.1.20) \qquad \sum_{|\gamma|\le m}\int_{\mathcal{G}} \varphi(x_n)^{2(|\gamma|-m)}\,|D_x^\gamma u|^2\,dx \le c\sum_{|\alpha|=m}\int_{\mathcal{G}} |D_x^\alpha u|^2\,dx.$$

Thus, u belongs to the space $\mathcal{W}^m_{2,0,0}(\mathcal{G})$, and

$$\|u\|_{\mathcal{W}^m_{2,0,0}(\mathcal{G})} \leq c\,\|u\|_{\overset{\circ}{L}{}^m_2(\mathcal{G})}\,.$$

The lemma is proved. ■

From Lemma 9.1.6 we conclude that $\overset{\circ}{L}{}^m_2(\mathcal{G}) \subset \mathcal{W}^0_{2,0,-m}(\mathcal{G})$ and, consequently,

$$(\overset{\circ}{L}{}^m_2(\mathcal{G}))^* \supset \mathcal{W}^0_{2,0,m}(\mathcal{G}) \supset \mathcal{W}^{l-2m}_{2,0,l-m}(\mathcal{G})$$

for $l \geq 2m$. It can be easily shown that

$$\varphi(x_n)^{l-m-\gamma}\, e^{-\beta y_n(x_n)} \longrightarrow 0$$

as $x_n \to +\infty$ if $\beta > 0$. Therefore, the space $\mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})$ is continuously imbedded into $\mathcal{W}^{l-2m}_{2,0,l-m}(\mathcal{G})$. Thus, every function $f \in \mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})$ can be considered as a linear and continuous functional on $\overset{\circ}{L}{}^m_2(\mathcal{G})$ if $l \geq 2m$, $\beta > 0$.

THEOREM 9.1.4. *Let $f \in \mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})$, $l \geq 2m$, $\beta > 0$, $\gamma \in \mathbb{R}$. We assume that no eigenvalues of problem (9.1.19) lie in the strip $-\beta \leq \operatorname{Re}\lambda \leq 0$. Then the solution $u \in \overset{\circ}{L}{}^m_2(\mathcal{G})$ belongs to the space $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ and satisfies the inequality*

$$\|u\|_{\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})} \leq c\|f\|_{\mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})}\,, \tag{9.1.21}$$

where the constant c is independent of f.

Proof: Let u be the uniquely determined weak solution in $\overset{\circ}{L}{}^m_2(\mathcal{G})$. By Theorem 4.3.3, this solution belongs to the space $W^l_2(\mathcal{G}')$ for every bounded subdomain $\mathcal{G}' \subset \mathcal{G}$. Let $v = v(y)$ be the function which arises from $\varphi^{n-m/2}u$ via the coordinate change $x \to y$. From the properties of the mapping (9.1.4) and from (9.1.20) it follows that $\varphi^{-m+n/2}v \in \mathcal{W}^m_2(\mathcal{C}_+)$. Since $\varphi^{-m+n/2}v$ is the solution of an elliptic equation in $\mathcal{C}_+$ with coefficients which stabilize for $y_n \to +\infty$, we obtain $\varphi^{-m+n/2}v \in W^l_2(\mathcal{C}_+)$ (cf. Lemma 5.5.3). This implies $u \in \mathcal{W}^l_{2,0,l-m}(\mathcal{G})$. Using the fact that

$$\varphi(x_n)^{\gamma-l+m}\, e^{-\varepsilon y_n(x_n)} \longrightarrow 0$$

as $x_n \to +\infty$ if $\varepsilon > 0$, we conclude that $u \in \mathcal{W}^l_{2,-\varepsilon,\gamma}(\mathcal{G})$ for arbitrary $\gamma \in \mathbb{R}$, $\varepsilon > 0$.

From Corollary 9.1.1 it follows that $u \in \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$. Furthermore, (9.1.21) holds. ■

9.2. Elliptic boundary value problems in cuspidal domains

Now we consider boundary value problems in a domain with a singularity of exterior cusp type. By means of a similar mapping as was used in the previous section, such domain can be transformed (in a neighbourhood of the singular point) onto a half-cylinder. Applying the results of Chapter 5, we obtain similar results as in the Section 9.1.

9.2.1. Description of the domain. Let φ be an infinitely differentiable positive function on the interval $(0,1]$ satisfying the conditions

(i) $\lim\limits_{t\to 0} \varphi(t)^{k-1}\,\varphi^{(k)}(t) < \infty \qquad$ for $k = 1, 2, \ldots\,,$

(ii) $\displaystyle\int_0^1 \frac{dt}{\varphi(t)} = +\infty$

These conditions are satisfied, for example, for the function $\varphi(t) = t^\alpha$ if $\alpha \geq 1$. Obviously, conditions (i) and (ii) imply $\varphi(0) = 0$.

Furthermore, we suppose that $\mathcal{G}$ is a bounded domain in $\mathbb{R}^n$, $\partial\mathcal{G}\backslash\{0\}$ is smooth, and there exist functions $y_j = h_j(x)$, $j = 1, \ldots, n-1$, such that

(iii) the mapping

$$y_j = \frac{h_j(x)}{\varphi(x_n)}, \quad j = 1, \ldots, n-1, \qquad y_n = \int_{x_n}^1 \frac{dt}{\varphi(t)}$$

transforms the set $\{x \in \mathcal{G} : x_n < 1\}$ onto the half-cylinder

$$\mathcal{C}_+ = \{y = (y', y_n) : y' = (y_1, \ldots, y_{n-1}) \in \Omega, \, y_n > 0\}.$$

(iv) $\det\left(\dfrac{\partial h_j}{\partial x_k}\right)_{j,k=1,\ldots,n-1} \geq \text{const} > 0$,

(v) there exist the finite limits (uniform with respect to $y' \in \Omega$)

$$\lim_{x_n \to 0} \varphi(x_n)^{|\alpha|-1} \, \partial_x^\alpha h_j(x), \qquad j = 1, \ldots, n-1;$$

for all multi-indices α.

Example: Let φ be an infinitely differentiable positive function on the interval $(0,1]$ satisfying conditions (i) and (ii). We assume that

$$\{x \in \mathcal{G} : \; x_n < 1\} = \{x \in \mathbb{R}^n : \; x_n < 1, |x'| < \varphi(x_n)\},$$

where $x' = (x_1, \ldots, x_{n-1})$ (see Figure 2).

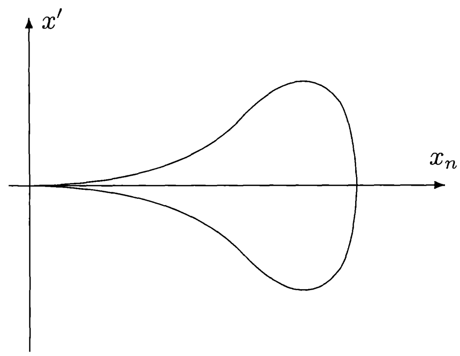


Figure 2: An exterior cusp

Then the mapping

$$y_j = \frac{x_j}{\varphi(x_n)}, \quad j = 1, \ldots, n-1, \qquad y_n = \int_{x_n}^1 \frac{dt}{\varphi(t)}$$

takes the set $\{x \in \mathcal{G} : x_n < 1\}$ onto the half-cylinder $\{y \in \mathbb{R}^n : |y'| < 1,\, y_n > 0\}$. Obviously conditions (iv) and (v) are satisfied for the functions $h_j(x) = x_j$, $j = 1, \dots, n-1$.

9.2.2. Solvability of the boundary value problem, structure of the solutions. We extend the function φ to an infinitely differentiable positive function on the interval $(0, +\infty)$. Analogously to the previous section, we define the space $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ as the closure of the set $C^\infty(\overline{\mathcal{G}}\backslash\{0\})$ with respect to the norm

$$\|u\|_{\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})} = \Big(\int\limits_{\mathcal{G}} \sum_{|\alpha|\le l} e^{2\beta y_n(x_n)}\, \varphi(x_n)^{2(\gamma-l+|\alpha|)}\, |D_x^\alpha u(x)|^2\, dx \Big)^{1/2},$$

where $y_n(x_n) = \int\limits_{x_n}^{1} \frac{dt}{\varphi(t)}$. Furthermore, we define the space $\mathcal{W}^{l-1/2}_{2,\beta,\gamma}(\mathcal{G})$ as the trace space for $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ if $l \ge 1$ and as the dual space of $\mathcal{W}^{-l+1/2}_{2,-\beta,-\gamma}(\mathcal{G})$ if $l \le 0$. The space $\tilde{\mathcal{W}}^{l,k}_{2,\beta,\gamma}(\mathcal{G})$ consists of all pairs $(u, \underline{\phi})$ such that

$$u \in \begin{cases} \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G}) & \text{if } l \ge 0, \\ \mathcal{W}^{-l}_{2,-\beta,-\gamma}(\mathcal{G})^* & \text{if } l < 0, \end{cases} \qquad \underline{\phi} = (\phi_1, \dots, \phi_k) \in \prod_{j=1}^k \mathcal{W}^{l-j+1/2}_{2,\beta,\gamma}(\partial\mathcal{G}),$$

and $\phi_j = D_\nu^{j-1} u|_{\partial\mathcal{G}\backslash\{0\}}$ for $j \le \min(k, l)$.

We consider the boundary value (9.1.8), (9.1.9) in the domain $\mathcal{G}$ described above. As in Section 9.1 we suppose that $\operatorname{ord} B_k < 2m$ for $k = 1, \dots, m+J$. Furthermore, the same conditions as in Section 9.1 are imposed on the coefficients of the operators L, B_k, and $C_{k,j}$. Then, analogously to Lemma 9.1.5, the coefficients of the operators

$$\begin{aligned} \hat{L}(y, \partial_y) &= \varphi(x_n)^{2m-\delta}\, L(x, \partial_x)\, \varphi(x_n)^\delta, \\ \hat{B}_k(y, \partial_y) &= \varphi(x_n)^{\mu_k-\delta}\, B_k(x, \partial_x)\, \varphi(x_n)^\delta, \\ \hat{C}_{k,j}(y, \partial_y) &= \varphi(x_n)^{\mu_k-\delta}\, C_{k,j}(x, \partial_x)\, \varphi(x_n)^{\tau_j+\delta} \end{aligned}$$

stabilize for $y_n \to +\infty$. Clearly, these operators and also the limit operators which are obtained if we replace the coefficients by there limits for $y_n \to +\infty$ depend on the number δ. We denote these limit operators for the number $\delta = l - \gamma - n/2$ by $\hat{L}^{(l-\gamma)}(y', \partial_y, \partial_{y_n})$, $\hat{B}_k^{(l-\gamma)}(y', \partial_y, \partial_{y_n})$, and $\hat{C}_{k,j}^{(l-\gamma)}(y', \partial_y, \partial_{y_n})$, respectively. Furthermore, we denote the operator of the parameter-depending boundary value problem

$$\hat{L}^{(l-\gamma)}(y', \partial_{y'}, \lambda)\, u = f \quad \text{in } \Omega,$$

$$\hat{B}_k^{(l-\gamma)}(y', \partial_{y'}, \lambda)\, u + \sum_{j=1}^J \hat{C}_{k,j}^{(l-\gamma)}(y', \partial_{y'}, \lambda)\, u_j = g_k \quad \text{on } \partial\Omega,\ k = 1, \dots, m+J,$$

by $\mathfrak{A}_{l-\gamma}(\lambda)$. Note that the operator pencil $\mathfrak{A}_{l-\gamma}$ is independent of $l-\gamma$ if $\varphi'(0) = 0$ (cf. Remark 9.1.1).

Repeating the proofs of Theorems 9.1.1 -9.1.3, we obtain the following theorems.

THEOREM 9.2.1. *Suppose that the boundary value problem (9.1.8), (9.1.9) is elliptic, the coefficients (9.1.11) stabilize for* $y_n \to +\infty$, *and the line* $\operatorname{Re}\lambda = -\beta$ *does* *not contain eigenvalues of the pencil $\mathfrak{A}_{l-\gamma}$. Then the operator $\mathcal{A}$ of the boundary value problem (9.1.8), (9.1.9) is Fredholm from from (9.1.12) into (9.1.13).*

THEOREM 9.2.2. *Let $(u, \underline{u}) \in \mathcal{W}^{l_1}_{2,\beta_1,\gamma_1}(\mathcal{G}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1,\gamma_1}(\partial\mathcal{G})$ be a solution of the boundary value problem (9.1.8), (9.1.9), where $(f, \underline{g})$ belongs to the intersection of the spaces (9.1.17), $i = 1, 2$, $\beta_1 < \beta_2$, $l_i \geq 2m$, $l_1 - \gamma_1 = l_2 - \gamma_2$. We assume that the boundary value problem (9.1.8), (9.1.9) is elliptic, the coefficients (9.1.11) stabilize for $y_n \to +\infty$, the lines $\mathrm{Re}\,\lambda = -\beta_i$ are free of eigenvalues of the pencil $\mathfrak{A}_{l_1-\gamma_1}$, and the strip $-\beta_2 < \mathrm{Re}\,\lambda < -\beta_1$ contains exactly s eigenvalues counting the algebraic multiplicity. Then the solution $(u, \underline{u})$ admits the representation*

$$(u, \underline{u}) = \sum_{k=1}^{s} c_j\, U_j + (w, \underline{w}),$$

for sufficiently small $x_n < \varepsilon$, where $(w, \underline{w}) \in \mathcal{W}^{l_2}_{2,\beta_2,\gamma_2}(\mathcal{G}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2,\gamma_2}(\partial\mathcal{G})$ and U_j are elements of the space $\mathcal{W}^{l_1}_{2,\beta_1,\gamma_1}(\mathcal{G}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1,\gamma_1}(\partial\mathcal{G})$ which are linearly independent modulo $\mathcal{W}^{l_2}_{2,\beta_2,\gamma_2}(\mathcal{G}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2,\gamma_2}(\partial\mathcal{G})$ and satisfy the homogeneous equations (9.1.8), (9.1.9) for $x_n < \varepsilon$.

In particular, Theorem 9.2.2 contains a regularity assertion for the solutions analogous to that given in Corollary 9.1.1.

Example: We consider the Dirichlet problem

$$(9.2.1) \qquad L(\partial_x)\, u = f \ \text{ in } \mathcal{G}, \qquad D_\nu^{k-1} u = 0 \ \text{ on } \partial\mathcal{G}\backslash\{0\},\ k = 1, \dots, m,$$

for a differential operator $L(\partial_x)$ of order $2m$ with constant coefficients. As in Section 9.1, we suppose that the inequality $|L(\xi)| \geq c\,|\xi|^{2m}$ with a positive constant c is satisfied for all $\xi \in \mathbb{R}^n$. Furthermore, we assume that the domain $\mathcal{G}$ is bounded, $\partial\mathcal{G}\backslash\{0\}$ is smooth, and the set $\{x \in \mathcal{G} : x_n < 1\}$ coincides with $\{x = (x', x_n) \in \mathbb{R}^n : x_n < 1,\ x'/\varphi(x_n) \in \Omega\}$, where φ is a smooth positive functions on the interval $(0, 1]$ satisfying conditions (i), (ii) and the condition $\varphi'(0) = 0$. Then analogously to Theorem 9.1.4, the following assertion holds:

Let $u \in \overset{\circ}{W}{}^m_2(\mathcal{G})$ be the solution of the Dirichlet problem (9.2.1), where $f \in \mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})$, $l \geq 2m$, $\beta > 0$, $\gamma \in \mathbb{R}$. If no eigenvalues of the problem

$$L(\partial_{x'}, \lambda)\, u = 0 \ \text{ in } \Omega, \quad u = D_\nu u = \cdots = D_\nu^{m-1} u = 0 \ \text{ on } \partial\Omega$$

are contained in the strip $-\beta \leq \mathrm{Re}\,\lambda \leq 0$, then $u \in \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$.

9.3. Variants and extensions

In this section we consider boundary value problems in domains which can be transformed by means of a more general mapping as was used in Section 9.1 onto a half-cylinder.

Moreover, we consider the case when the change of coordinates $x \to (r, \omega)$, where r, ω are spherical coordinates, leads to a boundary value problem in a quasicylindrical domain.

9.3.1. More general mappings.

Conditions on the domain. Let $\mathcal{G}$ be a domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{G}$ and let $\mathcal{G}_0$ a an open subset of $\mathcal{G}$ such that $\overline{\mathcal{G}}\backslash\mathcal{G}_0$ is compact. We suppose there exists a diffeomorphism κ mapping $\mathcal{G}_0$ onto the half-cylinder

$$\mathcal{C}_+ = \{y = (y', y_n) : y' = (y_1, \dots , y_{n-1}) \in \Omega,\ y_n > 0\}.$$

Here Ω is a bounded domain in $\mathbb{R}^{n-1}$ with smooth boundary $\partial\Omega$. Let the mapping $\kappa : x \to y$ satisfy the following conditions:

(i) $\Big|\det \dfrac{\partial x_j}{\partial y_k}\Big| \ge c\,\psi(y)^n$ on $\mathcal{C}_+$, where c is a positive constant and ψ is a positive infinitely differentiable function on $\overline{\mathcal{C}}_+$ such that the functions $\psi^{-1}\,\partial_y^\alpha\psi$ have finite limits as $y_n \to +\infty$ (uniform with respect to y'.)

(ii) The functions $\psi^{-1}\,\partial_y^\alpha x_j$ have finite limits als $y_n \to +\infty$ (uniform with respect to y'.)

Then, analogously to Lemma 9.1.2, the following lemma holds.

LEMMA 9.3.1. *The derivatives with respect to y of the coefficients of the differential operator*

$$P_\alpha(y, \partial_y) \stackrel{def}{=} \psi^{|\alpha|}\,\partial_x^\alpha$$

stabilize for $y_n \to +\infty$.

COROLLARY 9.3.1. *The functions $y \to \partial_y^\alpha(\psi^{|\gamma|}\,\partial_x^\gamma y_j)$ and $y \to \partial_y^\alpha(\psi^{|\gamma|-1}\,\partial_x^\gamma\psi)$ stabilize as $y_n \to +\infty$.*

Proof: It suffices to apply the operator P_α from Lemma 9.3.1 to the functions y_j and f and to use Lemma 9.3.1 and conditions (i), (ii). ■

Weighted Sobolev spaces. We introduce the weighted Sobolev space $\mathcal{W}^l_{2,\beta,\gamma}$. To this end, we extend the functions $\psi(y(x))$ and $y_n(x)$ with preservation of smoothness to $\overline{\mathcal{G}}$. Here the extension of ψ is assumed to be positive. Then $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ is defined for integer $l \ge 0$ and real β, γ as the space with the norm

$$\|u\|_{\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})} = \Big(\int_{\mathcal{G}} \sum_{|\alpha|\le l} e^{2\beta y_n(x)}\,\psi^{2(\gamma-l+|\alpha|)}\,|D_x^\alpha u(x)|^2\,dx\Big)^{1/2}.$$

Let $\mathcal{W}^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$ be the trace space for $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ if $l \ge 0$ and the dual space of $\mathcal{W}^{-l+1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$ if $l \le 0$. Finally, for arbitrary integer l and nonnegative integer k we define the space $\tilde{\mathcal{W}}^{l,k}_{2,\beta,\gamma}(\mathcal{G})$ as the set of all pairs $(u, \underline{\phi})$ such that

$$u \in \begin{cases} \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G}) & \text{if } l \ge 0, \\ \mathcal{W}^{-l}_{2,-\beta,-\gamma}(\mathcal{G})^* & \text{if } l < 0, \end{cases}$$

$$\underline{\phi} = (\phi_1, \dots , \phi_k) \in \prod_{j=1}^k \mathcal{W}^{l-j+1/2}_{2,\beta,\gamma}(\partial\mathcal{G}), \qquad \phi_j = D_\nu^{j-1}u \ \text{ for } j \le \min(l, k).$$

Solvability of elliptic boundary value problems, structure of the solutions. We consider the boundary value problem

$$L(x,\partial_x)\,u = f \quad \text{in } \mathcal{G}, \tag{9.3.1}$$

$$B(x,\partial_x)\,u + C(x,\partial_x)\,\underline{u} = \underline{g} \quad \text{on } \partial\mathcal{G}, \tag{9.3.2}$$

where L is an elliptic differential operator of order $2m$, B is a vector of differential operators B_k of order not greater than μ_k, and C is a matrix of tangential differential operators $C_{k,j}$ of order not greater than $\mu_k+\tau_j$, $k=1,\dots,m+J$, $j=1,\dots,J$. Suppose that the orders of the differential operators B_k are less than $2m$. We denote the coefficients of the differential operators L, B_k, and $C_{k,j}$, by a_α, $b_{k;\alpha}$, and $c_{k,j;\alpha}$, respectively, and assume that the functions

$$\left\{\begin{array}{l} \hat{a}_\alpha(y) = \psi(y)^{2m-|\alpha|}\,a_\alpha\big(x(y)\big) \\ \hat{b}_{k;\alpha}(y) = \psi(y)^{\mu_k-|\alpha|}\,b_{k;\alpha}\big(x(y)\big) \\ \hat{c}_{k,j;\alpha}(y) = \psi(y)^{\mu_k+\tau_j-|\alpha|}\,c_{k,j;\alpha}\big(x(y)\big) \end{array}\right. \tag{9.3.3}$$

stabilize for $y_n \to +\infty$. Then, analogously to Lemma 9.1.5, the following assertions hold.

LEMMA 9.3.2. *Let δ be an arbitrary complex number. Then the coefficients of the operators which arise from $\psi^{2m-\delta}\,L\,\psi^\delta$, $\psi^{\mu_k-\delta}B_k\,\psi^\delta$, and $\psi^{\mu_k-\delta}C_{k,j}\psi^{\tau_j+\delta}$ via the change of variables $x\to y$ stabilize for $y_n\to+\infty$.*

We set $\delta = l-\gamma-n/2$ and denote the operators introduced in the Lemma 9.3.2 by $L^{(l-\gamma)}(y,\partial_y)$ $B_k^{(l-\gamma)}(y,\partial_y)$, and $C_{k,j}^{(l-\gamma)}(y,\partial_y)$, respectively. Replacing the coefficients of these operators by their limits for $y_n\to+\infty$, we obtain the differential operators $\hat{L}^{(l-\gamma)}(y',\partial_{y'},\partial_{y_n})$, $\hat{B}_k^{(l-\gamma)}(y',\partial_{y'},\partial_{y_n})$, and $\hat{C}_{k,j}^{(l-\gamma)}(y',\partial_{y'},\partial_{y_n})$. By $\mathfrak{A}_{l-\gamma}(\lambda)$ we denote the operator of the parameter-depending boundary value problem

$$\hat{L}^{(l-\gamma)}(y',\partial_{y'},\lambda)\,u = f, \quad y'\in\Omega,$$

$$\hat{B}_k^{(l-\gamma)}(y',\partial_{y'},\lambda)\,u + \sum_{j=1}^{J}\hat{C}_{k,j}^{(l-\gamma)}(y',\partial_{y'},\lambda)\,u_j = g_k, \quad y'\in\partial\Omega, k=1,\dots,m+J.$$

Analogously to Theorem 9.1.1, it can be shown that the operator

$$\mathcal{A}\,:\; \big(u, \mathcal{D}u|_{\partial\mathcal{G}}, \underline{u}\big) \to \big(Lu,\; Bu_{\partial\mathcal{G}} + C\underline{u}\big)$$

continuously maps the space

$$\tilde{\mathcal{W}}^{l,2m}_{2,\beta,\gamma}(\mathcal{G}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G}) \tag{9.3.4}$$

into

$$\tilde{\mathcal{W}}^{l-2m,0}_{2,\beta,\gamma}(\mathcal{G}) \times \mathcal{W}^{l-\underline{\mu}-1/2}_{2,\beta,\gamma}(\partial\mathcal{G}) \tag{9.3.5}$$

for $l\ge 2m$ and that in the case $l<2m$ the operator

$$\mathcal{A}\,:\; \tilde{\mathcal{W}}^{2m,2m}_{2,\beta,\gamma-l+2m}(\mathcal{G}) \times \mathcal{W}^{l+\underline{\tau}-1/2}_{2,\beta,\gamma-l+2m}(\partial\mathcal{G}) \to \mathcal{W}^{0}_{2,\beta,\gamma-l+2m}(\mathcal{G}) \times \mathcal{W}^{2m-\underline{\mu}-1/2}_{2,\beta,\gamma-l+2m}(\partial\mathcal{G})$$

can be uniquely extended to a continuous mapping from (9.3.4) into (9.3.5). Furthermore, the following two theorems hold.

THEOREM 9.3.1. *Suppose that the coefficients (9.3.3) stabilize for $y_n \to \infty$, the boundary value problem (9.3.1), (9.3.2) is elliptic, and the line* $\operatorname{Re}\lambda = -\beta$ *does not contain eigenvalues of the pencil* $\mathfrak{A}_{l-\gamma}$. *Then* $\mathcal{A}$ *is a Fredholm operator from (9.3.4) into (9.3.5) for arbitrary integer* l.

THEOREM 9.3.2. *Suppose that the boundary value problem (9.3.1), (9.3.2) is elliptic, the coefficients (9.3.3) stabilize for $y_n \to +\infty$, the lines* $\operatorname{Re}\lambda = -\beta_i$ *are free of eigenvalues of the pencil* $\mathfrak{A}_{l_1-\gamma_1}$, *and the strip* $-\beta_2 < \operatorname{Re}\lambda < -\beta_1$ *contains exactly* s *eigenvalues counting the algebraic multiplicity. If* $(u,\underline{u}) \in \mathcal{W}^{l_1}_{2,\beta_1,\gamma_1}(\mathcal{G}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1,\gamma_1}(\partial\mathcal{G})$ *is a solution of the boundary value problem (9.3.1), (9.3.2) and* $(f,\underline{g})$ *belongs to the intersection of the spaces*

$$\mathcal{W}^{l_i-2m}_{2,\beta_i,\gamma_i}(\mathcal{G}) \times \mathcal{W}^{l_i-\underline{\mu}-1/2}_{2,\beta_i,\gamma_i}(\partial\mathcal{G})$$

$(i = 1,\, 2)$, $\beta_1 < \beta_2$, $l_i \geq 2m$, $l_1 - \gamma_1 = l_2 - \gamma_2$, *then* $(u,\underline{u})$ *admits the representation*

$$(u,\underline{u}) = \sum_{k=1}^{s} c_j\, U_j + (w,\underline{w}),$$

for sufficiently large $|\kappa(x)| > T$, *where* $(w,\underline{w}) \in \mathcal{W}^{l_2}_{2,\beta_2,\gamma_2}(\mathcal{G}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2,\gamma_2}(\partial\mathcal{G})$ *and* U_j *are elements of the space* $\mathcal{W}^{l_1}_{2,\beta_1,\gamma_1}(\mathcal{G}) \times \mathcal{W}^{l_1+\underline{\tau}-1/2}_{2,\beta_1,\gamma_1}(\partial\mathcal{G})$ *which are linearly independent modulo* $\mathcal{W}^{l_2}_{2,\beta_2,\gamma_2}(\mathcal{G}) \times \mathcal{W}^{l_2+\underline{\tau}-1/2}_{2,\beta_2,\gamma_2}(\partial\mathcal{G})$ *and satisfy the homogeneous equations (9.3.1), (9.3.2) for* $|\kappa(x)| > T$.

REMARK 9.3.1. The results of this subsection can also be applied to the case when $\mathcal{G}$ is a n-dimensional manifold. We assume that $\mathcal{G}_0$ is an open subset of $\mathcal{G}$ such that $\mathcal{G}\backslash\overline{\mathcal{G}}_0$ is compact and there exists a diffeomorphism κ which maps the subset $\mathcal{G}_0$ onto the half-cylinder $\mathcal{C}_+ = \Omega \times \mathbb{R}_+$. Let $y' = (y_1, \dots, y_{n-1})$ be local coordinates on Ω in a neighbourhood $\mathcal{U}$. Then the sets $\mathcal{U}\times\mathbb{R}_+$ are coordinate neighbourhoods on $\mathcal{C}_+$ with local coordinates (y', y_n). We consider the sets $\kappa^{-1}(\mathcal{U}\times\mathbb{R}_+)$ to be coordinate neighbourhoods on $\mathcal{G}_0$ and assume that in each such neighbourhood it is possible to introduce local coordinates $x = (x_1, \dots, x_n)$ satisfying conditions (i) and (ii). Then the assertions of Theorems 9.3.1, 9.3.2 are also valid for elliptic boundary value problems on the manifold $\mathcal{G}$ provided the coefficients (9.3.3) stabilize for $y_n \to +\infty$.

9.3.2. Boundary value problems in quasiconical domains. Now let $\mathcal{G}$ be a domain in $\mathbb{R}^n$ such that for sufficiently small (or sufficiently large) R the central projection of the set $\{x \in \mathbb{R}^n\backslash\mathcal{G} : |x| = R\}$ onto the sphere S^{n-1} contains a fixed neighbourhood $\mathcal{U}$ of a point P. On $S^{n-1}\backslash\mathcal{U}$ it is possible to introduce a regular coordinate grid ξ (for example, Cartesian coordinates of the stereographic projection with center P onto the tangential hyperplane to the sphere S^{n-1} through the point opposite to P.)

We assume that after passing from the spherical coordinates r,ω, $r > 0$, $\omega \in S^{n-1}$, to Cartesian coordinates (ξ, t), where $t = \pm\log r$, there arise a domain $\mathcal{G}'$ and a boundary value problem subject to the conditions in Sections 9.1. In the above described situation the function φ is bounded.

LEMMA 9.3.3. *Let u be a function in $\mathcal{G}$, and let v be the function which arises from u passing to the coordinates (ξ, t). Then $v \in \mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G}')$ if and only if*

$$\Big(\int\limits_{\mathcal{G}} \sum_{|\alpha|\le l} e^{2\beta\,\sigma(r)} r^{|\alpha|-n}\, \varphi(\pm\log r)^{2(\gamma-l+|\alpha|)}\, |D^\alpha_x u|^2\, dx\Big)^{1/2} < \infty, \tag{9.3.6}$$

where $\sigma(r) = \pm \int\limits_{e^{\pm 1}}^{r} \dfrac{d\rho}{\rho\,\varphi(\pm\log\rho)}$.

Proof: By the definition of the space $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G}')$ in Section 9.1, we have

$$\|v\|_{\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G}')} = \Big(\int\limits_{\mathcal{G}'} \sum_{j+|\gamma|\le l} e^{2\beta y_n(t)}\, \varphi(t)^{2(\gamma-l+j+|\gamma|)}\, |D^\gamma_\xi\, D^j_t v(\xi,t)|^2\, dt\, d\xi\Big)^{1/2}. \tag{9.3.7}$$

Using the inequalities

$$|D^\gamma_\xi\, D^j_t v(\xi,t)|^2 \le c \sum_{|\alpha|\le j+|\gamma|} r^{2|\alpha|}\, |D^\alpha_x u|^2\,, \qquad |\det \frac{\partial(\xi,t)}{\partial(x_1,\dots,x_n)}| \le c\, r^{-n},$$

and the boundedness of the function φ, the norm (9.3.7) can be estimated from above by the expression (9.3.6). Analogously, we can show that (9.3.7) can be estimated from below by (9.3.6). ∎

We define the space $V^l_{2,\beta,\gamma}(\mathcal{G})$ as the closure of $C^\infty_0(\overline{\mathcal{G}}\backslash\{0\})$ (or $C^\infty_0(\overline{\mathcal{G}})$) with respect to the norm (9.3.6). If the limit of the function φ at infinity is nonzero, then the term $\varphi(\pm\log r)^{2(\gamma-l+|\alpha|)}$ in the norm (9.3.6) can be omitted.

For $l \ge 1$ the space of the traces of functions from $V^l_{2,\beta,\gamma}(\mathcal{G})$ on the boundary $\partial\mathcal{G}$ is denoted by $V^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$.

Example. We consider the Dirichlet problem

$$-\Delta u = f \quad \text{in } \mathcal{G}, \qquad u = 0 \quad \text{on } \partial\mathcal{G} \tag{9.3.8}$$

in a domain $\mathcal{G}$ satisfying one of the following two conditions:

1) $\mathcal{G}$ is a bounded domain such that for sufficiently small $R_0 < 1$ the set $\{x \in \mathcal{G} : r < R_0\}$ coincides with $\{x \in \mathbb{R}^n : r < R_0,\, \theta < \vartheta(r)\}$, where θ denotes the angle to the x_n-axis, i.e., $\cos\theta = x_n/|x|$, and $\vartheta \in C^\infty((0,R_0))$ is a function satisfying the condition
$$\vartheta(r) \to \vartheta_0 \in (0,\pi), \quad r\,\vartheta'(r) \to 0, \quad r^2\,\vartheta''(r) \to 0 \ \text{ as } r \to 0.$$
The set $\partial\mathcal{G}\backslash\{0\}$ is assumed to be smooth (see Figure 3a).

2) $\mathcal{G}$ is situated outside the unit ball $|x| < 1$ and for large R_0 the set $\{x \in \mathcal{G} : r > R_0\}$ coincides with $\{x \in \mathbb{R}^n : r > R_0,\, \theta < \vartheta(r)\}$, where $\vartheta \in C^\infty([R_0,+\infty))$ is a function satisfying the conditions
$$\vartheta(r) \to \vartheta_0 \in (0,\pi), \quad r\,\vartheta'(r) \to 0, \quad r^2\,\vartheta''(r) \to 0 \ \text{ as } r \to +\infty.$$
The set $\{x \in \partial\mathcal{G} : |x| < R\}$ is assumed to be smooth and bounded (see Figure 3b).

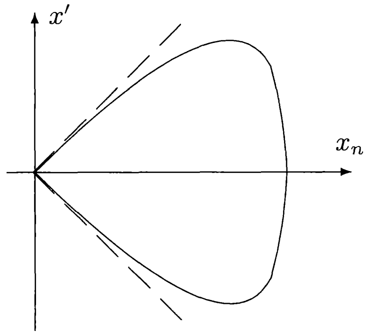


Figure 3a: A domain with a quasiconical point

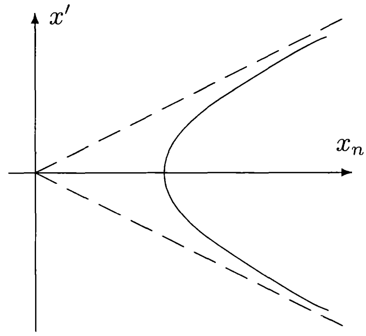


Figure 3b: An infinite quasicone

We introduce coordinates ξ on $\{\omega \in S^{n-1} : \theta > \pi - \varepsilon\}$ (ε is a sufficiently small positive number) such that $|\xi| = \theta$.

In case 1) the coordinate change $x \to (\omega, r) \to (\xi, t)$, where $t = -\log r$, takes the set $\{x \in \mathcal{G} : r < R_0\}$ onto $\{(\xi, t) : t > -\log R_0,\ |\xi| < \vartheta(e^{-t})\}$.

In case 2) the coordinate change $x \to (\omega, r) \to (\xi, t)$, where $t = \log r$, takes the set $\{x \in \mathcal{G} : r > R_0\}$ onto $\{(\xi, t) : t > \log R_0,\ |\xi| < \vartheta(e^{t})\}$.

It can be easily verified that the functions $\varphi(t) = \vartheta(e^{-t})$ (for case 1) and $\varphi(t) = \vartheta(e^{t})$ (for case 2) satisfy conditions (i), (ii) in Section 9.1.

Let β be a real number satisfying the inequality

$$-m(\vartheta_0) - n/2 < \beta/\vartheta_0 < m(\vartheta_0) - 2 + n/2\,,$$

where $m(\vartheta_0)$ is the least positive root of the equation $C_m^{(n-2)/2}(\cos\vartheta_0) = 0$ and $C_m^{(n-2)/2}$ is the Gegenbauer function. From our results in Section 9.1 it follows that for every f satisfying the condition

$$\int_{\mathcal{G}} e^{2\beta\sigma(r)}\, |f|^2\, dx < \infty,$$

where $\sigma(r) = \pm \int_{e^{\pm 1}}^{r} \frac{d\rho}{\rho\,\vartheta(\rho)}$, there exists a uniquely determined solution u with finite norm

$$\|u\| = \|e^{\beta\sigma(r)} u\|_{V_2^2(\mathcal{G})}\,,$$

where

$$\|u\|_{V_2^2(\mathcal{G})} = \Big(\int_{\mathcal{G}} \sum_{|\alpha| \le 2} r^{2|\alpha|-4}\, |D_x^\alpha u|^2\, dx \Big)^{1/2}.$$

9.4. Further results

9.4.1. Smoothness of solutions of the Dirichlet problem near cuspidal points. In Section 9.2 we have proved a regularity assertion for solutions of elliptic problems in a bounded domain $\mathcal{G}$ with a cuspidal point at the origin. In particular, we have shown that the solution $u \in \overset{\circ}{W}{}_2^m(\mathcal{G})$ of the Dirichlet problem (9.1.18) belongs to the space $\mathcal{W}^l_{2,\beta,\gamma}(\mathcal{G})$ if $f \in \mathcal{W}^{l-2m}_{2,\beta,\gamma}(\mathcal{G})$. Here l is an arbitrary integer, $l \geq 2m$, β is a certain positive real number, and γ is an arbitrary real number. If f is an infinitely differentiable function in $\overline{\mathcal{G}}$ vanishing near the origin, then it follows from this result that $u \in C^\infty(\overline{\mathcal{G}})$.

For the two-dimensional case M. Dauge [**56**] proved that the solution u is infinitely differentiable in $\overline{\mathcal{G}}$ if f is an arbitrary function from $C^\infty(\overline{\mathcal{G}})$.

9.4.2. Asymptotics of the solutions. Formulas for the main term in the asymptotics of harmonic functions in plane domains with cusps follow directly from S. E. Warschawski's asymptotic representation for conformal mappings of curvilinear strips [**254**]. Similar formulas for the n-dimensional case were obtained in 1967 by V. G. Maz'ya and G. M. Verzhbinskiĭ [**159, 160**].

We give here an asymptotic decomposition of the solution to the Dirichlet problem for an elliptic differential operator L of arbitrary order which was obtained by V. G. Maz'ya and B. A. Plamenevskiĭ [**146**] in 1977.

THEOREM 9.4.1. *Let $\mathcal{G}$ be a bounded in $\mathbb{R}^n$ such that $\partial\mathcal{G}\backslash\{0\}$ is smooth and the set $\{x \in \mathcal{G} : x_n < 1\}$ coincides with $\{x = (x', x_n) \in \mathbb{R}^n : x_n < 1,\ x'/x_n^\alpha \in \Omega\}$, where α is a real number, $\alpha > 1$. Furthermore, let u be a solution of the Dirichlet problem*

$$L(\partial_x)\, u = 0 \quad in\ \mathcal{G}, \qquad D_\nu^{k-1} u = 0 \quad on\ \partial\mathcal{G}\backslash\{0\},\ k = 1, \dots, m,$$

such that

$$e^{\frac{\beta_1}{1-\alpha}\, x_n^{1-\alpha}}\, u \in L_2(\mathcal{G}).$$

We assume that the strip $-\beta_2 < \operatorname{Re}\lambda \leq -\beta_1$ is free of eigenvalues of the pencil

$$L(\partial_{x'}, \lambda) \ :\ W_2^{2m}(\Omega) \cap \overset{\circ}{W}{}_2^m(\Omega) \to L_2(\Omega),$$

while the line $\operatorname{Re}\lambda = -\beta_2$ contains only the simple eigenvalue λ_0. The the following asymptotic formula for the solution u holds:

$$u(x) = c\, x_n^\gamma\, e^{\frac{\lambda_0}{1-\alpha}\, x_n^{1-\alpha}} \sum_{j=0}^{\infty} \varphi_j\Big(\frac{x'}{x_n^\alpha}\Big)\, x_n^{(\alpha-1)j}\,.$$

Here φ_0 is the eigenfunction of the pencil $L(\partial_{x'}, \lambda)$ to the eigenvalue λ_0, γ is a certain constant, and φ_j are smooth functions on Ω.

In the case of a conical point the eigenvalues of the operator pencil $\mathfrak{A}(\lambda)$ which is generated by the given boundary value problem on a subdomain of the sphere originate only finitely many terms in the asymptotics of the solutions. A feature of the asymptotic representations for cuspidal points is the occurrence of infinite series to the eigenvalues. This makes it difficult to interpret the formal asymptotic expansion which includes several eigenvalues. Recently, using methods of the so-called resurgent analysis [**238**], B.-W. Schulze, B. Sternin, and V. Shatalov [**227, 228**] constructed asymptotic representations for solutions of differential equations on manifolds without boundary which contain cuspidal points of integer order.

9.5. Notes

For boundary value problems without unknowns on the boundary the results of this chapter are contained in the paper of V. G. Maz'ya and B. A. Plamenevskiĭ [**147**]. In this paper also solutions in weighted L_p Sobolev spaces were considered. Applications to the Lamé and Stokes systems were given by the same authors in [**151**].

The solvability in weighted L_2 Sobolev spaces of elliptic problems in domains of cusp type (Figure 2) or in unbounded domains of the type shown in Figure 1 was studied also in papers of V. I. Feĭgin [**69**] and L. A. Bagirov, V. I. Feĭgin [**23**]. For the Dirichlet problem in plane domains with cuspidal points we refer also to the papers of J.-L. Steux [**239**] and M. Dauge [**56**]. Furthermore, we mention the papers of S. A. Nazarov [**180, 181**] and S. A. Nazarov, O. R. Polyakova [**183**], where boundary singularities of another types were considered. In particular, in [**180, 181**] the solutions near points of tangency of smooth components of the boundary were studied.

CHAPTER 10

Elliptic boundary value problems in domains with inside cusps

This chapter is devoted to the investigation of elliptic boundary-value problems in a domain $\mathcal{G} \subset \mathbb{R}^n$ with isolated singularities of the boundary. A common property of the considered singular points is the fact that in their neighbourhoods the complement to $\mathbb{R}^n$ or to a certain cylinder is "thin". For the singular point 0 this means that the central projection of the set $\{x \in \mathbb{R}^n \backslash \mathcal{G} : |x| = \rho\}$ onto the sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ shrinks to a point as $\rho \to 0$.

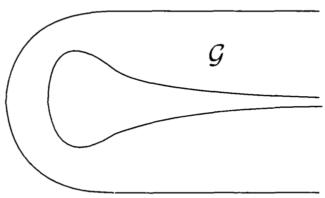


Figure 1a: A domain with singular point at infinity

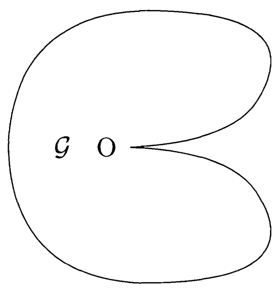


Figure 1b: A domain with singular point O

The situation, when the mentioned projection coincides in the limit with a punctured sphere or with a proper subdomain of it, has been studied in Chapters 6–9.

We prove theorems on the normal solvability of general elliptic boundary value problems in Sobolev spaces with weighted norms. Furthermore, we obtain coercive estimates for the solutions and investigate the properties of the solutions in dependence on the properties of the right-hand sides.

In order to illustrate the main results of this chapter, we consider the Dirichlet problem

$$L(x, \partial_x)\, u = f \;\text{ in } \mathcal{G}, \qquad D_\nu^{k-1} u = 0 \;\text{ on } \partial\mathcal{G}\backslash\{0\}, \quad k = 1, \dots, m,$$

in a bounded domain $\mathcal{G}$. We suppose that $\partial\mathcal{G}\backslash\{0\}$ is smooth and the set $\{x \in \mathcal{G} : |x| = \rho\}$ coincides with

$$\{x = (x', x_n) : |x| = \rho,\; x'/\rho\delta(\rho) \in \tilde{\Omega}\}$$

for small ρ, where $\tilde{\Omega}$ is a domain in $\mathbb{R}^{n-1}$ and δ is a positive function on $\mathbb{R}_+$ such that $\rho^j\delta^{(j)}(\rho) = o(\delta(\rho))$, $j \geq 1$, and $\delta(\rho) \leq \delta_0$, where δ_0 is a sufficiently small number. Furthermore, we suppose that L is an elliptic differential operator of order $2m$ in $\overline{\mathcal{G}}$ with smooth coefficients satisfying Gårding's inequality

$$\operatorname{Re}(Lu, u)_{L_2(\mathcal{G})} \geq c\,\|u\|^2_{W_2^m(\mathcal{G})}\,, \qquad u \in \overset{\circ}{W}{}_2^m(\mathcal{G}).$$

Let $\mathcal{V}^l_{2,\beta,\gamma}(\mathcal{G})$ be the weighted Sobolev space with the norm

$$\|u\|_{\mathcal{V}^l_{2,\beta,\gamma}(\mathcal{G})} = \Big(\sum_{|\alpha|\leq l}\int_{\mathcal{G}} |x|^{2(\gamma-l+|\alpha|)}\,\theta^{2(\beta-l+|\alpha|)}\,|D_x^\alpha u|^2\,dx\Big)^{1/2},$$

where θ denotes the angle between the vector $0x$ and the x_n-axis.

Then as a consequence of Theorem 10.5.4 the following statement holds:

1) *The operator of the Dirichlet problem for the equation* $Lu = f$ *realizes an isomorphism*

$$\mathcal{V}^{2m}_{2,\beta,\gamma}(\mathcal{G}) \to \mathcal{V}^{0}_{2,\beta,\gamma}(\mathcal{G}) \times \prod_{k=1}^{m} \mathcal{V}^{2m-k+1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$$

if $2m < n-1$, $\beta \in \big(2m-(n-1)/2\,,\,(n-1)/2\big)$, *and* $\gamma \in (2m-n/2\,,\,n/2)$.

2) *If* $u \in W_2^m(\mathcal{G})$ *is the generalized solution of the Dirichlet problem and* $f \in \mathcal{V}^0_{2,\beta,\gamma}(\mathcal{G})$, $g_k \in \mathcal{V}^{2m-k+1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$, *where* $2m < n-1$, $\beta \in \big(2m-(n-1)/2\,,\,(n-1)/2\big)$, *and* $\gamma \in (2m-n/2\,,\,n/2)$, *then* $u \in \mathcal{V}^{2m}_{2,\beta,\gamma}(\mathcal{G})$.

A large part of this chapter (Sections 10.1 - 10.4) is devoted to the study of general boundary value problems in a cylinder $\mathcal{C}$ from which one has removed a tube $\mathcal{D}$ with an infinitely small section at infinity. For these problems we construct the inverse operators "pasting together" the inverse operators of two "limit problems". This allows to obtain the Fredholm property for the operator of the boundary value problem in a domain which coincides with the domain $\mathcal{C}\backslash\overline{\mathcal{D}}$ for large x_n.

Other types of the considered domains can be reduced to this one by a change of variables. Problems in such domains are considered in the last subsection of Section 10.5.

10.1. Formulation of the problem

In this section we introduce the boundary value problem which will be studied in the following sections. We present the so-called auxiliary problem and the limit problems which are closely related to the starting problem. One of the limit problems arises in the cylinder after the freezing at infinity of the coefficients of the initial operators; in this case the tube is replaced by the x_n-axis and the boundary conditions given on the boundary of the tube are not taken into account. The second limit problem is a problem in the exterior of the cylinder obtained by "extension" of the tube. In this case the boundary conditions on the initial cylinder are disregarded, the coefficients of the elliptic equation are replaced by their limiting values at the point of infinity of the x_n-axis.

10.1.1. Assumptions on the domain and on the differential operators.

The domain $\mathcal{G}$. In the sequel, l is always an integer number $l \ge 2m$, $l > \max \mu_{s,k}$. We set $l_0 = l + \max(0, \max \tau_{s,j})$. (Here $\mu_{s,k}$, $\tau_{s,j}$ are the integer numbers determining the orders of the operators $B_{s,k}$, $C_{s;k,j}$ which will be introduced below.) Let Ω, $\tilde{\Omega}$ be bounded domains in $\mathbb{R}^{n-1}$ with boundaries of the class C^{l_0} such that the origin 0 is contained both in Ω and $\tilde{\Omega}$. Furthermore, let φ be a positive function on $\mathbb{R}$ with l_0 continuous derivatives such that

$$(10.1.1) \qquad \varphi(t)^{-1}\,|\varphi^{(j)}(t)| \le c < +\infty \quad \text{for } j = 1, 2, \dots, l_0$$

and $\varphi(t) \le \varphi_0$, where φ_0 is a sufficiently small constant. We set

$$\begin{aligned} \mathcal{C} &= \{x = (x', x_n) \in \mathbb{R}^n : x' \in \Omega,\ x_n \in \mathbb{R}\}, \\ \mathcal{D} &= \{x = (x', x_n) \in \mathbb{R}^n : x_n \in \mathbb{R},\ x'/\varphi(x_n) \in \tilde{\Omega}\}. \end{aligned}$$

Suppose that $\mathcal{G}$ is a domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{G}$ of class C^{l_0} such that the set $\{x \in \mathcal{G} : x_n < t\}$ is bounded for all t and

$$\{x \in \mathcal{G} : x_n > T\} = \{x \in \mathcal{C}\backslash\mathcal{D} : x_n > T\}$$

for sufficiently large T.

The boundary value problem in $\mathcal{G}$. We denote the connected components of the boundary $\partial\mathcal{G}$ by Γ_s, $s = 1, \dots, q$, and consider the boundary value problem

$$(10.1.2) \qquad L(x, \partial_x)\, u = f \quad \text{in } \mathcal{G},$$

$$(10.1.3) \qquad B_s(x, \partial_x)\, u + C_s(x, \partial_x)\, \underline{u}^{(s)} = \underline{g}^{(s)} \quad \text{on } \Gamma_s, \quad s = 1, \dots, q.$$

Here L is a uniformly elliptic differential operator of order $2m$, B_s is a vector of differential operators $B_{s;k}$, $k = 1, \dots, m + J$, ord $\mathrm{B}_{s;k} \le \mu_{s,k}$ and C_s is a matrix of differential operator $C_{s;k,j}$ which are tangential on Γ_s, $\operatorname{ord} C_{s;k,j} \le \mu_{s,k} + \tau_{s,j}$, such that problem (10.1.2), (10.1.3) is elliptic.

We assume that for arbitrary $T < \infty$ the coefficients of L have continuous derivatives up to order $l - 2m$ in $\{x \in \overline{\mathcal{G}} : x_n < T\}$, while the coefficients of $B_{s;k}$ and $C_{s;k,j}$ belong to the class $C^{l-\mu_{s,k}}$ in a neighbourhood of the set $\{x \in \Gamma_s : x_n < T\}$.

Moreover, we assume that the coefficients of L, $B_{s;k}$, and $C_{s;k,j}$ satisfy some stabilization condition for large x_n. In order to introduce this condition, we need the following definition.

DEFINITION 10.1.1. Let G be an open subset of the Euclidean space $\mathbb{R}^n$. The operator

$$P(x, r\,\partial_x) = \sum_{|\alpha| \le \mu} p_\alpha(x)\,(r\,\partial_x)^\alpha$$

belongs to the class $\mathcal{O}_k^\mu(G)$ if

$$|P|_{\mathcal{O}_k^\mu(G)} \stackrel{def}{=} \sum_{|\alpha| \le \mu} \sum_{|\gamma| \le k} \|(r\,\partial_x)^\gamma p_\alpha\|_{L_\infty(G)} < \infty.$$

Here $r = |x'| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$.

In the following, T is a sufficiently large positive number. Furthermore, let $\mathcal{U}$ be a cylindrical neighbourhood of the surface $\partial\mathcal{C}$ and $\mathcal{V}$ a set of points $x \in \mathbb{R}^n$ for which $x'/\varphi(x_n)$ belongs to some neighbourhood of the surface $\partial\tilde{\Omega}$. The sets

$\{x \in \mathcal{C} : x_n \geq T\}$, $\{x \in \mathcal{U} : x_n \geq T\}$, $\{x \in \mathcal{V} : x_n \geq T\}$ are denoted by $\mathcal{C}_T$, $\mathcal{U}_T$, and $\mathcal{V}_T$, respectively.

We suppose that there exists a differential operator

$$\mathcal{L}(x', \partial_x) = r^{-2m} \sum_{|\alpha| \leq 2m} a_\alpha(x') \, (r\,\partial_x)^\alpha \tag{10.1.4}$$

with coefficients $a_\alpha \in C^{l-2m}(\overline{\Omega}\backslash\{0\})$ such that the inequality

$$\left| r^{2m}(L - \mathcal{L}) \right|_{\mathcal{O}^{2m}_{l-2m}(\mathcal{C}_T)} < \varepsilon \tag{10.1.5}$$

is satisfied with a sufficiently small positive number ε and there exist functions $a_\alpha(\cdot, 0) \in C^{l-2m}(S^{n-2})$ such that

$$\lim_{r \to 0} \sum_{|\gamma|+j \leq l-2m} \sup_{\omega \in S^{n-2}} \left| \partial_\omega^\gamma (r\partial_r)^j \left(a_\alpha(\omega, r) - a_\alpha(\omega, 0) \right) \right| = 0.$$

Here r, ω are spherical coordinates in $\mathbb{R}^{n-1}$.

Furthermore, we assume that there exist differential operators

$$\mathcal{B}_k(x', \partial_x) = \sum_{|\alpha| \leq m_k} b_{k;\alpha}(x') \, \partial_x^\alpha, \tag{10.1.6}$$

$$\mathcal{B}'_k(x, \partial_x) = r^{-\mu_k} \sum_{|\alpha| \leq \mu_k} b'_{k;\alpha}(\varphi(x_n)^{-1}x') \, (r\partial_x)^\alpha \tag{10.1.7}$$

with coefficients $b_{k;\alpha} \in C^{l-m_k}$ in some neighbourhood of $\partial\Omega$ and $b'_{k;\alpha} \in C^{l-\mu_k}$ in some neighbourhood of $\partial\tilde{\Omega}$ satisfying the following conditions. In the case, when the component Γ_s is contained in $\partial\mathcal{C}$ for large x_n, we assume that $\mu_{s,k} = m_k$, $\tau_{s,j} = t_j$ and

$$\left| r^{m_k}(B_{s;k} - \mathcal{B}_k) \right|_{\mathcal{O}^{m_k}_{l-m_k}(\mathcal{U}_T)} < \varepsilon. \tag{10.1.8}$$

If, far from the origin, the component Γ_s coincides with $\partial\mathcal{D}$, then we require that $\mu_{s,k} = \mu_k$, $\tau_{s,j} = \tau_j$ and

$$\left| r^{\mu_k}(B_{s;k} - \mathcal{B}'_k) \right|_{\mathcal{O}^{\mu_k}_{l-\mu_k}(\mathcal{V}_T)} < \varepsilon. \tag{10.1.9}$$

Similar conditions are imposed on the operators $C_{s;k,j}$. We assume that there exist differential operators

$$\mathcal{C}_{k,j}(x', \partial_x) = \sum_{|\alpha| \leq m_k + t_j} c_{k,j;\alpha}(x') \, \partial_x^\alpha \tag{10.1.10}$$

with coefficients $c_{k,j;\alpha} \in C^{l-m_k}$ in a neighbourhood of $\partial\Omega$ such that (10.1.10) is tangential on $\mathcal{C}$ and

$$\left| r^{m_k+t_j}(C_{s;k,j} - \mathcal{C}_{k,j}) \right|_{\mathcal{O}^{m_k+t_j}_{l-m_k}(\mathcal{U}_T)} < \varepsilon \tag{10.1.11}$$

if Γ_s coincides with $\partial\mathcal{C}$ far from the origin. Furthermore, we suppose that for every index s for which Γ_s is contained in $\partial\mathcal{D}$ far from the origin and for all indices $j = 1, \ldots, J$, $k = 1, \ldots, m + J$ there exist differential operators $\mathcal{C}'_{k,j}$ of a special form (see formula (10.1.24) below) such that

$$\left| r^{\mu_k+\tau_j}(C_{s;k,j} - \mathcal{C}'_{k,j}) \right|_{\mathcal{O}^{\mu_k+\tau_j}_{l-\mu_k}(\mathcal{V}_T)} < \varepsilon. \tag{10.1.12}$$

One of the main goals of this chapter is the proof of the Fredholm property for the operator of the boundary value problem (10.1.2), (10.1.3) in certain classes of weighted Sobolev spaces. For this we need the invertibility of the operator of the auxiliary problem formed by the differential operators $\mathcal{L}$, $\mathcal{B}_k$, $\mathcal{B}'_k$, $\mathcal{C}_{k,j}$, and $\mathcal{C}'_{k,j}$.

10.1.2. The auxiliary and limit problems. The boundary value problem

$$\mathcal{L}(x',\partial_x)\,u = f \quad \text{in } \mathcal{C}\backslash\overline{\mathcal{D}}, \tag{10.1.13}$$

$$\mathcal{B}_k(x',\partial_x)\,u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(x',\partial_x)\,u_j = g_k \quad \text{on } \partial\mathcal{C},\ k=1,\dots,m+J, \tag{10.1.14}$$

$$\mathcal{B}'_k(x,\partial_x)\,u + \sum_{j=1}^{J} \mathcal{C}'_{k,j}(x,\partial_x)\,v_j = h_k \quad \text{on } \partial\mathcal{D},\ k=1,\dots,m+J, \tag{10.1.15}$$

is called the *auxiliary problem* for problem (10.1.2), (10.1.3). For the study of the auxiliary problem two so-called limit problems play an important role: a boundary value problem in the cylinder $\mathcal{C}^\circ = (\Omega\backslash\{0\})\times\mathbb{R}$ and another problem in the exterior of the cylinder $\tilde{\Omega}\times\mathbb{R}$.

The first limit problem. Since the function φ has small values, the quasicylinder $\mathcal{C}\backslash\mathcal{D}$ approximately coincides with the cylinder $(\Omega\backslash\{0\})\times\mathbb{R}$. For this reason, the boundary value problem

$$\mathcal{L}(x',\partial_x)\,u = f \quad \text{in } \mathcal{C}^\circ, \tag{10.1.16}$$

$$\mathcal{B}_k(x',\partial_x)\,u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(x',\partial_x)\,u_j = g_k \quad \text{on } \partial\mathcal{C},\ k=1,\dots,m+J, \tag{10.1.17}$$

plays a crucial role. This problem is called the *first limit problem.*

The second limit problem. The coordinate transformation

$$y_n = \chi(x_n) \stackrel{def}{=} \int_0^{x_n} \frac{dt}{\varphi(t)}, \qquad y_j = \frac{x_j}{\varphi(x_n)},\ j=1,\dots,n-1,$$

takes the exterior of $\overline{\mathcal{D}}$ onto

$$\tilde{\mathcal{D}} = \{y=(y',y_n)\ :\ y'\in\mathbb{R}^{n-1}\backslash\overline{\tilde{\Omega}},\ y_n\in\mathbb{R}\}. \tag{10.1.18}$$

We set

$$\mathcal{L}^{(0)}(x',\partial_x) = r^{-2m} \sum_{|\alpha|\le 2m} a_\alpha(\omega,0)\,(r\partial_x)^\alpha .$$

Since

$$r\,\frac{\partial}{\partial x_j} = \rho\,\frac{\partial}{\partial y_j} \ \text{ for } j=1,\dots,n-1, \quad r\,\frac{\partial}{\partial x_n} = \rho\,\frac{\partial}{\partial y_n} - \varphi'(x_n)\sum_{k=1}^{n-1} y_k\,\rho\,\frac{\partial}{\partial y_k},$$

where $\rho=|y'|$, the operator $\mathcal{L}^{(0)}(x',\partial_x)$ has the form

(10.1.19)

$$\varphi(x_n)^{-2m}\,\rho^{-2m} \sum_{|\alpha|\le 2m} a_\alpha(\omega,0)\,(\rho\,\partial_{y'})^{\alpha'}\,\Big(\rho\,\frac{\partial}{\partial y_n} - \varphi'(x_n)\sum_{k=1}^{n-1} y_k\,\rho\,\frac{\partial}{\partial y_k}\Big)^{\alpha_n}$$

in the coordinates $y = (y', y_n)$, where $x_n = \chi^{-1}(y_n)$. Here α' denotes the multi-index $(\alpha_1, \dots, \alpha_{n-1})$ if $\alpha = (\alpha_1, \dots, \alpha_n)$. By our assumptions on φ, the term

$$\varphi'(x_n) \sum_{k=1}^{n-1} y_k \, \rho \, \frac{\partial}{\partial y_k} \tag{10.1.20}$$

is small for $|x'| < \varphi(x_n)^\sigma$ (i.e., for $|y'| \le \varphi(x_n)^{\sigma-1}$) if σ is an arbitrary positive number less than one. Omitting this term in (10.1.19), we obtain the operator $\varphi(x_n)^{-2m}\, \mathcal{L}^{(0)}(y', \partial_y)$, where

$$\mathcal{L}^{(0)}(y', \partial_y) = \rho^{-2m} \sum_{|\alpha| \le 2m} a_\alpha(\omega, 0)\, (\rho \partial_y)^\alpha . \tag{10.1.21}$$

Analogously, passing to the coordinates y and omitting the term (10.1.20), the operator $\mathcal{B}'_k(x, \partial_x)$ will be transformed into the operator $\varphi(x_n)^{-\mu_k}\, \mathcal{P}_k(y', \partial_y)$, where

$$\mathcal{P}_k(y', \partial_y) = \rho^{-\mu_k} \sum_{|\alpha| \le \mu_k} b'_{k;\alpha}(y')\, (\rho \partial_y)^\alpha . \tag{10.1.22}$$

Let $\mathcal{Q}_{k,j}(y', \partial_y)$ be differential operators of order not greater than $\mu_k + \tau_j$ which are tangential on $\partial \tilde{\mathcal{D}}$. We assume that the coefficients of $\mathcal{Q}_{k,j}$ depend only on y' and belong to the class $C^{l-\mu_k}$ in a neighbourhood of $\partial \tilde{\Omega}$. Then these operators can be written in the form

$$\mathcal{Q}_{k,j}(y', \partial_y) = \rho^{-\mu_k - \tau_j} \sum_{|\alpha| \le \mu_k + \tau_j} c'_{k,j;\alpha}(y')\, (\rho \partial_y)^\alpha . \tag{10.1.23}$$

The differential operators

$$\begin{aligned} \mathcal{C}'_{k,j}(x, \partial_x) \;=\; & r^{-\mu_k - \tau_j} \sum_{|\alpha| \le \mu_k + \tau_j} c'_{k,j;\alpha}\Big(\frac{x'}{\varphi(x_n)}\Big)\, \big(r \partial_{x'}\big)^{\alpha'} \\ & \times \Big(r\partial_{x_n} + \frac{\varphi'(x_n)}{\varphi(x_n)} \sum_{k=1}^{n-1} x_k \, r \partial_{x_k} \Big)^{\alpha_n} \end{aligned} \tag{10.1.24}$$

which arise from $\varphi(x_n)^{\mu_k + \tau_j}\, \mathcal{Q}_{k,j}(y', \partial_y)$ via the coordinate change $y \to x$ are tangential on $\partial \mathcal{D}$. We suppose that for these operators the inequality (10.1.12) is satisfied.

The boundary value problem

$$\mathcal{L}^{(0)}(y', \partial_y)\, u = f \quad \text{in } \tilde{\mathcal{D}}, \tag{10.1.25}$$

$$\mathcal{P}_k(y', \partial_y)\, u + \sum_{j=1}^{J} \mathcal{Q}_{k,j}(y', \partial_y)\, v_j = g_k \quad \text{on } \partial \tilde{\mathcal{D}},\ k = 1, \dots, m + J, \tag{10.1.26}$$

is called the *second limit problem.*

The next two sections are devoted to the study of both limit problems.

10.2. The first limit problem

This section is dedicated to the limit problem (10.1.16), (10.1.17) in the cylinder $\mathcal{C}^\circ$. This limit problem is considered in weighted spaces $V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)$, where the index γ characterizes the exponential growth of the solution at infinity, while the index β characterizes the powerlike growth of the solution near the x_n-axis. The limit problem is solvable in the indicated scale of spaces not for all values of β, γ. The solvability depends on the properties of the operator pencil corresponding to a parameter depending boundary value problem in the domain $\Omega\backslash\{0\}$ which is obtained by applying the Laplace transformation with respect to the variable x_n to the limit problem. Such operator pencils have been studied (for a significantly more general situation) in Section 6.5. In our case the inner point 0 has to be considered as a singular point. The cone corresponding to this point is the set $\mathbb{R}^{n-1}\backslash\{0\}$. Therefore, the eigenvalues of a certain operator pencil on the sphere S^{n-2} are of importance for the solvability of the parameter-depending problem in $\Omega\backslash\{0\}$.

10.2.1. Weighted Sobolev spaces in the cylinder. Let $\mathcal{C}^\circ$ be the cylinder $(\Omega\backslash\{0\})\times\mathbb{R}$. We introduce the space $V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)$ of functions in $\mathcal{C}^\circ$ with finite norm

$$\|u\|_{V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)} = \Big(\sum_{|\alpha|\le l}\int_{\mathcal{C}} e^{2\gamma x_n}\, r^{2(\beta-l+|\alpha|)}\,|D^\alpha_x u|^2\,dx\Big)^{1/2}, \tag{10.2.1}$$

where $r = |x'|$, l is a nonnegative integer, and β, γ are arbitrary real numbers. Obviously, the norm in $V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)$ is equivalent to the norm

$$\|e^{\gamma x_n} r^\beta u\|_{V^l_{2,0,0}(\mathcal{C}^\circ)}\,.$$

By $V^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ we denote the space of traces on $\partial\mathcal{C}$ of functions from $V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)$, $l\ge 1$. Furthermore, we write $V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ and $V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ for the products of the spaces $V^{l+t_j-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ and $V^{l-m_k-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$, respectively.

Let $V^l_{2,\beta}(\Omega)$ be the space with the norm

$$\|u\|_{V^l_{2,\beta}(\Omega)} = \Big(\sum_{|\alpha'|\le l}\|r^{\beta-l+|\alpha'|}\,D^{\alpha'}_{x'}u\|^2_{L_2(\Omega)}\Big)^{1/2}.$$

Then the following lemma holds.

LEMMA 10.2.1. *The norm (10.2.1) is equivalent to the norm*

$$\|u\| = \Big(\frac{1}{2\pi i}\int_{Re\,\lambda=-\gamma}\big(\|\check{u}(\cdot,\lambda)\|^2_{V^l_{2,\beta}(\Omega)} + |\lambda|^{2l}\,\|\check{u}(\cdot,\lambda)\|^2_{V_{2,\beta}(\Omega)}\big)\,d\lambda\Big)^{1/2}, \tag{10.2.2}$$

where $\check{u}(x',\lambda) = \mathcal{L}_{x_n\to\lambda}\,u(x',x_n)$ is the Laplace transform of u. Analogously, the norm

$$\|u\| = \Big(\frac{1}{2\pi i}\int_{Re\,\lambda=-\gamma}\big(\|\check{u}(\cdot,\lambda)\|^2_{W^{l-1/2}_2(\partial\Omega)} + |\lambda|^{2l-1}\,\|\check{u}(\cdot,\lambda)\|^2_{L_2(\partial\Omega)}\big)\,d\lambda\Big)^{1/2}. \tag{10.2.3}$$

is equivalent to the norm in $V^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$.

Proof: By the Parseval equality (5.2.10), the norm (10.2.1) is equal to

$$\Big(\frac{1}{2\pi i}\int_{Re\,\lambda=-\gamma}\sum_{j=0}^{l}|\lambda|^{2j}\,\|\check{u}(\cdot,\lambda)\|^2_{V^{l-j}_{2,\beta}(\Omega)}\,d\lambda\Big)^{1/2}$$

Using Lemma 3.6.3 and Lemma 6.5.5 (where $\mathcal{K} = \mathbb{R}^{n-1}\backslash\{0\}$), it can be easily shown that this norm is equivalent to (10.2.2).

Obviously, the space $V^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ coincides with the space $\mathcal{W}^{l-1/2}_{2,\gamma}(\partial\mathcal{C})$ introduced in Section 5.2. Thus, the equivalence of the norm in $V^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ to (10.2.3) follows from the second part of Lemma 5.2.4. ∎

10.2.2. The parameter-depending problem corresponding to the first limit problem. We consider the limit problem (10.1.16), (10.1.17). It can be easily verified that the operator

$$\mathcal{A}_1 \,:\, V^l_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \to V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \tag{10.2.4}$$

of this problem is continuous for $l \geq 2m$, $l > \max m_k$. By our assumptions (see Section 10.1) on the boundary value problem (10.1.2), (10.1.3), problem (10.1.16), (10.1.17) is elliptic. The purpose of this section is the elucidation of the additional conditions under which the mapping (10.2.4) is an isomorphism.

Applying the Laplace transformation $x_n \to \lambda$ to the boundary value problem (10.1.16), (10.1.17), we obtain the problem

$$\mathcal{L}(x', \partial_{x'}, \lambda)\, u = f \quad \text{in } \Omega\backslash\{0\},$$

$$\mathcal{B}_k(x', \partial_{x'}, \lambda)\, u + \sum_{j=1}^{J} \mathcal{C}_{k,j}(x', \partial_{x'}, \lambda) u_j = g_k \quad \text{on } \partial\Omega,\ k = 1, \dots, m+J,$$

with the complex parameter λ. The operator $\mathfrak{A}(\lambda)$ of this boundary value problem realizes a continuous mapping

$$V^l_{2,\beta}(\Omega) \times \prod_{j=1}^{J} W_2^{l+t_j-1/2}(\partial\Omega) \to V^{l-2m}_{2,\beta}(\Omega) \times \prod_{k=1}^{m} W_2^{l-m_k-1/2}(\partial\Omega).$$

We introduce the parameter-depending differential operator

$$\mathcal{L}^{(0)}(x', \partial_{x'}, \lambda) = r^{-2m} \sum_{|\alpha|\leq 2m} a_\alpha(\omega, 0)\, (r\partial_{x'})^{\alpha'} (r\lambda)^{\alpha_n} \tag{10.2.5}$$

(here α' is the multi-index containing the first $n-1$ components of the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$) in $\mathbb{R}^{n-1}$. Analogously to the space $V^l_{2,\beta}(\Omega)$, we define $V^l_{2,\beta}(\mathbb{R}^{n-1})$ as the space with the norm

$$\|u\|_{V^l_{2,\beta}(\mathbb{R}^{n-1})} = \Big(\sum_{|\alpha'|\leq l} \|r^{\beta-l+|\alpha'|} D^{\alpha'}_{x'} u\|^2_{L_2(\mathbb{R}^{n-1})} \Big)^{1/2}.$$

Clearly, the mapping

$$\mathcal{L}^{(0)}(x', \partial_{x'}, 0) \,:\, V^l_{2,\beta}(\mathbb{R}^{n-1}) \to V^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}) \tag{10.2.6}$$

is continuous.

Passing to spherical coordinates r, ω in $\mathbb{R}^{n-1}$, we may write the differential operator $\mathcal{L}^{(0)}(x', \partial_{x'}, 0)$ in the form

$$\mathcal{L}^{(0)}(x', \partial_{x'}, 0) = r^{-2m}\, \mathfrak{M}(\omega, \partial_\omega, r\partial_r).$$

We consider the operator $\mathfrak{M}(\omega, \partial_\omega, \mu)$ with a complex parameter μ on the sphere S^{n-2}.

From the ellipticity of L it follows that the μ-spectrum of the operator pencil $\mathfrak{M} = \mathfrak{M}(\omega, \partial_\omega, \mu)$ is discrete and situated (with the possible exception of a finite number of values) inside a double angle, containing the real axis. The operator (10.2.6) is an isomorphism if and only if the line $\operatorname{Re}\mu = -\beta + l - (n-1)/2$ does not contain eigenvalues of the operator pencil $\mathfrak{M}$ (see Theorem 6.1.1).

Let $E^l_{2,\beta}(\mathbb{R}^{n-1})$ be the space with the norm

$$\|v\|_{E^l_{2,\beta}(\mathbb{R}^{n-1})} = \|v\|_{V^l_{2,\beta}(\mathbb{R}^{n-1})} + \|r^\beta v\|_{L_2(\mathbb{R}^{n-1})}.$$

If no eigenvalues of the pencil $\mathfrak{M}$ lie on the line $\operatorname{Re}\mu = -\beta + l - (n-1)/2$, then by Theorem 6.5.2 (in our situation the set $\mathbb{R}^{n-1}\backslash\{0\}$ plays the role of the cone $\mathcal{K}$), the operators

$$\mathcal{L}^{(0)}(x', \partial_{x'}, \pm i) \,:\, E^l_{2,\beta}(\mathbb{R}^{n-1}) \to E^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}) \tag{10.2.7}$$

are Fredholm. We suppose in the following that the operators (10.2.7) are isomorphisms for $l = l_0$, $\beta = \beta_0$, and that $\mu_- < \operatorname{Re}\mu < \mu_+$ is the widest strip containing the line $\operatorname{Re}\mu = -\beta_0 + l_0 - (n-1)/2$ which is free of eigenvalues of the pencil $\mathfrak{M}(\mu)$. From Theorem 6.5.3 it follows that the operators (10.2.7) are isomorphisms for arbitrary l, β satisfying the inequality $\mu_- < -\beta + l - (n-1)/2 < \mu_+$. Furthermore, the following theorem holds (cf. Theorem 6.5.5, Corollary 6.5.1).

THEOREM 10.2.1. *Suppose that the operators (10.2.7) are isomorphisms for $l = l_0$, $\beta = \beta_0$, and the numbers l, β satisfy the inequality $\mu_- < -\beta + l - (n-1)/2 < \mu_+$. Then the spectrum of the operator pencil $\mathfrak{A}$ is discrete. It consists of eigenvalues with finited algebraic multiplicities. All eigenvalues, with the exception of finitely many, are situated in a double angle containing the real axis.*

In addition, for λ situated outside the mentioned angle, sufficiently far from the origin, and $u \in V^l_{2,\beta}(\Omega)$ we have the estimate

$$\|u\|_{V^l_{2,\beta}(\Omega)} + |\lambda|^l\, \|u\|_{V^0_{2,\beta}(\Omega)} + \sum_{j=1}^{J} \Big(\|u_j\|_{W_2^{l+t_j-1/2}(\partial\Omega)} + |\lambda|^{l+t_j-1/2}\, \|u_j\|_{L_2(\partial\Omega)} \Big)$$
$$\leq c \Big(\|f\|_{V^{l-2m}_{2,\beta}(\Omega)} + |\lambda|^{l-2m}\, \|f\|_{V^0_{2,\beta}(\Omega)} + \sum_{k=1}^{m+J} \|g_k\|_{W_2^{l-m_k-1/2}(\partial\Omega)}$$
$$+ \sum_{k=1}^{m+J} |\lambda|^{l-m_k-1/2}\, \|g_k\|_{L_2(\partial\Omega)} \Big),$$

where

$$f = \mathcal{L}(x', \partial_{x'}, \lambda)u, \qquad g_k = \mathcal{B}_k(x', \partial_{x'}, \lambda)\, u\,\big|_{\partial\Omega} + \sum_{j=1}^{J} \mathcal{C}_{k,j}(x', \partial_{x'}, \lambda)\, u_j\,.$$

Here the constant c is independent of u and λ.

REMARK 10.2.1. As it was shown in Section 6.5 (cf. Remark 6.5.2), the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ do not depend on the choice of the numbers l and β satisfying the inequality $\mu_- < -\beta + l - (n-1)/2 < \mu_+$.

The special case, when $\mathcal{L}^{(0)}(x', \partial_{x'}, \pm i)$ are polynomials of $\partial_{x'}$ with constant coefficients. Often the differential operators $\mathcal{L}^{(0)}(x', \partial_{x'}, \pm i)$ which appear in the conditions of Theorem 10.2.1 have constant coefficients. We investigate now for which l, β the operators (10.2.7) are isomorphisms in this case.

THEOREM 10.2.2. *Let $\mathcal{L}^{(0)}(x', \partial_{x'}, \pm i) = \mathcal{L}^{(0)}(\partial_{x'}, \pm i)$ be polynomials in $\partial_{x'}$ with constant coefficients.*

1) *If $2m < n-1$, then the operators (10.2.7) are isomorphisms for all l and β satisfying the inequalities $2m - (n-1)/2 < l - \beta < (n-1)/2$.*

2) *If $2m = n-1$, then there do not exist numbers l and β for which the operators (10.2.7) are isomorphisms.*

3) *Let $2m > n-1$ and let the operator $\mathcal{L}^{(0)}(\partial_{x'}, \pm i)$ be strongly elliptic. If n is even, then the operators (10.2.7) are isomorphisms for $m - 1/2 < l - \beta < m + 1/2$, while for odd n there do no exist numbers l and β for which the operators (10.2.7) are isomorphisms.*

Proof: It suffices to prove the theorem for $l = 2m$. Since $\mathcal{L}^{(0)}(\partial_{x'}, 0)$ is a homogeneous differential operator of order $2m$ with constant coefficients, the spectrum of the operator pencil $\mathfrak{M}$ consists of the numbers k and $2m-n+1-k$ ($k = 0, 1, 2, \dots$). Therefore, the operator (10.2.7) is Fredholm for $l = 2m$ if and only if

$$\beta \neq 2m - (n-1)/2 - k, \quad \beta \neq (n-1)/2 + k, \quad k = 0, 1, \dots .$$

Elements of the kernel of the operator $\mathcal{L}^{(0)}(\partial_{x'}, \pm i)$ may be only linear combinations of the derivatives of the fundamental solution. At infinity the fundamental solutions decrease rapidly. Therefore, the fact that these combinations belong to the space $E^{2m}_{2,\beta}(\mathbb{R}^{n-1})$ is determined by their singularity at the origin.

In case 1) the principal term of the asymptotics of the fundamental solution at the origin has the form $|x'|^{2m-n+1}\Phi(x'/|x'|)$. Therefore, the kernels of the operators (10.2.7) are trivial for $-\beta + l - (n-1)/2 > 2m - n + 1$. Since the cokernel of the operator (10.2.7) is a subset of $E^{0}_{2,-\beta}(\mathbb{R}^{n-1})$ for $l = 2m$ and the singularity of the fundamental solution of the adjoint operator $\mathcal{L}^{(0)}(\partial_{x'}, \pm i)^*$ is a function of the form $|x'|^{2m-n+1}\Psi(x'/|x'|)$, it follows that the cokernels of the operators (10.2.7) are trivial for $l = 2m$, $\beta > 2m - (n-1)/2$.

In case 2) the principal term of the asymptotics of the fundamental solutions of the operators $\mathcal{L}^{(0)}(\partial_{x'}, \pm i)$ and $\mathcal{L}^{(0)}(\partial_{x'}, \pm i)^*$ has the form $c \log|x'| + \Phi(x'/|x'|)$, where c is a constant. Thus, the kernels of the operators (10.2.7) are nontrivial for $l = 2m$, $\beta < m$, and the cokernels of these operators are nontrivial for $l = 2m$, $\beta > m$, i.e., there do not exist numbers l and β for which the operator (10.2.7) an isomorphism.

We consider case 3). Let $\overset{\circ}{W}{}^m_2(\mathbb{R}^{n-1}\backslash\{0\})$ denote the closure of the set of all functions $u \in C_0^\infty(\mathbb{R}^{n-1})$ equal to zero near the origin in $W_2^m(\mathbb{R}^{n-1})$. From the strong ellipticity of $\mathcal{L}^{(0)}(\partial_x)$ it follows that

$$\begin{aligned}
&(-1)^m \operatorname{Re} \int\limits_{\mathbb{R}^{n-1}} \mathcal{L}^{(0)}(\partial_{x'}, \pm i)\, u(x') \cdot \overline{u(x')}\, dx' \\
&= (-1)^m \operatorname{Re} \int\limits_{\mathbb{R}^{n-1}} \mathcal{L}^{(0)}(i\xi', \pm i)\, |\hat{u}(\xi')|^2\, d\xi' \\
&= \int\limits_{\mathbb{R}^{n-1}} \operatorname{Re} \mathcal{L}^{(0)}(\xi', \pm 1)\, |\hat{u}(\xi')|^2\, d\xi' \geq c \int\limits_{\mathbb{R}^{n-1}} \left(1 + |\xi'|^2\right)^m |\hat{u}|^2\, d\xi' = c\, \|u\|^2_{W_2^m(\mathbb{R}^{n-1})}
\end{aligned}$$

for all $u \in C_0^\infty(\mathbb{R}^{n-1})$. Hence every sesquilinear form on $\overset{\circ}{W}{}^m_2(\mathbb{R}^{n-1}\backslash\{0\})^2$ corresponding to the operator $\mathcal{L}^{(0)}(\partial_{x'}, \pm i)$ is V-elliptic (see Definition 4.3.1). From this we conclude that the equation

$$\mathcal{L}^{(0)}(\partial_{x'}, \pm i)\, u = f \quad \text{in } \mathbb{R}^{n-1}$$

has a unique solution $u \in \overset{\circ}{W}{}^m_2(\mathbb{R}^{n-1}\backslash\{0\})$ for arbitrary $f \in C_0^\infty(\mathbb{R}^{n-1})$. If n is even, then according to Hardy's inequality (7.1.1), the solution $u \in \overset{\circ}{W}{}^m_2(\mathbb{R}^{n-1}\backslash\{0\})$ belongs also to the space $E^m_{2,0}(\mathbb{R}^{n-1})$ and, analogously to Lemma 6.5.2, we obtain

$u \in E^{2m}_{2,m}(\mathbb{R}^{n-1})$. Consequently, the kernels and cokernels of the operators (10.2.7) are trivial for $l = 2m$, $\beta = m$. Therefore, the operators (10.2.7) are isomorphisms for $l = 2m$ and $m - 1/2 < \beta < m + 1/2$.

Now we assume that n is an odd number. In this case $\lambda = m - (n-1)/2$ is an eigenvalue of the pencil $\mathfrak{M}$. We denote the corresponding eigenfunction by ϕ_0. Then the equation $\mathcal{L}^{(0)}(\partial_{x'}, 0)v = 0$ has the solution $v(x') = |x'|^{m-(n-1)/2}\, \phi_0(x'/|x'|)$. Let $\eta \in C_0^\infty(\mathbb{R}^{n-1})$ be a function equal to one in a neighbourhood of the origin. We set $h = \eta v + v_1$, where $v_1 \in \overset{\circ}{W}{}_2^m(\mathbb{R}^{n-1}\backslash\{0\})$ is a solution of the equation

$$\mathcal{L}^{(0)}(\partial_{x'}, \pm i)\, v_1 = -\mathcal{L}^{(0)}(\partial_{x'}, \pm i)\,(\eta v) \quad \text{in } \mathbb{R}^{n-1}\backslash\{0\}\,.$$

Since $v_1 = O(|x'|^{m-(n-1)/2+\varepsilon})$, the function h belongs to the kernel of the operator (10.2.7) for $l = 2m$, $\beta > m$.

In a similar manner, we can construct a nontrivial solution of the equation $\mathcal{L}^{(0)}(\partial_{x'}, \pm i)^*\, w = 0$ which has the asymptotics $|x'|^{m-(n-1)/2}\psi(x'/|x'|)$ as $|x'| \to 0$. Clearly, w belongs to the cokernel of the operator (10.2.7) for $l = 2m$, $\beta < m$. Thus, there do not exist numbers l, β for which (10.2.7) is an isomorphism. The theorem is proved. ■

Thus, under the assumption of Theorem 10.2.2, the assertions of Theorem 10.2.1 are valid for

$$\begin{aligned} &\mu_- = 2m - n + 1, \quad \mu_+ = 0 \quad \text{if } 2m < n - 1, \\ &\mu_- = m - n/2, \quad \mu_+ = m + 1 - n/2 \quad \text{if } 2m > n - 1 \text{ and } n \text{ is even.} \end{aligned}$$

In the other cases Theorem 10.2.1 is not applicable.

10.2.3. Solvability of the first limit problem. Using Theorem 10.2.1, we can prove the following theorem (cf. Theorem 5.2.2).

THEOREM 10.2.3. *Suppose that the operators (10.2.7) are isomorphisms for* $l = l_0$, $\beta = \beta_0$, *and the numbers* l, β *satisfy the inequalities* $\mu_- < -\beta + l - (n-1)/2 < \mu_+$. *Furthermore, we assume that no eigenvalues of the pencil*

(10.2.8)
$$\mathfrak{A}(\lambda) \; : \; V^l_{2,\beta}(\Omega) \times \prod_{j=1}^{J} W_2^{l+t_j-1/2}(\partial\Omega) \to V^{l-2m}_{2,\beta}(\Omega) \times \prod_{k=1}^{m+J} W_2^{l-m_k-1/2}(\partial\Omega)$$

lie on the line $\operatorname{Re}\lambda = -\gamma$. *Then the operator (10.2.4) is an isomorphism.*

Proof: From Lemma 10.2.1 and Theorem 10.2.1 it follows that every solution $(u, \underline{u}) \in V^l_{2,\beta,\gamma}(\mathcal{C}^\circ) \times \prod V^{l+t_j-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ of problem (10.1.16), (10.1.17) satisfies the estimate

$$\|u\|_{V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)} + \sum_{j=1}^{J} \|u_j\|_{V^{l+t_j-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})} \le c\Big(\|f\|_{V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ)} + \sum_{k=1}^{m} \|g_k\|_{V^{l-m_k-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})}\Big).$$

This proves the uniqueness of the solution. For arbitrary $f \in V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ)$, $\underline{g} \in V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ the solution of problem (10.1.16), (10.1.17) is given by the equality

$$(u, \underline{u}) = \mathcal{L}^{-1}_{\lambda\to x_n}\, \mathfrak{A}(\lambda)^{-1}\, \mathcal{L}_{x_n\to\lambda}\,(f, \underline{g})\,.$$

Here the inverse Laplace transformation $\mathcal{L}^{-1}_{\lambda\to x_n}$ is defined by the formula

$$(\mathcal{L}^{-1}_{\lambda\to x_n} v)(x_n) = \frac{1}{2\pi i} \int\limits_{Re\,\lambda=-\beta} e^{\lambda t}\, v(\lambda)\, d\lambda$$

(see Lemma 5.2.3). The theorem is proved. ■

10.2.4. Regularity assertions for the solution of the limit problem. The following assertion can be proved analogously to Corollary 5.4.1.

LEMMA 10.2.2. *Suppose that the operator (10.2.7) is an isomorphism for* $l = l_0$, $\beta = \beta_0$ *and the numbers* l, β *satisfy the inequality* $\mu_- < -\beta+l-(n-1)/2 < \mu_+$. *Furthermore, we assume that no eigenvalues of the pencil (10.2.8) lie in the closed strip between the lines* $\operatorname{Re}\lambda = -\gamma_1$ *and* $\operatorname{Re}\lambda = -\gamma_2$. *If* $(u, \underline{u}) \in V^l_{2,\beta,\gamma_1}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_1}(\partial\mathcal{C})$ *is a solution of problem (10.1.16), (10.1.17), where*

$$f \in V^{l-2m}_{2,\beta,\gamma_1}(\mathcal{C}^\circ) \cap V^{l-2m}_{2,\beta,\gamma_2}(\mathcal{C}^\circ), \qquad \underline{g} \in V^{l-\underline{m}-1/2}_{2,\beta,\gamma_1}(\partial\mathcal{C}) \cap V^{l-\underline{m}-1/2}_{2,\beta,\gamma_2}(\partial\mathcal{C}),$$

then $(u, \underline{u})$ *belongs to the space* $V^l_{2,\beta,\gamma_2}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_2}(\partial\mathcal{C})$.

As a consequence of Lemma 10.2.2 and Theorem 10.2.3, we obtain the following assertions (cf. Corollary 5.4.2).

COROLLARY 10.2.1. *Let the conditions of Lemma 10.2.2 be satisfied. Then the operator of problem (10.1.16), (10.1.17) is an isomorphism*

$$\bigcap_{i=1}^{2} \Big(V^l_{2,\beta,\gamma_i}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_i}(\partial\mathcal{C})\Big) \to \bigcap_{i=1}^{2} \Big(V^{l-2m}_{2,\beta,\gamma_i}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma_i}(\partial\mathcal{C})\Big)$$

and

$$\begin{aligned}&\Big(V^l_{2,\beta,\gamma_1}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_1}(\partial\mathcal{C})\Big) + \Big(V^l_{2,\beta,\gamma_2}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_2}(\partial\mathcal{C})\Big)\\ &\qquad \to \Big(V^{l-2m}_{2,\beta,\gamma_1}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma_1}(\partial\mathcal{C})\Big) + \Big(V^{l-2m}_{2,\beta,\gamma_2}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma_2}(\partial\mathcal{C})\Big)\end{aligned}$$

Let $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ)$ be the space with the norm

$$\|u\|_{V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ)} = \Big(\sum_{|\alpha|\le l} \int_{\mathcal{C}^\circ} e^{2x_n\gamma(x_n)}\, r^{2(\beta-l+|\alpha|)} |D^\alpha_x u(x)|^2\, dx\Big)^{1/2},$$

where γ is a real function from $C^\infty(\mathbb{R})$ such that $\gamma(x_n) = \gamma_1$ for $x_n < -1$ and $\gamma(x_n) = \gamma_2$ for $x_n > 1$, while β, γ_1, γ_2 are arbitrary real numbers. Obviously, the space $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ)$ coincides with $V^l_{2,\beta,\gamma_1}(\mathcal{C}^\circ) \cap V^l_{2,\beta,\gamma_2}(\mathcal{C}^\circ)$ if $\gamma_1 \le \gamma_2$ and with $V^l_{2,\beta,\gamma_1}(\mathcal{C}^\circ) + V^l_{2,\beta,\gamma_2}(\mathcal{C}^\circ)$ if $\gamma_1 \ge \gamma_2$.

Analogously to $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ)$, the spaces $V^{l+\underline{t}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$ and $V^{l-\underline{m}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$ are defined.

THEOREM 10.2.4. *Suppose that the operator (10.2.7) is an isomorphism for* $l = l_0$, $\beta = \beta_0$ *and the numbers* l, l', β, β' *satisfy the inequalities*

$$\mu_- < -\beta + l - (n-1)/2 < \mu_+, \quad \mu_- < -\beta' + l' - (n-1)/2 < \mu_+.$$

Furthermore, we assume that no eigenvalues of the pencil (10.2.8) lie in the strip $-\tau_2 \le \operatorname{Re}\lambda \le -\tau_1$, *where* $\tau_1 = \min(\gamma_1, \gamma_1')$, $\tau_2 = \max(\gamma_2, \gamma_2')$, $\gamma_1 \le \gamma_2$, $\gamma_1' \le \gamma_2'$. *If*

(10.2.9)
$$f \in V^{l-2m}_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ) \cap V^{l'-2m}_{2,\beta',\gamma_1',\gamma_2'}(\mathcal{C}^\circ), \quad \underline{g} \in V^{l-\underline{m}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C}) \cap V^{l'-\underline{m}-1/2}_{2,\beta',\gamma_1',\gamma_2'}(\partial\mathcal{C}),$$

then the uniquely determined solution $(u, \underline{u}) \in V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$ *of problem (10.1.16), (10.1.17) belongs to the space* $V^{l'}_{2,\beta',\gamma_1',\gamma_2'}(\mathcal{C}^\circ) \times V^{l'+\underline{t}-1/2}_{2,\beta',\gamma_1',\gamma_2'}(\partial\mathcal{C})$.

Proof: First let $l' = l$. We set $\sigma = \min(\beta, \beta')$. According to Remark 10.2.1, the spectrum of the operator pencil $\mathfrak{A}(\lambda)$ does not depend on the choice of the number β from the interval $l - \mu_+ - (n-1)/2 < \beta < l - \mu_- - (n-1)/2$. Therefore, it follows from the assumptions of the theorem that the strip $-\gamma_2' \le \operatorname{Re}\lambda \le -\gamma_1'$ is free of eigenvalues of the pencil

$$\mathfrak{A}(\lambda) \;:\; V^l_{2,\beta'}(\Omega) \times W_2^{l+\underline{t}-1/2}(\partial\Omega) \to V^{l-2m}_{2,\beta'}(\Omega) \times W_2^{l-\underline{m}-1/2}(\partial\Omega),$$

while the strip $-\tau_2 \le \operatorname{Re}\lambda \le -\tau_1$ is free of eigenvalues of the pencil

$$\mathfrak{A}(\lambda) \;:\; V^l_{2,\sigma}(\Omega) \times W_2^{l+\underline{t}-1/2}(\partial\Omega) \to V^{l-2m}_{2,\sigma}(\Omega) \times W_2^{l-\underline{m}-1/2}(\partial\Omega).$$

Consequently, according to Corollary 10.2.1, the problem (10.1.16), (10.1.17) has a unique solution in each of the spaces $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$, $V^l_{2,\beta',\gamma_1',\gamma_2'}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta',\gamma_1',\gamma_2'}(\partial\mathcal{C})$, and $V^l_{2,\sigma,\tau_1,\tau_2}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\sigma,\tau_1,\tau_2}(\partial\mathcal{C})$ for arbitrary functions f, g_k from $C_0^\infty((\overline{\Omega}\backslash\{0\}) \times \mathbb{R}))$, and $C_0^\infty(\partial\mathcal{C})$, respectively. Since

$$V^l_{2,\sigma,\tau_1,\tau_2}(\mathcal{C}^\circ) \subset V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ) \cap V^l_{2,\beta',\gamma_1',\gamma_2'}(\mathcal{C}^\circ),$$

it follows that all three solutions coincide.

We assume now that (10.2.9) is true and $\{(f^{(k)}, \underline{g}^{(k)})\}_{k\ge1}$ is a sequence in $C_0^\infty((\overline{\Omega}\backslash\{0\}) \times \mathbb{R})) \times C_0^\infty(\partial\mathcal{C})^m$ converging to $(f, \underline{g})$ in $V^{l-2m}_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$ and $V^{l-2m}_{2,\beta',\gamma_1',\gamma_2'}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta',\gamma_1',\gamma_2'}(\partial\mathcal{C})$. By Corollary 10.2.1, the corresponding sequence $\{(u^{(k)}, \underline{u}^{(k)})\}_{k\ge1}$ of solutions converges both in $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$ and $V^l_{2,\beta',\gamma_1',\gamma_2'}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta',\gamma_1',\gamma_2'}(\partial\mathcal{C})$. Since $u^{(k)} \to u$ in $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ)$ and $\underline{u}^{(k)} \to \underline{u}$ in $V^{l+\underline{t}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$, it follows that $u \in V^l_{2,\beta',\gamma_1',\gamma_2'}(\mathcal{C}^\circ)$ and $\underline{u} \in V^{l+\underline{t}-1/2}_{2,\beta',\gamma_1',\gamma_2'}(\partial\mathcal{C})$. Thus, the theorem is proved for $l' = l$.

The case $l' < l$ can be easily reduced to the case $l' = l$. If $l' > l$, then by the first part of the proof, we obtain $u \in V^l_{2,\beta+l-l',\gamma_1',\gamma_2'}(\partial\mathcal{C}^\circ)$ and $\underline{u} \in V^{l+\underline{t}-1/2}_{2,\beta+l-l',\gamma_1',\gamma_2'}(\partial\mathcal{C})$. From Theorem 10.2.3 we conclude that $u \in V^{l'}_{2,\beta,\gamma_1',\gamma_2'}(\partial\mathcal{C}^\circ)$, $\underline{u} \in V^{l'+\underline{t}-1/2}_{2,\beta,\gamma_1',\gamma_2'}(\partial\mathcal{C})$. Now it suffices to apply the above proved assertion. ■

10.3. The second limit problem

The goal of this section is to prove the solvability of the second limit problem in weighted Sobolev spaces $V^l_{2,\beta}$, where the index β characterizes the powerlike growth of the solution far from the x_n-axis. To this end, we study the parameter-depending problem which arises from the limit problem by applying the Laplace transformation with respect to the last variable.

10.3.1. The operator of the second limit problem. Let $\tilde{\mathcal{D}}$ be the set (10.1.18), where $\tilde{\Omega}$ is a domain in $\mathbb{R}^{n-1}$ with compact closure and with boundary $\partial\tilde{\Omega}$ of class C^l such that the origin 0 is contained in $\tilde{\Omega}$. We define the space $V^l_{2,\beta}(\tilde{\mathcal{D}})$ as the set of functions in $\tilde{\mathcal{D}}$ with finite norm

$$\|u\|_{V^l_{2,\beta}(\tilde{\mathcal{D}})} = \Big(\sum_{|\alpha|\le l}\int_{\tilde{\mathcal{D}}} \rho^{2(\beta-l+|\alpha|)}\,|D^\alpha_y u|^2\,dy\Big)^{1/2}, \tag{10.3.1}$$

where $\rho = |y'|$, $\beta\in\mathbb{R}$, $l = 0,1,2,\dots$. Obviously, the space of traces of functions from $V^l_{2,\beta}(\tilde{\mathcal{D}})$, $l\ge 1$, on $\partial\tilde{\mathcal{D}}$ coincides with the space $W_2^{l-1/2}(\partial\tilde{\mathcal{D}})$.

Let $\mathcal{L}^{(0)}(y',\partial_y)$, $\mathcal{P}_k(y',\partial_y)$, $\mathcal{Q}_{k,j}(y',\partial_y)$ be the operators (10.1.21), (10.1.22), and (10.1.23), respectively. From our assumptions on the boundary value problem (10.1.2), (10.1.3) it follows that the second limit problem (10.1.25), (10.1.26) is elliptic. The operator

$$\mathcal{A}_2\,:\, V^l_{2,\beta}(\tilde{\mathcal{D}})\times\prod_{j=1}^{J} W_2^{l+\tau_j-1/2}(\partial\tilde{\mathcal{D}}) \to V^{l-2m}_{2,\beta}(\tilde{\mathcal{D}})\times\prod_{k=1}^{m+J} W_2^{l-\mu_k-1/2}(\partial\tilde{\mathcal{D}}) \tag{10.3.2}$$

of this problem is continuous. Here and in what follows, l is always an integer, $l\ge 2m$, $l>\max\mu_k$.

The goal of this section is to obtain conditions under which the mapping (10.3.2) is an isomorphism.

10.3.2. The parameter-depending problem generated by the second limit problem. Applying the Laplace transformation (with respect to the variable y_n) to problem (10.1.25), (10.1.26), we obtain the parameter-depending problem

$$\mathcal{L}^{(0)}(y',\partial_{y'},\lambda)\,\check{u} = \check{f} \quad \text{in } \mathbb{R}^{n-1}\backslash\bar{\tilde{\Omega}},$$

$$\mathcal{P}_k(y',\partial_{y'},\lambda)\,\check{u} + \sum_{j=1}^{J}\mathcal{Q}_{k,j}(y',\partial_{y'},\lambda)\,\check{u}_j = \check{g}_k \quad \text{on } \partial\tilde{\Omega},\ k=1,\dots,m+J.$$

We denote the operator of this problem by $\mathfrak{B}(\lambda)$. For $\lambda\ne 0$ we consider the operators $\mathcal{L}^{(0)}(y',\partial_{y'},\lambda)$ and $\mathcal{P}_k(y',\partial_{y'},\lambda)$ on the space $W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$ with the norm

$$\|v\|_{W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} = \Big(\int_{\mathbb{R}^{n-1}\backslash\tilde{\Omega}}\sum_{|\alpha|\le l}\rho^{2\beta}\,|D^\alpha_{y'}u|^2\,dy'\Big)^{1/2}.$$

Note that the norm in $W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$ is equivalent to

$$\sum_{j=1}^{l}|\lambda|^j\,\|u\|_{V^{l-j}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})}$$

for every fixed $\lambda\ne 0$, where $V^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$ is the space with the norm

$$\|u\|_{V^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} = \Big(\int_{\mathbb{R}^{n-1}\backslash\tilde{\Omega}}\sum_{|\alpha|\le l}\rho^{2(\beta-l+|\alpha|)}\,|D^\alpha_{y'}u|^2\,dy'\Big)^{1/2}. \tag{10.3.3}$$

Clearly, for $\lambda\ne 0$, $l\ge 2m$, $l>\max\mu_k$ the mapping

$$\mathfrak{B}(\lambda)\,:\, W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})\times W_2^{l+\underline{\tau}-1/2}(\partial\tilde{\Omega}) \to W^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})\times W_2^{l-\underline{\mu}-1/2}(\partial\tilde{\Omega}) \tag{10.3.4}$$

is continuous. In addition, obviously, also the mapping

$$\mathfrak{B}(0) \,:\, V^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \times W_2^{l+\underline{\tau}-1/2}(\partial\tilde{\Omega}) \rightarrow V^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \times W_2^{l-\underline{\mu}-1/2}(\partial\tilde{\Omega}) \tag{10.3.5}$$

is continuous.

LEMMA 10.3.1. *If the line* $\operatorname{Re}\mu = -\beta+l-(n-1)/2$ *does not contain eigenvalues of the pencil* $\mathfrak{M}$ *(introduced before Theorem 10.2.1), then the mapping (10.3.4) is Fredholm for* $\lambda \neq 0$.

Proof: Let $u \in W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$ be a function with support contained in a bounded subset S_0 of $\mathbb{R}^{n-1}\backslash\tilde{\Omega}$ and let $\underline{u}$ be an arbitrary vector-function from $W_2^{l+\underline{\tau}-1/2}(\partial\tilde{\Omega})$. Then by Corollary 3.2.1, the estimate

$$\|u\|_{W_2^l(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} + \|\underline{u}\|_{W_2^{l+\underline{\tau}-1/2}(\partial\tilde{\Omega})} \;\leq\; c\Big(\|f\|_{W_2^{l-2m}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} + \|\underline{g}\|_{W_2^{l-\underline{\mu}-1/2}(\partial\tilde{\Omega})} + \|u\|_{L_2(S_0)} + \|\underline{u}\|_{L_2(\partial\tilde{\Omega})^J}\Big)\,,$$

where $f = \mathcal{L}^{(0)}(y',\partial_{y'},\lambda)u$, $\underline{g} = \mathcal{P}(y',\partial_{y'},\lambda)u|_{\partial\tilde{\Omega}} + \mathcal{Q}(y',\partial_{y'},\lambda)\underline{u}$, is valid with a constant c independent of u and $\underline{u}$. Since the W_2^l and $W^l_{2,\beta}$ norms are equivalent on the set of functions with support in S_0, the W_2^l and W_2^{l-2m} norms in the above estimate can be replaced by the $W^l_{2,\beta}$ and $W^{l-2m}_{2,\beta}$ norms, respectively.

If u is an arbitrary function from $W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$ equal to zero in a neighbourhood of $\partial\tilde{\Omega}$, then Theorem 6.5.1 (for $\mathcal{K} = \mathbb{R}^{n-1}\backslash\{0\}$) yields

$$\|u\|_{W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} \leq c\left(\|\mathcal{L}^{(0)}(y',\partial_{y'},\lambda)\,u\|_{W^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} + \|u\|_{L_2(S_1)}\right),$$

where S_1 is a compact subset of $\mathbb{R}^{n-1}\backslash\tilde{\Omega}$. Consequently, the inequality

$$\|u\|_{W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} + \|\underline{u}\|_{W_2^{l+\underline{\tau}-1/2}(\partial\tilde{\Omega})} \;\leq\; c\Big(\|f\|_{W^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})} + \|\underline{g}\|_{W_2^{l-\underline{\mu}-1/2}(\partial\tilde{\Omega})} + \|u\|_{L_2(S_2)} + \|\underline{u}\|_{L_2(\partial\tilde{\Omega})^J}\Big)\,,$$

holds for arbitrary $u \in W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$, where S_2 is a bounded subset of $\mathbb{R}^{n-1}\backslash\tilde{\Omega}$. From this inequality we conclude that the range of the operator (10.3.4) is closed and its kernel has finite dimension. The same fact is true for the operator of the formally adjoint boundary value problem. Hence the operator (10.3.4) is Fredholm. ■

Furthermore, analogously to Theorem 6.5.5, the following lemma holds.

LEMMA 10.3.2. *Let* $\mathcal{L}^{(0)}(x',\partial_{x'},\pm i)$ *be the operators realizing the mappings (10.2.7). If the line* $\operatorname{Re}\mu = -\beta + l - (n-1)/2$ *does not contain eigenvalues of the pencil* $\mathfrak{M}$ *and the subspaces* $\ker\mathcal{L}^{(0)}(x',\partial_{x'},\pm i)$, $\operatorname{coker}\mathcal{L}^{(0)}(x',\partial_{x'},\pm i)$ *are trivial, then for purely imaginary* λ *with sufficiently large modulus the mapping (10.3.4)* *is an isomorphism, and the estimate*

$$(10.3.6)\sum_{j=0}^{l}|\lambda|^j\|u\|_{V_{2,\beta}^{l-j}(\mathbb{R}^{n-1}\backslash\tilde\Omega)}+\sum_{j=1}^{J}\Big(\|u_j\|_{W_2^{l+\tau_j-1/2}(\partial\tilde\Omega)}+|\lambda|^{l+\tau_j-1/2}\|u_j\|_{L_2(\partial\tilde\Omega)}\Big)$$
$$\le c\bigg(\sum_{j=0}^{l-2m}|\lambda|^j\,\|f\|_{V_{2,\beta}^{l-2m-j}(\mathbb{R}^{n-1}\backslash\tilde\Omega)}$$
$$+\sum_{k=1}^{m+J}\Big(\|g_k\|_{W_2^{l-\mu_k-1/2}(\partial\tilde\Omega)}+|\lambda|^{l-\mu_k-1/2}\|g_k\|_{L_2(\partial\tilde\Omega)}\Big)\bigg)$$

is valid for all $u\in W_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega)$, $u_j\in W_2^{l+\tau_j-1/2}(\partial\tilde\Omega)$, $f=\mathcal{L}^{(0)}(y',\partial_{y'},\lambda)\,u$, $g_k=\mathcal{P}_k(y',\partial_{y'},\lambda)u|_{\partial\tilde\Omega}+\sum\mathcal{Q}_{k,j}(y',\partial_{y'},\lambda)u_j$. *Here the constant* c *is independent of* u, u_j, *and* λ.

We consider the operator $\mathfrak{B}(\lambda)$ for small $|\lambda|$, $\lambda\neq0$. Let $u\in W_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega)$ and let v be the function which arises from u via the transformation $y'\to|\lambda|\,y'$. This transformation takes the domain $\tilde\Omega$ onto

$$\tilde\Omega_\lambda=\{y'\,:\,|\lambda|^{-1}y'\in\tilde\Omega\}.$$

Furthermore, we have

$$|\lambda|^j\,\|u\|_{V_{2,\beta}^{l-j}(\mathbb{R}^{n-1}\backslash\tilde\Omega)}=|\lambda|^{l-\beta-(n-1)/2}\,\|v\|_{V_{2,\beta}^{l-j}((\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)}\quad\text{for } j=0,1,\dots,l,$$

where the norm in $V_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)$ is defined analogously to (10.3.3). We define $E_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)$ as the space with the norm

$$\|v\|_{E_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)}=\sum_{j=0}^{l}\|v\|_{V_{2,\beta}^{l-j}(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)}\,.$$

Furthermore, let $V_{2,\beta}^{l-1/2}(\partial\tilde\Omega_\lambda)$, $E_{2,\beta}^{l-1/2}(\partial\tilde\Omega_\lambda)$ be the trace spaces for $V_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)$ and $E_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)$, $l\ge1$, equipped with the norms

$$\|u\|_{V_{2,\beta}^{l-1/2}(\partial\tilde\Omega_\lambda)}=\inf\Big\{\|v\|_{V_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)}\,:\,v\in V_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda),\ v\big|_{\partial\tilde\Omega}=u\Big\},$$
$$\|u\|_{E_{2,\beta}^{l-1/2}(\partial\tilde\Omega_\lambda)}=\inf\Big\{\|v\|_{E_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda)}\,:\,v\in E_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde\Omega_\lambda),\ v\big|_{\partial\tilde\Omega}=u\Big\}.$$

LEMMA 10.3.3. *Suppose that the conditions of Lemma 10.3.2 are satisfied and the operator (10.3.5) realizes an isomorphism. Then for small* $|\lambda|$, $\operatorname{Re}\lambda=0$, *the mapping (10.3.4) realizes an isomorphism and the estimate (10.3.6) holds.*

Proof: The proof consists of two parts:

a) *Existence of a right inverse for* $\mathfrak{B}(\lambda)$. Let η be a function from $C_0^\infty(\mathbb{R}^{n-1})$ equal to one in the neighbourhood of the origin. We set

$$\eta_\sigma(y')=\eta(y'\,|\lambda|^{-\sigma}),\qquad \zeta_\sigma(y')=1-\eta_\sigma(y').$$

If $0<\sigma<1$ and $|\lambda|$ is small, then $\eta_\sigma=1$ in a neighbourhood of $\tilde\Omega_\lambda$, while $\operatorname{supp}\zeta_\sigma\subset\mathbb{R}^{n-1}\backslash\overline{\tilde\Omega}_\lambda$. In the sequel, θ ,σ, and τ are numbers satisfying the inequalities $0<\tau<\sigma<\theta<1$. Then the equalities $\zeta_\theta\,\zeta_\sigma=\zeta_\sigma$, $\zeta_\sigma\,\zeta_\tau=\zeta_\tau$ are valid for sufficiently small $|\lambda|$.

The mapping $y' \to |\lambda|\, y'$ transforms the differential operators $\mathcal{L}^{(0)}(y', \partial_{y'}, \lambda)$, $\mathcal{L}^{(0)}(y', \partial_{y'}, 0)$ into the operators $|\lambda|^{2m}\, \mathcal{L}^{(0)}(y', \partial_{y'}, \lambda/|\lambda|)$ and $|\lambda|^{2m}\, \mathcal{L}^{(0)}(y', \partial_{y'}, 0)$, respectively. Analogous assertions are true for the differential operators $\mathcal{P}_k$ and $\mathcal{Q}_{k,j}$. We denote the operators of the boundary value problems

$$\mathcal{L}^{(0)}(y', \partial_{y'}, \lambda/|\lambda|)\, v = f \quad \text{in } \mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda,$$
$$\mathcal{P}_k(y', \partial_{y'}, \lambda/|\lambda|)\, v + \sum_{j=1}^{J} \mathcal{Q}_{k,j}(y', \partial_{y'}, \lambda/|\lambda|)\, v_j = g_k \quad \text{on } \partial\tilde{\Omega}_\lambda,\ k = 1, \dots, m+J$$

and

$$\mathcal{L}^{(0)}(y', \partial_{y'}, 0)\, v = f \quad \text{in } \mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda,$$
$$\mathcal{P}_k(y', \partial_{y'}, 0)\, v + \sum_{j=1}^{J} \mathcal{Q}_{k,j}(y', \partial_{y'}, 0)\, v_j = g_k \quad \text{on } \partial\tilde{\Omega}_\lambda,\ k = 1, \dots, m+J$$

by $\hat{\mathfrak{B}}(\lambda)$ and $\hat{\mathfrak{B}}(0)$, respectively. The operator $\hat{\mathfrak{B}}(\lambda)$ continuously maps the space

$$\mathcal{E}^1_\lambda \stackrel{def}{=} E^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda) \times E^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\tilde{\Omega}_\lambda)$$

into

$$\mathcal{E}^2_\lambda \stackrel{def}{=} E^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda) \times E^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\tilde{\Omega}_\lambda)\,,$$

whereas $\hat{\mathfrak{B}}(0)$ continuously maps

$$\mathcal{V}^1_\lambda \stackrel{def}{=} V^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda) \times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\tilde{\Omega}_\lambda)$$

into

$$\mathcal{V}^2_\lambda \stackrel{def}{=} V^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda) \times V^{l-\underline{\mu}-1/2}_{2,\beta}(\partial\tilde{\Omega}_\lambda).$$

Under the assumptions of the lemma, the operator $\hat{\mathfrak{B}}(0)$ is invertible and the norm of its inverse is independent of λ.

We introduce the operator

$$\mathfrak{S}_r(\lambda)\, :\, (f, \underline{g}) \to \big(\zeta_\theta\, (\mathcal{L}^{(0)}(\lambda/|\lambda|))^{-1}\, \zeta_\sigma\, f,\, 0\big) + \eta_\tau\, \hat{\mathfrak{B}}(0)^{-1}\, \eta_\sigma(f, \underline{g}), \tag{10.3.7}$$

where $\mathcal{L}^{(0)}(\lambda) = \mathcal{L}^{(0)}(y', \partial_{y'}, \lambda)$, $(f, \underline{g}) \in \mathcal{E}^2_\lambda$. Clearly, for $\lambda \neq 0$ the operator (10.3.7) continuously maps $\mathcal{E}^2_\lambda$ into $\mathcal{E}^1_\lambda$, and its norm is uniformly bounded with respect to λ.

We show that $\hat{\mathfrak{B}}(\lambda)\mathfrak{S}_r(\lambda) - I$ has a small operator norm in $\mathcal{E}^2_\lambda$. Let $u^{(1)}$ and $(u^{(2)}, \underline{u}^{(2)})$ be defined as follows:

$$u^{(1)} = \mathcal{L}^{(0)}(\lambda/|\lambda|)^{-1}\, \zeta_\sigma\, f, \qquad (u^{(2)}, \underline{u}^{(2)}) = \hat{\mathfrak{B}}(0)^{-1} \eta_\sigma\, (f, \underline{g}).$$

Then $\mathfrak{S}_r(\lambda)\, (f, \underline{g}) = \zeta_\theta(u^{(1)}, 0) + \eta_\tau(u^{(2)}, \underline{u}^{(2)})$. Furthermore, for sufficiently small λ we have

$$\mathcal{L}^{(0)}(\lambda/|\lambda|)\, \zeta_\theta u^{(1)} = \zeta_\theta\, \mathcal{L}^{(0)}(\lambda/|\lambda|)\, u^{(1)} + [\mathcal{L}^{(0)}(\lambda/|\lambda|), \zeta_\theta]\, u^{(1)}\,.$$

Here $[\mathcal{L}^{(0)}(\lambda/|\lambda|), \zeta_\theta]$ denotes the commutator of $\mathcal{L}^{(0)}(\lambda/|\lambda|)$ and ζ_θ. Since $\zeta_\theta\, \zeta_\sigma = \zeta_\sigma$, we get $\zeta_\theta\, \mathcal{L}^{(0)}(\lambda/|\lambda|)\, u^{(1)} = \zeta_\sigma f$. Therefore,

$$\mathcal{L}(\lambda/|\lambda|)\, \zeta_\theta u^{(1)} = \zeta_\sigma f + [\mathcal{L}^{(0)}(\lambda/|\lambda|), \zeta_\theta]\, u^{(1)}\,. \tag{10.3.8}$$

Moreover, we have $\mathcal{P}_k(\lambda/|\lambda|)\,\zeta_\theta u^{(1)} = 0$ for $k = 1, \dots, m+J$. Obviously, the operator

$$\mathcal{Q}_\varepsilon^{(1)}(\lambda) \stackrel{def}{=} (\rho^{-\varepsilon}\,\mathcal{L}^{(0)}(\lambda/|\lambda|)\,\rho^\varepsilon)^{-1} \,:\, E_{2,\beta}^{l-2m}(\mathbb{R}^{n-1}) \to E_{2,\beta}^{l}(\mathbb{R}^{n-1})$$

has a norm uniformly bounded with respect to small positive ε. Using the equality

$$[\mathcal{L}^{(0)}(\lambda/|\lambda|),\zeta_\theta]\,u^{(1)} = [\mathcal{L}^{(0)}(\lambda/|\lambda|),\zeta_\theta]\,(\rho/|\lambda|^\theta)^\varepsilon\,\mathcal{Q}_\varepsilon^{(1)}(\lambda)\,\zeta_\sigma(|\lambda|^\sigma/\rho)^\varepsilon\,|\lambda|^{(\theta-\sigma)\varepsilon}\,f$$

and the fact that the norms of the operators

$$\begin{aligned}
&[\mathcal{L}^{(0)}(\lambda/|\lambda|),\zeta_\theta]\,(\rho/|\lambda|^\theta)^\varepsilon \,:\, E_{2,\beta}^{l}(\mathbb{R}^{n-1}) \to E_{2,\beta}^{l-2m}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda),\\
&\zeta_\sigma\,(|\lambda|^\sigma/\rho)^\varepsilon \,:\, E_{2,\beta}^{l-2m}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda) \to E_{2,\beta}^{l-2m}(\mathbb{R}^{n-1})
\end{aligned}$$

are uniformly bounded, we obtain the inequality

$$\|[\mathcal{L}^{(0)}(\lambda/|\lambda|),\zeta_\theta]\,u^{(1)}\|_{E_{2,\beta}^{l-2m}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda)} \le c\,|\lambda|^{(\theta-\sigma)\varepsilon}\,\|(f,\underline{g})\|_{\mathcal{E}_\lambda^2}\,.$$

From this and from (10.3.8) it follows that the norm of the operator

$$(f,\underline{g}) \to \hat{\mathfrak{B}}(\lambda)\,\zeta_\theta(u^{(1)},0) - \zeta_\sigma\,(f,\underline{g}) = \big([\mathcal{L}^{(0)}(\lambda/|\lambda|),\zeta_\theta]\,u^{(1)},\,0\big)$$

in the space $\mathcal{E}_\lambda^2$ is small for small $|\lambda|$.

Now we consider the second term on the right-hand side of (10.3.7). We have

$$\hat{\mathfrak{B}}(\lambda)\,\eta_\tau(u^{(2)},\underline{u}^{(2)}) = \big(\hat{\mathfrak{B}}(\lambda) - \hat{\mathfrak{B}}(0)\big)\,\eta_\tau(u^{(2)},\underline{u}^{(2)}) + \hat{\mathfrak{B}}(0)\,\eta_\tau(u^{(2)},\underline{u}^{(2)})\,. \tag{10.3.9}$$

Since the principal parts of the operators $\hat{\mathfrak{B}}(\lambda)$ and $\hat{\mathfrak{B}}(0)$ (with respect to differentiation) coincide and $\rho \le c\,|\lambda|^\tau$ on $\operatorname{supp}\eta_\tau$, we get

$$\begin{aligned}
\|\big(\hat{\mathfrak{B}}(\lambda) - \hat{\mathfrak{B}}(0)\big)\,\eta_\tau(u^{(2)},\underline{u}^{(2)})\|_{\mathcal{E}_\lambda^2} &\le c\,|\lambda|^\tau\,\|\eta_\tau(u^{(2)},\underline{u}^{(2)})\|_{\mathcal{E}_\lambda^1}\\
&\le c\,|\lambda|^\tau\,\|\eta_\tau(u^{(2)},\underline{u}^{(2)})\|_{\mathcal{V}_\lambda^1}\,.
\end{aligned}$$

Due to the fact that $\hat{\mathfrak{B}}(0)$ is an isomorphism, it follows from this inequality that

$$\|\big(\hat{\mathfrak{B}}(\lambda) - \hat{\mathfrak{B}}(0)\big)\,\eta_\tau(u^{(2)},\underline{u}^{(2)})\|_{\mathcal{E}_\lambda^2} \le c\,|\lambda|^\tau\,\|\eta_\sigma(f,\underline{g})\|_{\mathcal{V}_\lambda^2} \le c\,|\lambda|^\tau\,\|(f,\underline{g})\|_{\mathcal{E}_\lambda^2}\,. \tag{10.3.10}$$

Furthermore, since $\eta_\tau\eta_\sigma = \eta_\sigma$, we have

$$\hat{\mathfrak{B}}(0)\,\eta_\tau(u^{(2)},\underline{u}^{(2)}) = \eta_\sigma\,(f,\underline{g}) + [\hat{\mathfrak{B}}(0),\eta_\tau]\,(u^{(2)},\underline{u}^{(2)})\,.$$

Obviously, the norm of the operator

$$\mathcal{Q}_\varepsilon^{(2)}(\lambda) \stackrel{def}{=} (\rho^\varepsilon\hat{\mathfrak{B}}(0)\rho^{-\varepsilon})^{-1} \,:\, \mathcal{V}_\lambda^2 \to \mathcal{V}_\lambda^1$$

is uniformly bounded with respect to small $|\lambda|$ and $\varepsilon > 0$. Using the equality

$$[\hat{\mathfrak{B}}(0),\eta_\tau]\,(u^{(2)},\underline{u}^{(2)}) = [\hat{\mathfrak{B}}(0),\eta_\tau]\,(|\lambda|^\tau/\rho)^\varepsilon\,\mathcal{Q}_\varepsilon^{(2)}(\lambda)\,\eta_\sigma\,(\rho/|\lambda|^\sigma)^\varepsilon\,|\lambda|^{(\sigma-\tau)\varepsilon}\,(f,\underline{g})$$

and the fact that the norms of the operators

$$[\hat{\mathfrak{B}}(0),\eta_\tau]\,(|\lambda|^\tau/\rho)^\varepsilon \,:\, \mathcal{V}_\lambda^1 \to \mathcal{E}_\lambda^2\,, \qquad \eta_\sigma\,(\rho/|\lambda|^\sigma)^\varepsilon \,:\, \mathcal{E}_\lambda^2 \to \mathcal{V}_\lambda^2$$

are uniformly bounded, we get

$$\|[\hat{\mathfrak{B}}(0),\eta_\tau]\,(u^{(2)},\underline{u}^{(2)})\|_{\mathcal{E}_\lambda^2} \le c\,|\lambda|^{(\sigma-\tau)\varepsilon}\|(f,\underline{g})\|_{\mathcal{E}_\lambda^2}\,.$$

From this and from (10.3.9), (10.3.10) we obtain that the norm of the operator $(f,\underline{g}) \to \hat{\mathfrak{B}}(\lambda)\eta_\tau(u^{(2)},\underline{u}^{(2)}) - \eta_\sigma(f,\underline{g})$ in the space $\mathcal{E}^2_\lambda$ is small for small $|\lambda|$. Thus, the norm of the operator $\hat{\mathfrak{B}}(\lambda)\,\mathfrak{S}_r - I$ in $\mathcal{E}^2_\lambda$ is small and

$$\mathfrak{S}_r \sum_{k=0}^{\infty} \left(I - \hat{\mathfrak{B}}(\lambda)\,\mathfrak{S}_r\right)^k .$$

is a right inverse to the operator $\hat{\mathfrak{B}}(\lambda)$. Consequently, there exists also a right inverse to the operator $\mathfrak{B}(\lambda)$.

b) *Existence of a left inverse to* $\mathfrak{B}(\lambda)$. We introduce the operator

$$\mathfrak{S}_l \,:\, (f,\underline{g}) \to \left(\zeta_\sigma\left(\mathcal{L}^{(0)}(\lambda/|\lambda|)\right)^{-1}\zeta_\theta f,\, 0\right) + \eta_\sigma \hat{\mathfrak{B}}(0)^{-1}\,\eta_\tau(f,\underline{g}).$$

Obviously,

$$(10.3.11) \qquad \mathfrak{S}_l\,\hat{\mathfrak{B}}(\lambda)\,(u,\underline{u}) = \left(\zeta_\sigma\left(\mathcal{L}^{(0)}(\lambda/|\lambda|)\right)^{-1}\zeta_\theta\,\mathcal{L}^{(0)}(\lambda/|\lambda|)\,u,\, 0\right) + \eta_\sigma\,\hat{\mathfrak{B}}(0)^{-1}\,\eta_\tau\,\hat{\mathfrak{B}}(\lambda)\,(u,\underline{u}).$$

We consider the first term on the right-hand side. Since $\zeta_\sigma\zeta_\theta = \zeta_\sigma$, we obtain

$$(10.3.12) \qquad \zeta_\sigma\big(\mathcal{L}^{(0)}(\lambda/|\lambda|)\big)^{-1}\zeta_\theta\,\mathcal{L}^{(0)}(\lambda/|\lambda|)\,u = \zeta_\sigma\big(\mathcal{L}^{(0)}(\lambda/|\lambda|)\big)^{-1}[\zeta_\theta, \mathcal{L}^{(0)}(\lambda/|\lambda|)]\,u + \zeta_\sigma\,u.$$

The norm of the operator

$$\mathcal{Q}^{(1)}_{-\varepsilon}(\lambda) \stackrel{def}{=} \left(\rho^\varepsilon\,\mathcal{L}^{(0)}(\lambda/|\lambda|)\,\rho^{-\varepsilon}\right)^{-1} \,:\, E^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}) \to E^l_{2,\beta}(\mathbb{R}^{n-1})$$

is uniformly bounded with respect to small $\varepsilon > 0$. Since

$$\zeta_\sigma\left(\mathcal{L}^{(0)}(\lambda/|\lambda|)\right)^{-1}[\zeta_\theta, \mathcal{L}^{(0)}(\lambda/|\lambda|)]u = \zeta_\sigma\left(|\lambda|^\sigma/\rho\right)^\varepsilon \mathcal{Q}^{(1)}_{-\varepsilon}\left(\rho/|\lambda|^\theta\right)^\varepsilon [\zeta_\theta, \mathcal{L}^{(0)}(\lambda/|\lambda|)]\,|\lambda|^{(\theta-\sigma)\varepsilon}\,u$$

and the norms of the operators

$$(\rho/|\lambda|^\theta)^\varepsilon\,[\zeta_\theta, \mathcal{L}^{(0)}(\lambda/|\lambda|)] \,:\, E^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda) \to E^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}),$$
$$\zeta_\sigma\,(|\lambda|^\sigma/\rho)^\varepsilon \,: E^l_{2,\beta}(\mathbb{R}^{n-1}) \to E^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda)$$

are uniformly bounded, we get

$$\|\zeta_\sigma\left(\mathcal{L}^{(0)}(\lambda/|\lambda|)\right)^{-1}[\zeta_\theta, \mathcal{L}^{(0)}(\lambda/|\lambda|)]\,u\|_{E^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda)} \le c\,|\lambda|^{(\theta-\sigma)\varepsilon}\,\|u\|_{E^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda)}\,.$$

From this and from (10.3.12) it follows that the norm of the operator

$$(10.3.13) \qquad u \to \zeta_\sigma\left(\mathcal{L}^{(0)}(\lambda/|\lambda|)\right)^{-1}\zeta_\theta\,\mathcal{L}^{(0)}(\lambda/|\lambda|)\,u - \zeta_\sigma\,u$$

is small in the space $E^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda)$ for small $|\lambda|$.

Now we consider the second term on the right-hand side of (10.3.11). We have

$$\eta_\sigma\,\hat{\mathfrak{B}}(0)^{-1}\,\eta_\tau\,\hat{\mathfrak{B}}(\lambda)\,(u,\underline{u}) = \eta_\sigma\,\hat{\mathfrak{B}}(0)^{-1}\,\eta_\tau\left(\hat{\mathfrak{B}}(\lambda) - \hat{\mathfrak{B}}(0)\right)(u,\underline{u}) + \eta_\sigma\,\hat{\mathfrak{B}}(0)^{-1}\,\eta_\tau\,\hat{\mathfrak{B}}(0)\,(u,\underline{u}).$$

By the same considerations as in the first part of the proof, we obtain

$$(10.3.14) \qquad \|\eta_\sigma\hat{\mathfrak{B}}(0)^{-1}\,\eta_\tau\left(\hat{\mathfrak{B}}(\lambda) - \hat{\mathfrak{B}}(0)\right)(u,\underline{u})\|_{\mathcal{E}^1_\lambda} \le c\,\|\eta_\sigma\,\hat{\mathfrak{B}}(0)^{-1}\,\eta_\tau\left(\hat{\mathfrak{B}}(\lambda) - \hat{\mathfrak{B}}(0)\right)(u,\underline{u})\|_{\mathcal{V}^1_\lambda} \le c\,\|\eta_\tau\big(\hat{\mathfrak{B}}(\lambda) - \hat{\mathfrak{B}}(0)\big)\,(u,\underline{u})\|_{\mathcal{E}^2_\lambda} \le c\,|\lambda|^\tau\,\|(u,\underline{u})\|_{\mathcal{E}^1_\lambda}\,.$$

Now, since $\eta_\sigma \eta_\tau = \eta_\sigma$, we get

$$\eta_\sigma \hat{\mathfrak{B}}(0)^{-1} \eta_\tau \hat{\mathfrak{B}}(0)(u, \underline{u}) = \eta_\sigma (u, \underline{u}) + \eta_\sigma \hat{\mathfrak{B}}(0)^{-1} [\eta_\lambda, \mathfrak{B}(0)] (u, \underline{u}). \tag{10.3.15}$$

The norm of the operator

$$Q^{(2)}_{-\varepsilon}(\lambda) \stackrel{def}{=} \left(\rho^{-\varepsilon} \hat{\mathfrak{B}}(0) \rho^{\varepsilon}\right)^{-1} : \mathcal{V}^2_\lambda \to \mathcal{V}^1_\lambda$$

is uniformly bounded for small $|\lambda|$ and $\varepsilon > 0$. Furthermore,

$$\begin{aligned} &\eta_\sigma \hat{\mathfrak{B}}(0)^{-1} [\eta_\tau, \hat{\mathfrak{B}}(0)] (u, \underline{u}) \\ &= \eta_\sigma \left(\rho/|\lambda|^\sigma\right)^\varepsilon Q^{(2)}_{-\varepsilon}(\lambda) \left(|\lambda|^\tau/\rho\right)^\varepsilon [\eta_\tau, \hat{\mathfrak{B}}(0)] \, |\lambda|^{\varepsilon(\sigma-\tau)} (u, \underline{u}). \end{aligned} \tag{10.3.16}$$

Here the norms of the operators

$$\left(|\lambda|^\tau/\rho\right)^\varepsilon [\eta_\tau, \hat{\mathfrak{B}}(0)] : \mathcal{E}^1_\lambda \to \mathcal{V}^2_\lambda, \qquad \eta_\sigma \left(\rho/|\lambda|^\sigma\right)^\varepsilon : \mathcal{V}^1_\lambda \to \mathcal{E}^1_\lambda$$

are uniformly bounded. Therefore, from (10.3.14)-(10.3.16) it follows that the norm of the operator $(u, \underline{u}) \to \eta_\sigma \hat{\mathfrak{B}}(0)^{-1} \eta_\tau \hat{\mathfrak{B}}(\lambda) (u, \underline{u}) - \eta_\sigma(u, \underline{u})$ is small in the space $\mathcal{E}^1_\lambda$. Hence the norm of the operator $\mathfrak{S}_l \hat{\mathfrak{B}}(\lambda) - I$ in the space $\mathcal{E}^1_\lambda$ is small for small $|\lambda|$, i.e., there exists a left inverse to the operator $\hat{\mathfrak{B}}(\lambda)$. This implies the existence of a left inverse to the operator $\mathfrak{B}(\lambda)$. The lemma is proved. ■

Combining Lemmas 10.3.1 - 10.3.3, we obtain the following result.

THEOREM 10.3.1. *Suppose that the following conditions are satisfied:*

(i) *The line* $\mathrm{Re}\,\mu = -\beta + l - (n-1)/2$ *does not contain eigenvalues of the pencil* $\mathfrak{M}(\omega, \partial_\omega, \mu)$*, and the kernels and cokernels of the operators (10.2.7) are trivial.*

(ii) *The subspaces* $\ker \mathfrak{B}(\lambda)$ *and* $\operatorname{coker} \mathfrak{B}(\lambda)$ *are trivial for purely imaginary* $\lambda \neq 0$.

(iii) *The mapping (10.3.5) is an isomorphism.*

Then for all purely imaginary λ *the operator* $\mathfrak{B}(\lambda)$ *realizes an isomorphism and estimate (10.3.6) holds with a constant* c *independent of* λ.

REMARK 10.3.1. The following example shows that conditions (i) and (iii) are not sufficient for the validity of condition (ii) in Theorem 10.3.1.

Example. We consider the following problem in the exterior of the unit ball B in $\mathbb{R}^3$:

$$\Delta_{y'} u + \lambda^2 u = f \quad \text{outside } B, \tag{10.3.17}$$

$$\partial u/\partial \rho - cu = g \quad \text{on } \partial B, \tag{10.3.18}$$

where $c \in (-\infty, -1)$. Let $\mathfrak{B}(\lambda)$ denote the operator of this problem.

We verify condition (i) of Theorem 10.3.1. For $l = 2$ condition (i) means that the mapping

$$\Delta - 1 : E^2_{2,\beta}(\mathbb{R}^3) \to E^0_{2,\beta}(\mathbb{R}^3) \tag{10.3.19}$$

is an isomorphism. Since the spectrum of the operator pencil

$$\mathfrak{M}(\omega, \partial_\omega, \mu) = \delta + \mu(\mu + 1)$$

consists of the numbers $\mu = 0, \pm 1, \dots$, the operator (10.3.19) is Fredholm if and only if the number $\beta - 1/2$ is not an integer. Clearly, elements of the kernel and cokernel of the operator may be only the fundamental solution $r^{-1}e^{-r}$ and its derivatives. Therefore, for $l=2$, $1/2<\beta<3/2$ these subspaces are trivial. Thus, condition (i) is satisfied for $1/2<l-\beta<3/2$.

In order to verify condition (iii) of the theorem, we have to show that the operator

$$(10.3.20)\quad \mathfrak{B}(0)=\big(\Delta,\,(\partial/\partial\rho-cI)|_{\partial B}\big)\,:\; V^2_{2,\beta}(\mathbb{R}^3\backslash B)\to V^0_{2,\beta}(\mathbb{R}^3\backslash B)\times W^{3/2}_2(\partial B)$$

realizes an isomorphism. For noninteger $\beta-1/2$ this operator is Fredholm. From the imbeddings $\ker\mathfrak{B}(0)\subset V^2_{2,\beta}(\mathbb{R}^3\backslash B)$, $\operatorname{coker}\mathfrak{B}(0)\subset V^0_{2,-\beta}(\mathbb{R}^3\backslash B)$ it follows that each of these subspaces can contain perhaps only harmonic functions vanishing at infinity. By means of expansion in a series of spherical functions, we find that the unique (within the accuracy of a constant factor) bounded harmonic function satisfying the homogeneous boundary condition (10.3.18) has the form $c+1-c\rho^{-1}$. This function does not belong to $V^2_{2,\beta}(\mathbb{R}^3\backslash B)$ and $V^0_{2,-\beta}(\mathbb{R}^3\backslash B)$ if $c\neq-1$ and $1/2<\beta<3/2$. Hence the subspaces $\ker\mathfrak{B}(0)$ and $\operatorname{coker}\mathfrak{B}(0)$ are trivial and the operator (10.3.20) realizes an isomorphism, i.e., condition (iii) of Theorem 10.3.1 is satisfied for $l=2$, $1/2<\beta<3/2$.

However, for $\lambda=\pm(c+1)i$ the subspaces $\ker\mathfrak{B}(\lambda)$, $\operatorname{coker}\mathfrak{B}(\lambda)$ contain the function $u=\rho^{-1}e^{-|\lambda|\rho}$. Thus, condition (ii) is not satisfied.

10.3.3. Solvability of the second limit problem. Theorem 10.3.1 implies the following statement (cf. Theorem 10.2.3).

THEOREM 10.3.2. *If conditions (i) - (iii) of Theorem 10.3.1 are satisfied, then the mapping (10.3.2) is an isomorphism.*

Analogously to Theorem 10.2.4 the following regularity assertion for the solution of problem (10.1.25), (10.1.26) holds.

COROLLARY 10.3.1. *Suppose that conditions (i) - (iii) of Theorem 10.3.1 are satisfied for the pairs (β,l) and (β',l'). If $(u,\underline{u})\in V^l_{2,\beta}(\tilde{\mathcal{D}})\times V^{l+\underline{\tau}-1/2}_{2,\beta}(\partial\tilde{\mathcal{D}})$ is a solution of problem (10.1.25), (10.1.26), where $f\in V^{l'-2m}_{2,\beta'}(\tilde{\mathcal{D}})$, $\underline{g}\in V^{l'-\underline{\mu}-1/2}_{2,\beta'}(\partial\tilde{\mathcal{D}})$, then $u\in V^{l'}_{2,\beta'}(\tilde{\mathcal{D}})$ and $\underline{u}\in V^{l'+\underline{\tau}-1/2}_{2,\beta'}(\partial\tilde{\mathcal{D}})$. Furthermore, the estimate*

$$\|u\|_{V^{l'}_{2,\beta'}(\tilde{\mathcal{D}})}+\|\underline{u}\|_{V^{l'+\underline{\tau}-1/2}_{2,\beta'}(\partial\tilde{\mathcal{D}})}\le c\left(\|f\|_{V^{l'}_{2,\beta'}(\tilde{\mathcal{D}})}+\|\underline{g}\|_{V^{l'-\underline{\mu}-1/2}_{2,\beta'}(\partial\tilde{\mathcal{D}})}\right)$$

is valid with a constant c independent of u and $\underline{u}$.

10.4. The auxiliary problem

This section deals with the auxiliary problem (10.1.13) - (10.1.15) in the domain $\mathcal{C}\backslash\mathcal{D}$, where $\mathcal{C}$ is an infinite cylinder and $\mathcal{D}$ is an infinite tube contained in $\mathcal{C}$ (see Section 10.1). Using the invertibility of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ of the limit problems considered in the previous sections, we prove the unique solvability of problem (10.1.13) - (10.1.15) in the weighted Sobolev space $V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})$. Here, as in Section 10.2, the index γ characterizes the exponential growth at infinity, while β characterizes the powerlike growth near the x_n-axis.

10.4.1. Auxiliary Statements.

Differential operators of the class $\mathcal{O}^\mu_k$. First we study the class $\mathcal{O}^\mu_k(G)$ of operators introduced in Definition 10.1.1. It can be easily seen that the following assertion is true.

LEMMA 10.4.1. *If* $\mathcal{P} \in \mathcal{O}_k^p(G)$, $\mathcal{Q} \in \mathcal{O}_l^q(G)$, $k \geq q$, *then* $\mathcal{Q}\mathcal{P} \in \mathcal{O}_s^{p+q}(G)$, *where* $s = \min(k-q, l)$, *and*

$$|\mathcal{Q}\mathcal{P}|_{\mathcal{O}_s^{p+q}(G)} \leq c\,|\mathcal{Q}|_{\mathcal{O}_l^q(G)}\ |\mathcal{P}|_{\mathcal{O}_k^p(G)}\,.$$

Let φ be a bounded positive infinitely differentiable function on $\mathbb{R}$ satisfying condition (10.1.1). Furthermore, let $\mathcal{C}^\circ = (\Omega\backslash\{0\})\times\mathbb{R}$, where Ω is an open bounded subset of $\mathbb{R}^{n-1}$ containing the origin.

LEMMA 10.4.2. *If* $\mathcal{P} \in \mathcal{O}_{l-k}^k(\mathcal{C}^\circ)$, $l \geq k$, *then*

$$|r^{-\lambda}\,\mathcal{P}\,r^\lambda - \mathcal{P}|_{\mathcal{O}_{l-k}^{k-1}(\mathcal{C}^\circ)} + |\varphi^{-\lambda}\,\mathcal{P}\,\varphi^\lambda - \mathcal{P}|_{\mathcal{O}_{l-k}^{k-1}(\mathcal{C}^\circ)} = O(|\lambda|), \quad |\lambda| \leq 1.$$

This assertion has to be verified first for the operator $r\partial_x$ and then one has to apply Lemma 10.4.1.

LEMMA 10.4.3. *The operator of multiplication by the function* $x \to \varphi(x_n)^\lambda$, $\lambda \geq 0$, *belongs to the class* $\mathcal{O}_l^0(\mathcal{C}^\circ)$. *Furthermore,*

$$|\varphi^\lambda|_{\mathcal{O}_l^0(\mathcal{C}^\circ)} \leq c\left(\sup_t \varphi(t)\right)^\lambda.$$

Proof: For $|\gamma| \leq l$ we have

$$\|(r\partial_x)^\gamma\,\varphi^\lambda\|_{L_\infty(\mathcal{C}^\circ)} \leq c\sum_{j=0}^{l}\|(\varphi^\lambda)^{(j)}\|_{L_\infty(\mathbb{R})}$$
$$\leq c\sum_{j=0}^{l}\Big\|\varphi^\lambda\sum_{k=0}^{j}\prod_{i_1+\cdots+i_k=j}|\varphi^{(i_1)}\varphi^{-1}|\cdots|\varphi^{(i_k)}\varphi^{-1}|\Big\|_{L_\infty(\mathbb{R})} \leq c\,\varphi_0^\lambda\,.$$

This proves the lemma. ∎

LEMMA 10.4.4. *If* $\eta \in \mathcal{O}_l^0(\mathbb{R}^{n-1})$, *then the operator of multiplication by the function* $x \to \eta(x'/\varphi(x_n)^\mu)$, $\mu \in \mathbb{R}$, *belongs to the class* $\mathcal{O}_l^0(\mathcal{C}^\circ)$.

Proof: For $l = 0$ the statement is obvious. Suppose that it is proved for $l = 0, 1, \dots, k-1$. We have

$$r\,\partial_{x_j}\,\eta(x'/\varphi^\mu) = \sum_{q=1}^{n-1}(x_q/r)\,(x_q/\varphi^\mu)\,(\partial_{x_j}\eta)(x'/\varphi^\mu), \quad j \leq n-1, \tag{10.4.1}$$

$$r\,\partial_{x_n}\,\eta(x'/\varphi^\mu) = -\mu\sum_{q=1}^{n-1}(\partial_{x_q}\eta)(x'/\varphi^\mu)\,(x_q/\varphi^\mu)\,(r\varphi'/\varphi). \tag{10.4.2}$$

By virtue of the induction hypothesis, the operator of multiplication by the function $x \to (\partial_{x_j}\eta)(x'/\varphi(x_n)^\mu)\,x_q/\varphi(x_n)^\mu$ belongs to the class $\mathcal{O}_{l-1}^0(\mathcal{C}^\circ)$ for $j, q = 1, \dots, n-1$. Consequently, the same is true for the left-hand sides of equalities (10.4.1), (10.4.2). The lemma is proved. ∎

LEMMA 10.4.5. *If* $\mathcal{P} \in \mathcal{O}_{l-k}^k(\mathcal{C}^\circ)$, $l \geq k$, *and* $\eta \in \mathcal{O}_l^0(\mathbb{R}^{n-1})$, *then the commutator* $[\mathcal{P}, \eta(x'/\varphi(x_n)^\mu)]$ *belongs to the class* $\mathcal{O}_{l-k}^{k-1}(\mathcal{C}^\circ)$.

For the operator $r\partial_x$ the assertion follows from Lemma 10.4.4. In the general case we obtain this assertion using Lemma 10.4.1.

Change of Variables. We consider the mapping $\kappa : (x', x_n) \to (y', y_n)$, where

$$y' = \frac{x'}{\varphi(x_n)}, \qquad y_n = \chi(x_n) \stackrel{def}{=} \int_0^{x_n} \frac{dt}{\varphi(t)}.$$

The Jacobi matrix of this mapping is

$$\kappa' = \varphi(x_n)^{-1} I - \varphi(x_n)^{-2} \varphi'(x_n) \begin{pmatrix} 0 & \cdots & 0 & x_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & x_{n-1} \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

where I is the identity matrix, and the inverse mapping has the form

$$x' = \frac{y'}{\psi(y_n)}, \qquad x_n = \chi^{-1}(y_n) = \int_0^{y_n} \frac{dt}{\psi(t)},$$

where $\psi(y_n) = 1/\varphi(\chi^{-1}(y_n))$. By induction in l, it can be easily shown that

$$\sup_{t \in \mathbb{R}} \psi(t)^{j-1} \, |\psi^{(j)}(t)| < \infty \qquad \text{for } j = 1, 2, \dots, l.$$

Thus, the function ψ satisfies also the relation (10.1.1).

LEMMA 10.4.6. *If $a \in \mathcal{O}^0_k(\mathcal{C}^\circ)$, $k \le l$, then $a \circ \chi^{-1} \in \mathcal{O}^0_k(\kappa(\mathcal{C}^\circ))$ and*

$$c_1 \, |a|_{\mathcal{O}^0_k(\mathcal{C}^\circ)} \le |a \circ \kappa^{-1}|_{\mathcal{O}^0_k(\kappa(\mathcal{C}^\circ))} \le c_2 \, |a|_{\mathcal{O}^0_k(\mathcal{C}^\circ)}, \tag{10.4.3}$$

where c_1, c_2 are positive constants independent of a.

Proof: We show the left inequality of (10.4.3). Let κ' be the Jacobi matrix of the mapping κ. Then

$$|a|_{\mathcal{O}^0_k(\mathcal{C}^\circ)} \le c \Big(|r \, (\kappa')^* \, (\partial_y(a \circ \kappa^{-1})) \circ \kappa|_{\mathcal{O}^0_{k-1}(\mathcal{C}^\circ)} + |a \circ \kappa^{-1}|_{L_\infty(\kappa(\mathcal{C}^\circ))} \Big).$$

Using the explicit form of the mapping κ, we can estimate the first term on the right-hand side from above by the quantity

$$c \Big(|\rho \partial_y(a \circ \kappa^{-1}) \circ \kappa|_{\mathcal{O}^0_{k-1}(\mathcal{C}^\circ)} + \sum_{j=1}^{n-1} |(\varphi' x_j/\varphi) \, (\rho \partial_y(a \circ \kappa^{-1})) \circ \kappa|_{\mathcal{O}^0_{k-1}(\mathcal{C}^\circ)} \Big),$$

where $\rho(y) = |y'|$. Since the operators of multiplication by the functions $x_j r^{-1}$ and $r\varphi'/\varphi$ belong to the class $\mathcal{O}^0_{k-1}(\mathcal{C}^\circ)$, the last sum does not exceed

$$c \, |(\rho \partial_y(a \circ \kappa^{-1})) \circ \kappa|_{\mathcal{O}^0_{k-1}(\mathcal{C}^\circ)}. \tag{10.4.4}$$

We assume that the left inequality of (10.4.3) is proved for the spaces $\mathcal{O}^0_{k-1}$. Then the expression (10.4.4) does not exceed the quantity

$$c \, |\rho \partial_y(a \circ \kappa^{-1})|_{\mathcal{O}^0_{k-1}(\kappa(\mathcal{C}^\circ))}.$$

This proves the left inequality of (10.4.3). Analogously, the right inequality can be proved. ∎

LEMMA 10.4.7. *The operator* $v \to (r\,\partial_x(v \circ \kappa)) \circ \kappa^{-1}$ *belongs to the class* $\mathcal{O}^1_{l-1}(\kappa(\mathcal{C}^\circ))$ *and the operator* $u \to (\rho\,\partial_y(u \circ \kappa^{-1})) \circ \kappa$ *belongs to the class* $\mathcal{O}^1_{l-1}(\mathcal{C}^\circ)$, $k = 1, 2, \dots$.

Proof: We have

$$r\,\partial_x(v \circ \kappa) = r \cdot (\kappa')^*\,(\partial_y v) \circ \kappa = \big((\varphi \cdot (\kappa')^*)\, \circ \kappa^{-1}\,\rho\,\partial_y v\big) \circ \kappa. \tag{10.4.5}$$

From the form of the mapping κ it follows that the elements of the matrix $\varphi \cdot (\kappa')^*$ belong to the space $\mathcal{O}^0_{l-1}(\mathcal{C}^\circ)$. Hence by virtue of Lemma 10.4.1, we obtain the first statement of the lemma. The second statement can be proved in a similar manner. ■

Let $\mathcal{P}_\kappa$ be the operator defined by the equality $\mathcal{P}_\kappa v = (\mathcal{P}(v \circ \kappa)) \circ \kappa^{-1}$. Lemmas 10.4.1, 10.4.6, and 10.4.7 lead to the following statement.

LEMMA 10.4.8. *The operator* $\mathcal{P}$ *belongs to the class* $\mathcal{O}^k_{l-k}(\mathcal{C}^\circ)$, $l \ge k$, *if and only if* $\mathcal{P}_\kappa \in \mathcal{O}^k_{l-k}(\kappa(\mathcal{C}^\circ))$. *Furthermore,*

$$c_1\,|\mathcal{P}|_{\mathcal{O}^k_{l-k}(\mathcal{C}^\circ)} \le |\mathcal{P}_\kappa|_{\mathcal{O}^k_{l-k}(\kappa(\mathcal{C}^\circ))} \le c_2\,|\mathcal{P}|_{\mathcal{O}^k_{l-k}(\mathcal{C}^\circ)}\,.$$

Let $\mathcal{Q}$ be a differential operator in $\kappa(\mathcal{C}^\circ)$,

$$\mathcal{Q}(y, \rho\partial_y) = \sum_{|\alpha| \le k} q_\alpha(y)\,(\rho\partial_y)^\alpha,$$

and let $\mathcal{T}$ be the operator in $\mathcal{C}^\circ$ defined by the equality

$$\mathcal{T}(x, r\partial_x) = \sum_{|\alpha| \le k} q_\alpha(\kappa(x))\,(r\partial_x)^\alpha\,.$$

LEMMA 10.4.9. *If* $0 < \varphi \le \varphi_0 = const \le 1$ *and* $\lambda \ge 0$, *then*

$$|\mathcal{T} - \mathcal{Q}_{\kappa^{-1}}|_{\mathcal{O}^k_{l-k}(\mathcal{C}_\lambda)} \le c\,\varphi_0^\lambda, \tag{10.4.6}$$

where $\mathcal{C}_\lambda = \{x \in \mathbb{R}^n : x'/\varphi(x_n)^\lambda \in \Omega\}$.

Proof: From the explicit form of the mapping κ it follows that

$$|\varphi\,(\kappa')^* - I|_{\mathcal{O}^0_{l-1}(\mathcal{C}_\lambda)} \le c\,\varphi_0^\lambda\,,$$

where I is the identity matrix. This inequality and (10.4.5) yield the assertion of the lemma for the operator $r\partial_x$ and, consequently, also for all operators of class $\mathcal{O}^1_{l-1}(\mathcal{C}_\lambda)$.

We assume that the lemma is proved for operators of the class $\mathcal{O}^{k-1}_{l-k+1}(\mathcal{C}_\lambda)$. For an arbitrary multi-index α of order k we have

$$\begin{aligned}|(r\partial_x)^\alpha - (\rho\partial_y)^\alpha_{\kappa^{-1}}|_{\mathcal{O}^k_{l-k}(\mathcal{C}_\lambda)} &\le \sum_{|\beta| \le k-1} |r\,\partial_x\big((r\partial_x)^\beta - (\rho\partial_y)^\beta_{\kappa^{-1}}\big)|_{\mathcal{O}^k_{l-k}(\mathcal{C}_\lambda)} \\ &+ \sum_{|\beta| \le k-1} |\big(r\partial_x - (\rho\partial_y)_{\kappa^{-1}}\big)\,(\rho\partial_y)^\beta_{\kappa^{-1}}|_{\mathcal{O}^k_{l-k}(\mathcal{C}_\lambda)}\,.\end{aligned}$$

The first sum on the right can be estimated by means of the induction hypothesis and of Lemma 10.4.1, while the second sum can be estimated using the basis of the induction and Lemma 10.4.8. Thus,

$$|(r\partial_x)^\alpha - (\rho\partial_y)^\alpha_{\kappa^{-1}}|_{\mathcal{O}^k_{l-k}(\mathcal{C}_\lambda)} \le c\,\varphi_0^\lambda\,.$$

Using the fact that the left-hand side of the inequality (10.4.6) does not exceed

$$c \sum_{|\alpha|\le k} |q_\alpha|_{\mathcal{O}^0_{l-k}(\kappa(\mathcal{C}^\circ))} \, |(r\partial_x)^\alpha - (\rho\partial_y)^\alpha_{\kappa^{-1}}|_{\mathcal{O}^k_{l-k}(\mathcal{C}_\lambda)},$$

we obtain the assertion of the lemma. ∎

As a consequence of Lemma 10.4.8, the following statement holds.

LEMMA 10.4.10. *For all $u \in V^l_{2,\beta,0}(\mathcal{C}^\circ)$, $v \in V^l_{2,\beta}(\kappa(\mathcal{C}^\circ))$ the inequalities*

$$c_1 \|u\|_{V^l_{2,\beta,0}(\mathcal{C}^\circ)} \le \|(\varphi^{\beta-l+n/2}u)\circ\kappa^{-1}\|_{V^l_{2,\beta}(\kappa(\mathcal{C}^\circ))} \le c_2 \|u\|_{V^l_{2,\beta,0}(\mathcal{C}^\circ)},$$

$$c_1 \|v\|_{V^l_{2,\beta}(\kappa(\mathcal{C}^\circ))} \le \|\varphi^{-\beta+l-n/2}(v\circ\kappa)\|_{V^l_{2,\beta,0}(\mathcal{C}^\circ)} \le c_2 \|v\|_{V^l_{2,\beta}(\kappa(\mathcal{C}^\circ))}$$

are valid with constants c_1, c_2 independent of u an v. Here the norm in $V^l_{2,\beta}(\kappa(\mathcal{C}^\circ))$ is defined by (10.3.1) (replacing $\tilde{\mathcal{D}}$ by $\kappa(\mathcal{C}^\circ)$).

The following lemma is obvious.

LEMMA 10.4.11. *If $\mathcal{P} \in \mathcal{O}^k_{l-k}$, $l \ge k$, then the mapping $r^{-k}\mathcal{P} : V^l_{2,\beta,\gamma}(\mathcal{C}^\circ) \to V^{l-k}_{2,\beta,\gamma}(\mathcal{C}^\circ)$ is continuous and the estimate*

$$\|r^{-k}\mathcal{P}\|_{V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)\to V^{l-k}_{2,\beta,\gamma}(\mathcal{C}^\circ)} \le c\,|\mathcal{P}|_{\mathcal{O}^k_{l-k}(\mathcal{C}^\circ)}$$

is valid with a constant c independent of $\mathcal{P}$.

10.4.2. Solvability of the auxiliary problem. Now we consider problem (10.1.13) - (10.1.15), where $\mathcal{L}$ is the operator (10.1.4), $\mathcal{B}_k$, $\mathcal{B}'_k$ are the operators (10.1.6) and (10.1.7), while $\mathcal{C}_{k,j}$, $\mathcal{C}'_{k,j}$ are the operators (10.1.10) and (10.1.24), respectively.

Analogously to the space $V^l_{2,\beta,\gamma}(\mathcal{C}^\circ)$, we define $V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})$ as the weighted Sobolev space with the norm

$$\|u\|_{V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})} = \Big(\sum_{|\alpha|\le l}\ \int_{\mathcal{C}\backslash\mathcal{D}} e^{2\gamma x_n}\, r^{2(\beta-l+|\alpha|)}\, |D^\alpha_x u|^2\, dx\Big)^{1/2}.$$

The spaces of traces of functions from $V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})$, $l \ge 1$, on $\partial\mathcal{C}$ and $\partial\mathcal{D}$ are denoted by $V^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ and $V^{l-1/2}_{2,\beta,\gamma}(\partial\mathcal{D})$, respectively.

Using the fact that the operator ∂_x continuously maps the space $V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})$ into $V^{l-1}_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})$, the operator of multiplication by r continuously maps the space $V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})$ into $V^l_{2,\beta-1,\gamma}(\mathcal{C}\backslash\mathcal{D})$, we obtain that the operator of the auxiliary problem (10.1.13) - (10.1.15) continuously maps the space

$$\mathfrak{D}^l_{\beta,\gamma} \stackrel{def}{=} V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D}) \times \prod_{j=1}^{J} V^{l+t_j-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \times \prod_{j=1}^{J} V^{l+\tau_j-1/2}_{2,\beta,\gamma}(\partial\mathcal{D}) \tag{10.4.7}$$

into

$$\mathfrak{R}^l_{\beta,\gamma} \stackrel{def}{=} V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D}) \times \prod_{k=1}^{m+J} V^{l-m_k-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \times \prod_{k=1}^{m+J} V^{l-\mu_k-1/2}_{2,\beta,\gamma}(\partial\mathcal{D}) \tag{10.4.8}$$

for $l \ge 2m$, $l > \max m_k$, $l > \max \mu_k$. We denote this operator by $\mathcal{A}_0$, while the operators of the limit problems are denoted by $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

THEOREM 10.4.1. *Suppose that the operators*

$$\mathcal{A}_1 \ : \ V^l_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \to V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}),$$
$$\mathcal{A}_2 \ : \ V^l_{2,\beta}(\tilde{\mathcal{D}}) \times W^{l+\underline{\tau}-1/2}_2(\partial\tilde{\mathcal{D}}) \to V^{l-2m}_{2,\beta}(\tilde{\mathcal{D}}) \times W^{l-\underline{\mu}-1/2}_2(\partial\tilde{\mathcal{D}})$$

of the limit problems are isomorphisms. Furthermore, we assume that the function φ satisfies the inequalities (10.1.1) and $\varphi \le \varphi_0$, where φ_0 is a sufficiently small positive constant. Then the operator $\mathcal{A}_0$ is an isomorphism from $\mathfrak{D}^l_{\beta,\gamma}$ onto $\mathfrak{R}^l_{\beta,\gamma}$.

Proof: The proof consists of two parts.

a) *Existence of a right inverse.* Let η be a function from $C_0^\infty(\mathbb{R}^{n-1})$ equal to one near the origin. We set

$$\eta_\sigma(x) = \eta(x'\,\varphi(x_n)^{-\sigma}), \qquad \zeta_\sigma = 1-\eta_\sigma,$$

where $\sigma \in (0,1)$. For sufficiently small φ_0 the function η_σ is equal to one in a neighbourhood of $\mathcal{D}$ and the support of η_σ is contained in $\mathcal{C}$. In the sequel, θ, σ, and τ are real numbers satisfying the inequalities $0 < \tau < \sigma < \theta < 1$. Then $\zeta_\theta\zeta_\sigma = \zeta_\sigma$ and $\zeta_\sigma\,\zeta_\tau = \zeta_\tau$ if φ_0 is sufficiently small.

Furthermore, we set $q_l(x_n) = e^{\gamma x_n}\,\varphi(x_n)^{\beta-l+n/2}$ and introduce the operators

$$\mathcal{H}_{l,\underline{\tau}} \ : \ (u, v_1, \dots, v_J) \to \big(q_l u, q_{l+\tau_1} v_1, \dots, q_{l+\tau_J} v_J\big)$$
$$\mathcal{H}_{l-2m,-\underline{\mu}} \ : \ \big(f, h_1, \dots, h_{m+J}\big) \to \big(q_{l-2m} f, q_{l-\mu_1} h_1, \dots, q_{l-\mu_{m+J}} v_{m+J}\big).$$

If $F = (f, \underline{g}, \underline{h})$ is an arbitrary element of the space $\mathfrak{R}^l_{\beta,\gamma}$, then, according to Lemma 10.4.10, $(\eta_\sigma\, q_{l-2m}\, f)\circ\kappa^{-1} \in V^{l-2m}_{2,\beta}(\tilde{\mathcal{D}})$ and $(q_{l-\mu_k}\, h_k)\circ\kappa^{-1} \in V^{l-\mu_k-1/2}_{2,\beta}(\partial\tilde{\mathcal{D}})$. We introduce the operator $\mathfrak{S}_r$ as follows:

$$\mathfrak{S}_r \ : F = \big(\zeta_\theta\, u^{(1)} + \eta_\tau u^{(2)},\, \underline{u}, \underline{v}\big), \tag{10.4.9}$$

where

$$(u^{(1)}, \underline{u}) \ = \ \mathcal{A}_1^{-1}\,(\zeta_\sigma f, \underline{g}), \tag{10.4.10}$$
$$(u^{(2)}, \underline{v}) \ = \ \mathcal{H}^{-1}_{l,\underline{\tau}}\,(\mathcal{A}_2)^{-1}_{\kappa^{-1}}\,\mathcal{H}_{l-2m,-\underline{\mu}}\,(\eta_\sigma f, \underline{h}). \tag{10.4.11}$$

Here $(\mathcal{A}_2)^{-1}_{\kappa^{-1}}\,\Phi = \big(\mathcal{A}_2^{-1}(\Phi\circ\kappa^{-1})\big)\circ\kappa$.

From Lemma 10.4.4 and from the continuity of the operator $\mathcal{A}_1^{-1}$ it follows that the mapping $F \to (\zeta_\theta u^{(1)}, \underline{u}, 0)$ is continuous from $\mathfrak{R}^l_{\beta,\gamma}$ into $\mathfrak{D}^l_{\beta,\gamma}$. Lemmas 10.4.4, 10.4.10 and the continuity of the operator $\mathcal{A}_2^{-1}$ ensure the continuity of the operator $F \to (\eta_\tau u^{(2)},\, 0, \underline{v})$ (from $\mathfrak{R}^l_{\beta,\gamma}$ into $\mathfrak{D}^l_{\beta,\gamma}$). Thus, the operator $\mathfrak{S}_r$ continuously maps $\mathfrak{R}^l_{\beta,\gamma}$ into $\mathfrak{D}^l_{\beta,\gamma}$.

We show that the operator $\mathcal{A}_0\,\mathfrak{S}_r - I$ has a small norm in $\mathfrak{R}^l_{\beta,\gamma}$. Obviously,

$$\mathcal{A}_0\,(\zeta_\theta u^{(1)}, \underline{u}, 0) = \big(f^{(0)}, \underline{g}^{(0)}, 0\big),$$

where

$$\begin{aligned}(f^{(0)}, \underline{g}^{(0)}) \ &= \ \mathcal{A}_1\,\zeta_\theta\,(u^{(1)}, \underline{u}) = [\mathcal{A}_1, \zeta_\theta]\,(u^{(1)}, \underline{u}) + \zeta_\theta\,\mathcal{A}_1\,(u^{(1)}, \underline{u}) \\ &= \ [\mathcal{A}_1, \zeta_\theta]\,(u^{(1)}, \underline{u}) + (\zeta_\sigma f, \underline{g}).\end{aligned}$$

Here $[\mathcal{A}_1, \zeta_\theta] = \mathcal{A}_1\zeta_\theta - \zeta_\theta\mathcal{A}_1$ is the commutator of $\mathcal{A}_1$ and ζ_θ. By virtue of Lemma 10.4.2, for sufficiently small positive ε there exists the operator

$$Q_\varepsilon \overset{def}{=} \big((\varphi^\theta/r)^\varepsilon\,\mathcal{A}_1\,(\varphi^\theta/r)^{-\varepsilon}\big)^{-1}$$

continuously mapping $V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ into $V^{l}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$. By the definition of Q_ε and $(u^{(1)}, \underline{u})$, we have

$$[\mathcal{A}_1, \zeta_\theta]\,(u^{(1)}, \underline{u}) = [\mathcal{A}_1, \zeta_\theta]\,(r/\varphi^\theta)^\varepsilon\, Q_\varepsilon\, \zeta_\sigma\,(\varphi^\sigma/r)^\varepsilon\, \varphi^{(\theta-\sigma)\varepsilon}\,(f, \underline{g}).$$

From Lemmas 10.4.4, 10.4.5, and 10.4.11 it follows that the operators

$$[\mathcal{A}_1, \zeta_\theta]\,(r/\varphi^\theta)^\varepsilon \;:\; V^{l}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \to V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D}) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$$

and

$$\zeta_\sigma\,(\varphi^\sigma/r)^\varepsilon \;:\; V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D}) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \to V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$$

are continuous.

In what follows, for the sake of brevity, we use the notation $v \approx w$ if one of the inequalities

$$\|v-w\|_{V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})} \le \delta\,\|F\|_{\mathfrak{R}^l_{\beta,\gamma}}, \qquad \|v-w\|_{\mathfrak{R}^l_{\beta,\gamma}} \le \delta\,\|F\|_{\mathfrak{R}^l_{\beta,\gamma}},$$

is satisfied, where δ is a small constant independent of F.

Due to Lemmas 10.4.3, 10.4.11, the norm of the operator of multiplication by $\varphi^{(\theta-\sigma)\varepsilon}$ is small. Therefore, $\big([\mathcal{A}_1, \zeta_\theta]\,(u^{(1)}, \underline{u})\,,\,0\big) \approx 0$ and

$$\mathcal{A}_0\,(\zeta_\theta u^{(1)}, \underline{u}, 0) \approx (\zeta_\sigma f, \underline{g}, 0). \tag{10.4.12}$$

Now we show that $\mathcal{A}_0\,(\eta_\tau u^{(2)}, 0, \underline{v}) \approx (\eta_\sigma f, 0, \underline{h})$. We consider the function

$$\mathcal{L}(x',\partial_x)(\eta_\tau u^{(2)}) = \big(\mathcal{L}(x',\partial_x) - \mathcal{L}^{(0)}(x',\partial_x)\big)\,(\eta_\tau u^{(2)}) + \mathcal{L}^{(0)}(x',\partial_x)\,(\eta_\tau u^{(2)}).$$

From Lemma 10.4.4 and from the definition of the operator $\mathcal{L}^{(0)}$ it follows that

$$(\mathcal{L} - \mathcal{L}^{(0)})\,(\eta_\tau u^{(2)}) \approx 0. \tag{10.4.13}$$

Furthermore,

$$\begin{aligned}\mathcal{L}^{(0)}(x',\partial_x)\,(\eta_\tau u^{(2)}) \;&=\; \big(\mathcal{L}^{(0)}(x',\partial_x) - \varphi^{-2m}\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\big)\,(\eta_\tau u^{(2)})\\ &\quad + \varphi^{-2m}\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\,(\eta_\tau u^{(2)}),\end{aligned}$$

where $(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\,u = \big(\mathcal{L}^{(0)}(y',\partial_y)\,(u\circ\kappa^{-1})\big)\circ\kappa$. According to Lemma 10.4.9, we have

$$\big(\mathcal{L}^{(0)}(x',\partial_x) - \varphi^{-2m}\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\big)\,(\eta_\tau u^{(2)}) \approx 0.$$

Using Lemmas 10.4.2 and 10.4.4, we get

$$\begin{aligned}&\big(\varphi^{-2m}\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}} - q^{-1}_{l-2m}\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\,q_l\big)\,\eta_\tau u^{(2)}\\ &\quad = \varphi^{-2m}\,\big((\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}} - q_l^{-1}\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\,q_l\big)\,\eta_\tau u^{(2)} \approx 0.\end{aligned}$$

Consequently, from (10.4.13) we obtain

$$\begin{aligned}\mathcal{L}(x',\partial_x)\,(\eta_\tau u^{(2)}) &\approx \mathcal{L}^{(0)}(x',\partial_x)\,(\eta_\tau u^{(2)}) \approx q^{-1}_{l-2m}\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\,(q_l\,\eta_\tau\,u^{(2)})\\ &= q^{-1}_{l-2m}\Big(\eta_\tau\,(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}} + [(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\,,\eta_\tau]\Big)\,(q_l\,u^{(2)}).\end{aligned}$$

By (10.4.11), we have $\mathcal{L}^{(0)}(y',\partial_y)_{\kappa^{-1}}\,(q_l u^{(2)}) = q_{l-2m}\eta_\sigma f$. Since $\eta_\tau\eta_\sigma = \eta_\sigma$, it holds

$$\mathcal{L}(x',\partial_x)\,(\eta_\tau u^{(2)}) = \eta_\sigma f + q^{-1}_{l-2m}\,[(\mathcal{L}^{(0)}(y',\partial_y))_{\kappa^{-1}}\,,\eta_\tau]\,(q_l u^{(2)}). \tag{10.4.14}$$

Let h be an infinitely differentiable function on $\mathbb{R}_+$ such that $h(t) = 1$ on $[2,\infty)$ and $h(t) = t$ on $[0,1]$. We set

$$h_{\tau,\varepsilon}(x) = h(r^\varepsilon/\varphi(x_n)^{\tau\varepsilon})$$

and rewrite the second term on the right of (10.4.14) in the form

$$q_{l-2m}^{-1}\,[(\mathcal{L}^{(0)})_{\kappa^{-1}}\,,\eta_\tau]\,q_l\,h_{\tau,\varepsilon}^{-1}\cdot h_{\tau,\varepsilon}\,u^{(2)}\,. \tag{10.4.15}$$

Since $h_{\tau,\varepsilon}(x)=r^\varepsilon/\varphi(x_n)^{\tau\varepsilon}$ on the support of the function η_σ, we have

$$\begin{aligned} h_{\tau,\varepsilon}\,(u^{(2)},\underline{v}) &= \mathcal{H}_{l,\underline{\tau}}^{-1}\,h_{\tau,\varepsilon}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}\,(\eta_\sigma f,\underline{h})\\ &= \mathcal{H}_{l,\underline{\tau}}^{-1}\,h_{\tau,\varepsilon}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,h_{\tau,\varepsilon}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}\,\big(r/\varphi^\sigma\big)^\varepsilon\,\varphi^{(\sigma-\tau)\varepsilon}\,(\eta_\sigma f,\underline{h})\,. \end{aligned}$$

From the inequality $\varphi\le\varphi_0$, where φ_0 is small, and from Lemmas 10.4.3 and 10.4.11 it follows that the operator of multiplication by $\varphi^{(\sigma-\tau)\varepsilon}$ has a small norm in $\mathfrak{R}^l_{\beta,\gamma}$. Due to Lemma 10.4.4, the norm of the operator

$$(r^\varepsilon/\varphi^{\sigma\varepsilon})\,\eta_\sigma\;:\;V_{2,\beta,\gamma}^{l-2m}(\mathcal{C}\backslash\mathcal{D})\times V_{2,\beta,\gamma}^{l-\underline{\mu}-1/2}(\partial\mathcal{D})\to V_{2,\beta,\gamma}^{l-2m}(\mathbb{R}^n\backslash\mathcal{D})\times V_{2,\beta,\gamma}^{l-\underline{\mu}-1/2}(\partial\mathcal{D})$$

is bounded by a constant independent of $\varepsilon\in(0,\varepsilon_0)$. We prove that the norm of the operator

$$\begin{aligned} &\mathcal{H}_{l,\underline{\tau}}^{-1}\,h_{\tau,\varepsilon}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,h_{\tau,\varepsilon}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}\;:\\ &\qquad V_{2,\beta,\gamma}^{l-2m}(\mathbb{R}^n\backslash\mathcal{D})\times V_{2,\beta,\gamma}^{l-\underline{\mu}-1/2}(\partial\mathcal{D})\to V_{2,\beta,\gamma}^{l}(\mathbb{R}^n\backslash\mathcal{D})\times V_{2,\beta,\gamma}^{l+\underline{\tau}-1/2}(\partial\mathcal{D}) \end{aligned} \tag{10.4.16}$$

is bounded by a constant for small ε. For this it suffices to show that the norm of the operator

$$\begin{aligned} &\mathcal{H}_{l-2m,-\underline{\mu}}^{-1}\,\big(h_{\tau,\varepsilon}\,(\mathcal{A}_2)_{\kappa^{-1}}\,h_{\tau,\varepsilon}^{-1}-(\mathcal{A}_2)_{\kappa^{-1}}\big)\,\mathcal{H}_{l,\underline{\tau}}\;:\\ &\qquad V_{2,\beta,\gamma}^{l}(\mathbb{R}^n\backslash\mathcal{D})\times V_{2,\beta,\gamma}^{l+\underline{\tau}-1/2}(\partial\mathcal{D})\to V_{2,\beta,\gamma}^{l-2m}(\mathbb{R}^n\backslash\mathcal{D})\times V_{2,\beta,\gamma}^{l-\underline{\mu}-1/2}(\partial\mathcal{D}) \end{aligned} \tag{10.4.17}$$

is small for small ε. We consider the operator

$$q_{l-2m}^{-1}\,\big(h_{\tau,\varepsilon}\,(\mathcal{L}^{(0)})_{\kappa^{-1}}\,h_{\tau,\varepsilon}^{-1}-(\mathcal{L}^{(0)})_{\kappa^{-1}}\big)\,q_l\;:\;V_{2,\beta,\gamma}^{l}(\mathbb{R}^n\backslash\mathcal{D})\to V_{2,\beta,\gamma}^{l-2m}(\mathbb{R}^n\backslash\mathcal{D}).$$

By Lemma 10.4.9, the norms of this operator and of the operator

$$q_l^{-1}\,\big(h_{\tau,\varepsilon}\,\mathcal{L}^{(0)}\,h_{\tau,\varepsilon}^{-1}-\mathcal{L}^{(0)}\big)\,q_l=q_l^{-1}\,h_{\tau,\varepsilon}\,[\mathcal{L}^{(0)},\,h_{\tau,\varepsilon}^{-1}]\,q_l$$

are small. It can be easily verified that $[\mathcal{L}^{(0)},\,h_{\tau,\varepsilon}^{-1}]=\varepsilon\,z(r^\varepsilon/\varphi(x_n)^{\tau\varepsilon})\,\mathcal{T}(x,\partial_x)$, where $\mathcal{T}$ is a differential operator such that the mapping

$$q_l^{-1}\,\mathcal{T}\,q_l\;:\;V_{2,\beta,\gamma}^{l}(\mathbb{R}^n\backslash\mathcal{D})\to V_{2,\beta,\gamma}^{l-2m}(\mathbb{R}^n\backslash\mathcal{D})$$

is continuous, while z is an infinitely differentiable function on $\mathbb{R}_+$ such that $z(t)=t^{-1}$ for $t\in[0,1]$, $z(t)=0$ for $t>2$. Moreover, the function $h_{\tau,\varepsilon}(x)\,z(r^\varepsilon/\varphi^{\tau\varepsilon})$ is a multiplier in $V_{2,\beta,\gamma}^{l-2m}(\mathbb{R}^n\backslash\mathcal{D})$ (by virtue of Lemma 10.4.4). This an analogous assertions for the operators $\mathcal{B}'_k$ and $\mathcal{C}'_{k,j}$ prove that the operator (10.4.17) has a small norm. Consequently, the norm of the operator (10.4.16) is bounded by a constant for small ε. This proves that $h_{\tau,\varepsilon}\,u^{(2)}$ has a small norm in $V_{2,\beta,\gamma}^{l}(\mathbb{R}^n\backslash\mathcal{D})$ for small ε.

Furthermore, the norm of the operator

$$q_{l-2m}^{-1}\,[(\mathcal{L}^{(0)})_{\kappa^{-1}},\eta_\tau]\,q_l\,h_{\tau,\varepsilon}^{-1}\;:\;V_{2,\beta,\gamma}^{l}(\mathbb{R}^n\backslash\mathcal{D})\to V_{2,\beta,\gamma}^{l-2m}(\mathcal{C}\backslash\mathcal{D}).$$

is bounded by a constant for small ε. From this we conclude that the expression (10.4.15) is approximately equal to zero (in the sense of the above defined relation $\approx$). Thus, according to (10.4.14), it holds $\mathcal{L}\,(\eta_\tau u^{(2)})\approx\eta_\sigma f$. Analogously, we obtain that $\mathcal{B}'(\eta_\tau u^{(2)})+\mathcal{C}'\underline{v}$ is approximately equal to $\underline{h}$. Therefore,

$$\mathcal{A}_0\,(\eta_\tau u^{(2)},0,\underline{v})\approx(\eta_\sigma f,0,\underline{h}).$$

Together with (10.4.12) this relation implies $\mathcal{A}_0 \mathfrak{S}_r F \approx F$. Hence

$$(\mathcal{A}_0)_r^{-1} = \mathfrak{S}_r \sum_{k=0}^{\infty} (I - \mathcal{A}_0 \mathfrak{S}_r)^k .$$

is a right inverse of the operator $\mathcal{A}_0$.

b) *Existence of a left inverse.* Let $(u, \underline{u}, \underline{v})$ be an arbitrary element of the space $\mathfrak{D}^l_{\beta,\gamma}$ and $(f, \underline{g}, \underline{h}) = \mathcal{A}_0\,(u, \underline{u}, \underline{v})$. We introduce the operator

$$\mathfrak{S}_l\,(f, \underline{g}, \underline{h}) = \big(\zeta_\sigma\, u^{(1)} + \eta_\sigma u^{(2)},\ \underline{u}^{(1)}, \underline{v}^{(2)}\big)\,,$$

where

$$(u^{(1)}, \underline{u}^{(1)}) = \mathcal{A}_1^{-1}\,(\zeta_\theta f, \underline{g}) \quad \text{and} \quad (u^{(2)}, \underline{v}^{(2)}) = \mathcal{H}^{-1}_{l,\underline{\tau}}\,(\mathcal{A}_2)^{-1}_{\kappa^{-1}}\,\mathcal{H}_{l-2m,-\underline{\mu}}(\eta_\tau f, \underline{h})$$

In the sequel, the notation $v \sim w$ is used in the case when

$$\|v - w\|_{\mathfrak{D}^l_{\beta,\gamma}} \le \delta\, \|(u, \underline{u}, \underline{v})\|_{\mathfrak{D}^l_{\beta,\gamma}}\,,$$

where δ is a small constant. We show that the norm of the operator $\mathfrak{S}_l\,\mathcal{A}_0 - I$ is small in $\mathfrak{D}^l_{\beta,\gamma}$, i.e.,

$$\mathfrak{S}_l\,\mathcal{A}_0\,(u, \underline{u}, \underline{v}) \sim (u, \underline{u}, \underline{v}).$$

Since $\zeta_\theta \zeta_\sigma = \zeta_\sigma$, we have

$$\begin{aligned}(\zeta_\sigma u^{(1)}, \underline{u}^{(1)}) &= \zeta_\sigma \mathcal{A}_1^{-1}\,(\zeta_\theta f, \underline{g}) = \zeta_\sigma\, \mathcal{A}_1^{-1}\, \zeta_\theta\, \mathcal{A}_1\,(u, \underline{u}) \\ &= \zeta_\sigma\, \mathcal{A}_1^{-1}\,[\zeta_\theta, \mathcal{A}_1]\,(u, \underline{u}) + \zeta_\sigma (u, \underline{u}).\end{aligned}$$

By virtue of Lemma 10.4.2, for sufficiently small positive ε there exists the operator

$$Q_{-\varepsilon} \stackrel{def}{=} \big((\varphi^\theta/r)^{-\varepsilon}\, \mathcal{A}_1\,(\varphi^\theta/r)^\varepsilon\big)^{-1}$$

continuously mapping $V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$ into $V^l_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$. Then

$$\zeta_\sigma \mathcal{A}_1^{-1}\,[\zeta_\theta, \mathcal{A}_1]\,(u, \underline{u}) = \zeta_\sigma\,(\varphi^\sigma/r)^\varepsilon\, \varphi^{(\theta-\sigma)\varepsilon}\, Q_{-\varepsilon}\,(r/\varphi^\theta)^\varepsilon\,[\mathcal{A}_1, \zeta_\theta]\,(u, \underline{u}).$$

Here, according to Lemmas 10.4.4 and 10.4.5, the operators

$$(r/\varphi^\theta)^\varepsilon\,[\mathcal{A}_1, \zeta_\theta]\ :\ V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D}) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \to V^{l-2m}_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}),$$

$$\zeta_\sigma\,(\varphi^\sigma/r)^\varepsilon\ :\ V^l_{2,\beta,\gamma}(\mathcal{C}^\circ) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C}) \to V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D}) \times V^{l+\underline{t}-1/2}_{2,\beta,\gamma}(\partial\mathcal{C})$$

have bounded norms for small ε. Since the norm of the operator of multiplication by $\varphi^{(\theta-\sigma)\varepsilon}$ is small, we obtain $\zeta_\sigma \mathcal{A}_1^{-1}\,[\zeta_\theta, \mathcal{A}_1]\,(u, \underline{u}) \sim 0$, and, therefore,

$$(\zeta_\sigma u^{(1)}, \underline{u}^{(1)}, 0) \sim (\zeta_\sigma u, \underline{u}, 0). \tag{10.4.18}$$

We consider the term $(\eta_\sigma u^{(2)}, \underline{v}^{(2)})$. As it was shown in the first part of the proof, there is the relation

$$\eta_\tau f = \eta_\tau\, \mathcal{L} u \approx \eta_\tau\, \varphi^{-2m}\,(\mathcal{L}^{(0)})_{\kappa^{-1}}\, u \approx \eta_\tau\, q^{-1}_{l-2m}\,(\mathcal{L}^{(0)})_{\kappa^{-1}}\, q_l u\,.$$

An analogous assertion is true for the vector-function $\underline{h}$. Hence

$$(\eta_\tau f, 0, \underline{h}) \approx (\eta_\tau f^{(1)}, 0, \underline{h}^{(1)}), \tag{10.4.19}$$

where

$$(f^{(1)}, \underline{h}^{(1)}) = \mathcal{H}^{-1}_{l-2m,-\underline{\mu}}\,(\mathcal{A}_2)_{\kappa^{-1}}\,\mathcal{H}_{l+\underline{\tau}}\,(u, \underline{v}).$$

From (10.4.19) it follows that

$$\eta_\sigma\,(u^{(2)},0,\underline{v}^{(2)}) \sim \eta_\sigma\,(u^{(3)},0,\underline{v}^{(3)})\,, \tag{10.4.20}$$

where

$$\begin{aligned}(u^{(3)},\underline{v}^{(3)}) &= \mathcal{H}_{l,\underline{\tau}}^{-1}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}\,(\eta_\tau f^{(1)},\underline{h}^{(1)})\\ &= \mathcal{H}_{l,\underline{\tau}}^{-1}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,\eta_\tau\,(\mathcal{A}_2)_{\kappa^{-1}}\,\mathcal{H}_{l,\underline{\tau}}\,(u,\underline{v})\,.\end{aligned}$$

Let $h_{\tau,\varepsilon}$ be the same function as in the first part of the proof. Then

$$\begin{aligned}\eta_\sigma\,\mathcal{H}_{l,\underline{\tau}}^{-1}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,\eta_\tau\,(\mathcal{A}_2)_{\kappa^{-1}}\,\mathcal{H}_{l,\underline{\tau}} &= \eta_\sigma + \eta_\sigma\,\mathcal{H}_{l,\underline{\tau}}^{-1}(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,[\eta_\tau,(\mathcal{A}_2)_{\kappa^{-1}}]\,\mathcal{H}_{l,\underline{\tau}}\\ &= \eta_\sigma + \eta_\sigma\,(r/\varphi^\sigma)^\varepsilon\,\varphi^{(\sigma-\tau)\varepsilon}\cdot h_{\tau,\varepsilon}^{-1}\,\mathcal{H}_{l,\underline{\tau}}^{-1}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}\,h_{\tau,\varepsilon}\\ &\quad\cdot h_{\tau,\varepsilon}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}^{-1}\,[\eta_\tau,(\mathcal{A}_2)_{\kappa^{-1}}]\,\mathcal{H}_{l,\underline{\tau}}\,.\end{aligned}$$

Using the fact that the norm of the operator of multiplication by the function $\varphi^{(\sigma-\tau)\varepsilon}$ in the space $\mathfrak{D}^l_{\beta,\gamma}$ is small for small ε and the boundedness of the operators

$$\eta_\sigma\,(r/\varphi^\sigma)^\varepsilon\,,\quad h_{\tau,\varepsilon}^{-1}\,\mathcal{H}_{l,\underline{\tau}}^{-1}\,(\mathcal{A}_2)_{\kappa^{-1}}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}\,h_{\tau,\varepsilon}\,,\quad h_{\tau,\varepsilon}^{-1}\,\mathcal{H}_{l-2m,-\underline{\mu}}^{-1}\,[\eta_\tau,(\mathcal{A}_2)_{\kappa^{-1}}]\,\mathcal{H}_{l,\underline{\tau}}$$

in corresponding function spaces, we can conclude, analogously to the first part of the proof, that

$$(\eta_\sigma u^{(3)},0,\underline{u}^{(3)}) \sim (\eta_\sigma u,0,\underline{v})\,.$$

Together with (10.4.18), (10.4.20) this implies $(\zeta_\sigma u^{(1)}+\eta_\sigma u^{(2)},\underline{u}^{(1)},\underline{u}^{(2)}) \sim (u,\underline{u},\underline{v})$. Hence the norm of $\mathfrak{S}_l\mathcal{A}_0 - I$ is small and the left inverse $(\mathcal{A}_0)_l^{-1}$ of $\mathcal{A}_0$ has the form

$$(\mathcal{A}_0)_l^{-1} = \sum_{k=0}^{\infty}(I-\mathfrak{S}_l\mathcal{A}_0)^k\,\mathfrak{S}_l\,.$$

The theorem is proved. ■

Comparing the last theorem with Theorems 10.2.3 and 10.3.2, we obtain the following statement.

THEOREM 10.4.2. *Suppose that the following conditions are satisfied:*

(i) *The operators (10.2.7) are isomorphisms for* $l=l_0$, $\beta=\beta_0$, *and the numbers* l, β *satisfy the inequalities* $\mu_- < -\beta+l-(n-1)/2 < \mu_+$. *(The numbers* μ_-, μ_+ *were introduced before Theorem 10.2.1.)*

(ii) *On the line* $\operatorname{Re}\lambda = -\gamma$ *there are no points of the spectrum of the operator pencil* $\mathfrak{A}(\lambda)$ *defined on the space* $V^l_{2,\beta}(\Omega)$.

(iii) *The operator* $\mathfrak{B}(\lambda)$ *considered in the Section 10.3 is such that*

a) *the mapping*

$$\begin{aligned}\mathfrak{B}(\lambda)\,:\,W^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde\Omega)\times W_2^{l+\underline{\tau}-1/2}(\partial\tilde\Omega)\\ \to W^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde\Omega)\times W_2^{l-\underline{\mu}-1/2}(\partial\tilde\Omega)\end{aligned}$$

has trivial kernel and cokernel for purely imaginary $\lambda\neq 0$.

b) *the mapping*

$$\begin{aligned}\mathfrak{B}(0)\,:\,V^l_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde\Omega)\times W_2^{l+\underline{\tau}-1/2}(\partial\tilde\Omega)\\ \to V^{l-2m}_{2,\beta}(\mathbb{R}^{n-1}\backslash\tilde\Omega)\times W_2^{l-\underline{\mu}-1/2}(\partial\tilde\Omega)\end{aligned}$$

is an isomorphism.

Then the operator $\mathcal{A}_0$ realizes an isomorphism from (10.4.7) onto (10.4.8).

10.4.3. Regularity assertions for the solution. Analogously to the space $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ)$, we define the space $V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}\backslash\mathcal{D})$ as the weighted Sobolev space with the norm

$$\|u\|_{V^l_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}\backslash\mathcal{D})} = \Big(\sum_{|\alpha|\le l}\ \int\limits_{\mathcal{C}\backslash\mathcal{D}} e^{2x_n\gamma(x_n)}\, r^{2(\beta-l+|\alpha|)}|D^\alpha_x u(x)|^2\,dx\Big)^{1/2},$$

where γ is a real function from $C^\infty(\mathbb{R})$ such that $\gamma(x_n) = \gamma_1$ for $x_n < -1$ and $\gamma(x_n) = \gamma_2$ for $x_n > 1$. Furthermore, we denote by $\mathfrak{D}^l_{\beta,\gamma_1,\gamma_2}$ and $\mathfrak{R}^l_{\beta,\gamma_1,\gamma_2}$ the spaces which are obtained if one replaces $V^l_{2,\beta,\gamma}$ by $V^l_{2,\beta,\gamma_1,\gamma_2}$ in the definition of the space $\mathfrak{D}^l_{\beta,\gamma}$ and $\mathfrak{R}^l_{\beta,\gamma}$, respectively.

THEOREM 10.4.3. *Suppose that conditions (i) and (iii) of Theorem 10.4.2 are satisfied. Furthermore, we assume that the closed strip $-\gamma_2 \le \operatorname{Re} \le -\gamma_1$ does not contain points of the spectrum of the operator pencil $\mathfrak{A}(\lambda)$ defined on the space $V^l_{2,\beta}(\Omega)$. Then the operator*

$$\mathcal{A}_0 \,:\, \mathfrak{D}^l_{\beta,\gamma_1,\gamma_2} \to \mathfrak{R}^l_{\beta,\gamma_1,\gamma_2}$$

of problem (10.1.13) - (10.1.15) realizes an isomorphism.

The proof is obtained by the word by word repetition of the proof of Theorems 10.4.1 if the function q_l (introduced at the beginning of part a) of the proof of Theorem 10.4.1) is replaced by the function $x_n \to e^{x_n\gamma(x_n)}\,\varphi(x_n)^{\beta-l+n/2}$ Moreover, instead of Theorem 10.2.3 one has to make use of Corollary 10.2.1. (The latter guarantees the existence of a continuous inverse operator $\mathcal{A}_1^{-1}$ defined on $V^{l-2m}_{2,\beta,\gamma_1,\gamma_2}(\mathcal{C}^\circ) \times V^{l-\underline{m}-1/2}_{2,\beta,\gamma_1,\gamma_2}(\partial\mathcal{C})$.)

THEOREM 10.4.4. *Suppose that $l' > l$ and the smoothness of the domain $\mathcal{C}\backslash\mathcal{D}$ and of the coefficients of problem (10.1.13) - (10.1.15) is determined by the number l'. Furthermore, we assume that the following conditions are satisfied:*

(i) *The operators (10.2.7) are isomorphisms for $l = l_0$, $\beta = \beta_0$, and the numbers l, l', β, β' satisfy the inequalities*

$$\mu_- < -\beta + l - (n-1)/2 < \mu_+, \quad \mu_- < -\beta' + l' - (n-1)/2 < \mu_+.$$

(ii) *There are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ (defined on the space $V^l_{2,\beta}(\Omega) \times W_2^{l+\underline{t}-1/2}(\partial\Omega)$) in the strip $-\tau_2 \le \operatorname{Re}\lambda \le -\tau_1$, where $\tau_1 = \min(\gamma_1,\gamma_1')$, $\tau_2 = \max(\gamma_2,\gamma_2')$, $\gamma_1 \le \gamma_2$, $\gamma_1' \le \gamma_2'$.*

(iii) *The requirement (iii) of Theorem 10.4.2 is valid for each of the pairs (l,β) and (l',β').*

If the vector function $(f,\underline{g},\underline{h})$ belongs to the intersection $\mathfrak{R}^l_{\beta,\gamma_1,\gamma_2} \cap \mathfrak{R}^{l'}_{\beta',\gamma_1',\gamma_2'}$, then the solution $(u,\underline{u},\underline{v}) \in \mathfrak{D}^l_{\beta,\gamma_1,\gamma_2}$ of problem (10.1.13) - (10.1.15) belongs to the space $\mathfrak{D}^{l'}_{\beta',\gamma_1',\gamma_2'}$ and satisfies the estimate

$$\|(u,\underline{u},\underline{v})\|_{\mathfrak{D}^{l'}_{\beta',\gamma_1',\gamma_2'}} \le c\|(f,\underline{g},\underline{h})\|_{\mathfrak{R}^{l'}_{\beta',\gamma_1',\gamma_2'}}\,.$$

10.5. Elliptic problems in domains of the exterior of a cusp type

In the first two subsections of this section we consider elliptic boundary value problems in a domain which coincides for large x_n with $\mathcal{C}\backslash\mathcal{D}$. We show that the assumptions imposed in the previous section on the auxiliary problem ensure the Fredholm property of the operator of the boundary value problem (10.1.2), (10.1.3) in the weighted Sobolev space $V^l_{2,\beta,\gamma}(\mathcal{G})$. Here the indices β and γ have to belong to certain intervals which depend on the spectrum of the operator pencils $\mathfrak{A}(\lambda)$ and $\mathfrak{B}(\lambda)$ generated by the limit problems. One of the goals of this section is to calculate these intervals for special boundary value problems (Dirichlet problem, V-elliptic boundary value problems).

The last subsection is dedicated to V-elliptic problems in a domain which coincides with the exterior of a cusp in a neighbourhood of the origin. In this case we pass to spherical coordinates ρ, ω and substitute $t = \log|x|^{-1}$. This leads to a boundary value problem in a domain with singularity at infinity.

10.5.1. Solvability and regularity assertions for solutions of elliptic problems in a domain which is "quasicylindrical" at infinity. We consider the problem (10.1.2), (10.1.3)

$$L(x, \partial_x)\, u = f \quad \text{in } \mathcal{G},$$
$$B_s(x, \partial_x)\, u + C_s(x, \partial_x)\, \underline{u}^{(s)} = \underline{g}^{(s)} \quad \text{on } \Gamma_s\,,\ s = 1, \dots, q,$$

with the same assumptions on the domain $\mathcal{G}$ and on the operators L, B_s, and C_s as in Section 10.1.

Let $V^l_{2,\beta,\gamma}(\mathcal{G})$ be the weighted Sobolev space with the norm

$$\|u\|_{V^l_{2,\beta,\gamma}(\mathcal{G})} = \|(1-\eta_T)u\|_{W^l_2(\mathcal{G})} + \|\eta_T u\|_{V^l_{2,\beta,\gamma}(\mathcal{C}\backslash\mathcal{D})}\,.$$

Here $\eta_T = \eta_T(x_n)$ is a smooth function on $\mathbb{R}$, $\eta_T(x_n) = 0$ for $x_n < T$, $\eta_T(x_n) = 1$ for $x_n > T+1$. Analogously to Lemma 6.2.1, it can be shown that the space $V^l_{2,\beta,\gamma}(\mathcal{G})$ is compactly imbedded into $V^{l'}_{2,\beta',\gamma'}(\mathcal{G})$ if $l > l'$, $\gamma > \gamma'$, and $\beta - l < \beta' - l'$. The space of traces of functions from $V^l_{2,\beta,\gamma}(\mathcal{G})$, $l \geq 1$, on Γ_s is denoted by $V^{l-1/2}_{2,\beta,\gamma}(\Gamma_s)$. Furthermore, we set

$$\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G}) = V^l_{2,\beta,\gamma}(\mathcal{G}) \times \prod_{s=1}^{q}\prod_{j=1}^{J} V^{l+\tau_{s,j}-1/2}_{2,\beta,\gamma}(\Gamma_s)$$

and

$$\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G}) = V^{l-2m}_{2,\beta,\gamma}(\mathcal{G}) \times \prod_{s=1}^{q}\prod_{k=1}^{m+J} V^{l-\mu_{s,k}-1/2}_{2,\beta,\gamma}(\Gamma_s).$$

Let

$$\mathcal{A}\,:\, \mathfrak{D}^l_{\beta,\gamma}(\mathcal{G}) \to \mathfrak{R}^\ell_{\beta,\gamma}(\mathcal{G}) \tag{10.5.1}$$

be the operator of the boundary value problem (10.1.2), (10.1.3). From conditons (10.1.5), (10.1.8), (10.1.9), (10.1.11), (10.1.12) it follows that

$$\|(\mathcal{A}-\mathcal{A}_0)\eta_T\|_{\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})\to\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G})} < \varepsilon'\,, \tag{10.5.2}$$

where $\mathcal{A}_0$ is the operator of the auxiliary boundary value problem (10.1.13) - (10.1.15) and ε' is a small number.

In contrast to the previous chapters, we prove the Fredholm property for the operator of the boundary value problem constructing left and right regularizers (see Definition 3.4.2) for this operator.

THEOREM 10.5.1. *Suppose the operator $\mathcal{A}_0$ of the auxiliary proplem (10.1.13) - (10.1.15) satisfies conditions (i) - (iii) of Theorem 10.4.2. Moreover, we assume that the inequalities (10.1.5), (10.1.8), (10.1.9), (10.1.11), and (10.1.12) are satisfied for certain T with a sufficiently small number ε. Then the operator (10.5.1) is Fredholm and every solution of the boundary value problem (10.1.2), (10.1.3) satisfies the estimate*

$$(10.5.3)\quad \|(u,\underline{u}^{(1)},\dots,\underline{u}^{(q)})\|_{\mathcal{D}^l_{\beta,\gamma}(\mathcal{G})} \le c\Big(\|\mathcal{A}\,(u,\underline{u}^{(1)},\dots,\underline{u}^{(q)})\|_{\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G})} + \|u\|_{L_2(\mathcal{G}_T)} + \sum_{s=1}^{q}\|\underline{u}^{(s)}\|_{L_2(\Gamma_{s,T})^J}\Big),$$

where $\mathcal{G}_T = \{x\in\mathcal{G} : x_n < T\}$, $\Gamma_{s,T} = \{x\in\Gamma_s : x_n < T\}$.

Proof: Existence of a right regularizer. Let T be sufficiently large and let $\{\mathcal{U}_\nu\}_{1\le\nu\le N}$ be a sufficiently fine covering of the set $\{x\in\overline{\mathcal{G}} : x_n < T\}$. Furthermore, let ζ_ν, η_ν, $\nu = 0,1,\dots,N$, be C^∞ functions satisfying the conditions

$$\begin{aligned}&\operatorname{supp}\zeta_0 \subset \operatorname{supp}\eta_0 \subset \{x\in\overline{\mathcal{G}} : x_n > T\},\\ &\operatorname{supp}\zeta_\nu \subset \operatorname{supp}\eta_\nu \subset \mathcal{U}_\nu \quad \text{for } \nu = 1,\dots,N,\\ &\zeta_\nu\,\eta_\nu = \zeta_\nu \quad \text{for } \nu = 0,1,\dots,N, \qquad \sum_{\nu=0}^{N}\zeta_\nu = 1 \ \text{ in } \overline{\mathcal{G}}.\end{aligned}$$

If $F = (f,\underline{g}^{(1)},\dots,\underline{g}^{(q)})$ is an arbitrary element of $\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G})$, then let $\eta_\nu F$ denote simultaneously the elements $(\eta_\nu f,\eta_\nu\underline{g}^{(1)},\dots,\underline{g}^{(q)})$ and $(\eta_\nu f,\eta_\nu\underline{g}^{(i_1)},\dots,\eta_\nu\underline{g}^{(i_\mu)})$, where the last one contains only those vector-functions $\eta_\nu\underline{g}^{(s)}$ for which $\operatorname{supp}\eta_\nu \cap \Gamma_s \ne \emptyset$. The same notation will be used for $\zeta_\nu U$ if $U = (u,\underline{u}^{(1)},\dots,\underline{u}^{(q)})$ is an arbitrary element of $\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$.

For every index $\nu = 0,1,\dots,N$ we introduce linear operators R_ν such that $\zeta_\nu\,R_\nu\,\eta_\nu$ continuously map $\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G})$ into $\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$ and

$$(10.5.4)\qquad \mathcal{A}\,\zeta_\nu\,R_\nu\,\eta_\nu = \zeta_\nu\,(I+T_\nu) + K_\nu\,,$$

where T_ν are linear and continuous operators in $\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G})$ with small norms and K_ν are linear and continuous operators from $\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G})$ into $\mathfrak{R}^{l+1}_{\beta,\gamma'}(\mathcal{G})$ with arbitrary real γ'.

For $\nu \ge 1$ such operators R_ν were constructed in the proof of Theorem 3.4.3. In the case $\nu = 0$ we set $R_0 = \mathcal{A}_0^{-1}$. Then

$$(10.5.5)\qquad \mathcal{A}\,\zeta_0\,R_0\,\eta_0 F = \zeta_0\,\big(F + (\mathcal{A}-\mathcal{A}_0)\,\eta_0 F\big) + [\mathcal{A},\zeta_0]\,R_0\,\eta_0 F,$$

where $[\mathcal{A},\zeta_0]$ denotes the commutator of $\mathcal{A}$ and ζ_0. Consequently, by (10.5.2), the representation (10.5.4) is valid for $\nu = 0$.

As in the proof of Theorem 3.4.3, we define the operator $\mathcal{R}_r : \mathfrak{R}^l_{\beta,\gamma}(\mathcal{G}) \to \mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$ as follows:

$$\mathcal{R}_r = \sum_{\nu=0}^{N}\zeta_\nu\,R_\nu\,\eta_\nu\,(I+T_\nu)^{-1}\,.$$

Then by (10.5.4), we have

$$\mathcal{A}\,\mathcal{R}_r = I + \sum_{\nu=0}^{N} K_\nu\,(I+T_\nu)^{-1}\,, \tag{10.5.6}$$

i.e., $\mathcal{R}_r$ is a right regularizer for the operator $\mathcal{A}$.

Existence of a left regularizer. Let R_ν be the operators introduced in the first part of the proof. For these operators also the representation

$$\zeta_\nu\,R_\nu\,\eta_\nu\,\mathcal{A} = \zeta_\nu\,(I + T_\nu + K_\nu) \tag{10.5.7}$$

holds, where T_ν is a linear and continuous operator in $\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$ with small norm and K_ν is a compact operator in $\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$. For example, for the index $\nu=0$ we get

$$\zeta_0\,R_0\,\eta_0\,\mathcal{A} = \zeta_0\,\mathcal{A}_0^{-1}\,(\mathcal{A}\,\eta_0 - [\mathcal{A},\eta_0]) = \zeta_0\,\big(I + \mathcal{A}_0^{-1}\,(\mathcal{A}-\mathcal{A}_0)\,\eta_0 - \mathcal{A}_0^{-1}\,[\mathcal{A},\eta_0]\big),$$

where, according to (10.5.2), the operator $\mathcal{A}_0^{-1}\,(\mathcal{A}-\mathcal{A}_0)\,\eta_0$ has a small norm. Furthermore, by Theorem 10.4.4, the operator $\mathcal{A}_0^{-1}\,[\mathcal{A},\eta_0]$ continuously maps the space $\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$ into $\mathfrak{D}^{l+1}_{\beta',\gamma'}(\mathcal{G})$ with certain $\beta' < \beta+1$, $\gamma' > \gamma$. Therefore, the operator $\mathcal{A}_0^{-1}\,[\mathcal{A},\eta_0]$ is compact in $\mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$. Analogously, the representation (10.5.7) can be shown for the indices $\nu \ge 1$. Thus,

$$\mathcal{R}_l = \sum_{\nu=0}^{N} (I+T_\nu)^{-1}\,\zeta_\nu\,R_\nu\,\eta_\nu$$

is a left regularizer for the operator $\mathcal{A}$.

c) The inequality (10.5.3) is an immediate consequence of the estimate (10.5.2), Theorem 10.4.2, and Corollary 3.2.1. ■

THEOREM 10.5.2. *Let $l' \ge l > \max(2m-1, \mu_{s,k})$. Suppose that the smoothness of the boundary $\partial\mathcal{G}$ and of the coefficients of the operators L, $B_{s;k}$, and $C_{s;k,j}$ is determined by the number l' and the inequalities (10.1.5), (10.1.8), (10.1.9), (10.1.11), (10.1.12) are valid (with the replacement of l by l'). Moreover, we assume that the following conditions are satisfied:*

(i) *The operators (10.2.7) are isomorphisms for $l=l_0$, $\beta=\beta_0$, and the numbers l, l', β, β' satisfy the inequalities*

$$\mu_- < -\beta + l - (n-1)/2 < \mu_+, \quad \mu_- < -\beta' + l' - (n-1)/2 < \mu_+$$

(the numbers μ_-, μ_+ were introduced before Theorem 10.2.1).

(ii) *The closed strip between the lines* $\operatorname{Re}\lambda = -\gamma$ *and* $\operatorname{Re}\lambda = \gamma'$ *does not contain eigenvalues of the operator pencil $\mathfrak{A}(\lambda)$ defined on the space $V^l_{2,\beta}(\Omega)$.*

(iii) *The requirement (iii) of Theorem 10.4.2 is satisfied for each of the pairs (l,β) and (l',β').*

If $(f, \underline{g}^{(1)},\dots,\underline{g}^{(q)})$ belongs to the intersection $\mathfrak{R}^l_{\beta,\gamma}(\mathcal{G}) \cap \mathfrak{R}^{l'}_{\beta',\gamma'}(\mathcal{G})$, then the solution $(u,\underline{u}^{(1)},\dots,\underline{u}^{(q)}) \in \mathfrak{D}^l_{\beta,\gamma}(\mathcal{G})$ of problem (10.1.2), (10.1.3) belongs to the space $\mathfrak{D}^{l'}_{\beta',\gamma'}(\mathcal{G})$.

Proof: Let η_T be the truncating function occurring in the definition of the space $V^l_{2,\beta,\gamma}(\mathcal{G})$. Then for $U = (u,\underline{u}^{(1)},\dots,\underline{u}^{(q)})$ we have

$$\mathcal{A}(\eta_T U) = \eta_T\,AU + [\mathcal{A},\eta_T]\,U. \tag{10.5.8}$$

Obviously, $\eta_T AU \in \mathfrak{R}^l_{\beta'+l-l',\gamma'}(\mathcal{G})$. Furthermore, $[\mathcal{A},\eta_T]\,U \in \mathfrak{R}^l_{\beta-1,\gamma'}(\mathcal{G})$.

We assume that $|\beta' - l' - (\beta - l)| \le 1$. Then the right-hand side of (10.5.8) is an element of the space $\mathfrak{R}^l_{\beta'+l-l',\gamma'}(\mathcal{G})$. Since the norm of the operator $(\mathcal{A} - \mathcal{A}_0)\,\eta_T$ is small (see (10.5.2)), it follows from (10.5.8) and Theorems 10.4.3, 10.4.4 that $\eta_T U \in \mathfrak{D}^l_{\beta'+l-l',\gamma'}(\mathcal{G})$.

In the case $|\beta' - l' - (\beta - l)| > 1$ one can arrive at the same result, improving several times the index β.

It remains to show that the inclusion $U \in \mathfrak{D}^l_{\beta'+l-l',\gamma'}(\mathcal{G})$ implies $U \in \mathfrak{D}^{l'}_{\beta',\gamma'}(\mathcal{G})$ (cf. Lemma 6.3.1). In order to show this we construct a covering of finite multiplicity of the set $\{x : x' \neq 0\}$ by open balls B_j with centers $x^{(j)} = (x'^{(j)}, x_n^{(j)})$ and radii r_j proportional to $|x'^{(j)}|$. We introduce a partition of unity $\{\zeta_j\}$ subordinate to this covering such that $|(r\partial_x)^\alpha \zeta_j| \le c_\alpha$ for all multiindices α, where c_α are constants independent of j. Furthermore, let η_j be functions satisfying the conditions $\operatorname{supp} \eta_j \subset B_j$, $\eta_j \zeta_j = \zeta_j$ and $|(r\partial_x)^\alpha \eta_j| \le const.$

By a similarity transformation with coefficient r_j, from the local $W_2^{l'}$-estimate (see Corollary 3.2.1) we obtain

$$\begin{aligned} \|\zeta_j U\|^2_{\mathfrak{D}^{l'}_{l',0}(\mathcal{G})} \le\; & c\Big(\|\eta_j \mathcal{A} U\|^2_{\mathfrak{R}^{l'}_{l',0}(\mathcal{G})} + \|\eta_j u\|^2_{L_2(\mathcal{G})} \\ & + \sum_{s=1}^{q}\sum_{j=1}^{J} r_j^{-2\tau_{s,j}+1}\, \|\eta_j u_j^{(s)}\|^2_{L_2(\Gamma_s)}\Big). \end{aligned} \tag{10.5.9}$$

Multiplying this inequality by $e^{2\gamma' x_n^{(j)}}\, r_j^{2(\beta'-l')}$ and summing up over all j, we obtain the finiteness of the norm of U in $\mathfrak{D}^{l'}_{\beta',\gamma'}(\mathcal{G})$. The theorem is proved. ■

REMARK 10.5.1. Without essential modification in the proofs, the assertions of Theorems 10.5.1, 10.5.2 can be extended to a domain of the same type whose section is a $(n-1)$-dimensional manifold. Then the tube $\mathcal{D}$ thinning at infinity can be described in the following manner. Let κ be a diffeomorphism of a neighbourhood of the point $0 \in \Omega$ onto an open subset $\mathcal{U} \subset \mathbb{R}^{n-1}$ such that $\kappa(0) = 0$. Furthermore, let $\tilde{\Omega}$ be a domain in $\mathbb{R}^{n-1}$ which contains the origin. We set $\mathcal{D} = \{(x', x_n) \in \Omega \times \mathbb{R} : x' = \kappa^{-1}(\varphi(x_n)\tilde{\Omega})\}$, where φ is the same function as before.

10.5.2. The Dirichlet problem. Let $\mathcal{G}$ be the same domain as in the previous subsection. We consider the Dirichlet problem

$$u = f \quad \text{in } \mathcal{G}, \tag{10.5.10}$$

$$D_\nu^{k-1} u = g_k \quad \text{on } \partial\mathcal{G},\ k = 1, \dots, m, \tag{10.5.11}$$

for the $2m$ order elliptic differential operator

$$L = \sum_{|\alpha| \le 2m} a_\alpha(x)\, \partial_x^\alpha .$$

Suppose that Gårding's inequality

$$|\mathrm{Re}\,(Lu, u)_{\mathcal{G}}| \ge c\, \|u\|^2_{W_2^m(\mathcal{G})} \tag{10.5.12}$$

is satisfied for all $u \in W_2^m(\mathcal{G})$ and the coefficients a_α stabilize at infinity, i.e.,

$$\lim_{x_n \to +\infty} \sum_{j=0}^{l-2m} \|\partial_{x_n}^j (a_\alpha(\cdot, x_n) - a_\alpha(\cdot, +\infty))\|_{C^{l-2m-j}(\Omega)} = 0,$$

where $a_\alpha(\cdot,\infty)$ are smooth functions on $\overline{\Omega}$. We set

$$\mathcal{L}(x',\partial_x) = \sum_{|\alpha|\le 2m} a_\alpha(x',\infty)\,\partial_x^\alpha\,.$$

Then the first limit problem has the form

$$\mathcal{L}(x',\partial_x)\,u = f \quad \text{in } \mathcal{C}^\circ, \tag{10.5.13}$$

$$D_\nu^{k-1}u = f_k \quad \text{on } \partial\mathcal{C},\ k=1,\dots,m. \tag{10.5.14}$$

We consider only the cases $2m > n-1$, n even, and $2m < n-1$ (see Theorem 10.2.2).

Inequality (10.5.12) yields a similar inequality for the operator $\mathcal{L}(x',\partial_x)$ and for functions from $C_0^\infty((\overline{\Omega}\backslash\{0\})\times\mathbb{R})$. Therefore, problem (10.5.13), (10.5.14) is uniquely solvable in the closure of $C_0^\infty((\overline{\Omega}\backslash\{0\})\times\mathbb{R})$ with respect to the W_2^m-norm, i.e., in $V_{2,0,0}^m(\mathcal{C}^\circ)$. Applying a local coercive estimate analogous to (10.5.9), we obtain that the operator of the problem (10.5.13), (10.5.14) realizes an isomorphism

$$V_{2,m,0}^{2m}(\mathcal{C}^\circ) \to V_{2,m,0}^{0}(\mathcal{C}^\circ) \times \prod_{k=1}^{m} V_{2,m,0}^{2m-k+1/2}(\partial\mathcal{C}). \tag{10.5.15}$$

In the case $2m > n-1$, n even, it follows from Gårding's inequality for the operator $\mathcal{L}(x',\partial_x)$ that problem (10.5.13), (10.5.14) with the additional condition

$$D_x^\alpha u = 0 \quad \text{on } \{0\}\times\mathbb{R},\ |\alpha| < m-(n-1)/2,$$

is uniquely solvable in the Sobolev space $W_2^m(\mathcal{C}^\circ)$. From this, as in the case $2m < n-1$, we conclude that the operator of problem (10.5.13), (10.5.14) realizes the isomorphism (10.5.15).

Thus, there are no eigenvalues of the operator pencil

$$\mathfrak{A}(\lambda) = \big(\mathcal{L}(x',\partial_{x'},\lambda), 1, D_\nu,\dots,D_\nu^{m-1}\big)$$

(defined on the space $V_{2,m}^{2m}(\Omega)$) on the line $\operatorname{Re}\lambda = 0$ (cf. Lemma 5.2.5). Let γ_- and γ_+ be real numbers, $\gamma_- < 0 < \gamma_+$, such that $-\gamma_+ < \operatorname{Re}\lambda < -\gamma_-$ is the widest strip containing the line $\operatorname{Re}\lambda = 0$ which is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

Under our conditions on the coefficients a_α, the operator $\mathcal{L}^{(0)}(x',\partial_x)$ coincides with the operator

$$\mathcal{L}^\circ(0,\partial_x) = \sum_{|\alpha|=2m} a_\alpha(0,\infty)\,\partial_x^\alpha\,.$$

Consequently, according to Theorem 10.2.2, in the case $2m < n-1$ the operators

$$\mathcal{L}^{(0)}(x',\partial_{x'},\pm i)\,:\,E_{2,\beta}^l(\mathbb{R}^{n-1}) \to E_{2,\beta}^{l-2m}(\mathbb{R}^{n-1})$$

are isomorphisms for $2m-(n-1)/2 < l-\beta < (n-1)/2$. If $2m > n-1$, n even, then these operators are isomorphisms for $m-1/2 < l-\beta < m+1/2$.

Hence by Theorem 10.2.3, the operator of the first limit problem (10.5.13), (10.5.14) realizes the isomorphism

$$V_{2,\beta,\gamma}^l(\mathcal{C}^\circ) \to V_{2,\beta,\gamma}^{l-2m}(\mathcal{C}^\circ) \times \prod_{k=1}^{m} V_{2,\beta,\gamma}^{l-k+1/2}(\partial\mathcal{C})$$

if $\gamma_- < \gamma < \gamma_+$ and

$$l-\beta \in \begin{cases} (2m-(n-1)/2\,,\,(n-1)/2) & \text{for } \ 2m < n-1, \\ (m-1/2\,,\,m+1/2) & \text{for } \ 2m > n-1,\ n \text{ even}. \end{cases} \tag{10.5.16}$$

The second limit problem has the form

$$\mathcal{L}^\circ(0,\partial_y)\,u = h \quad \text{in } \tilde{\mathcal{D}},$$
$$D_\nu^{k-1}u = h_k \quad \text{on } \partial\tilde{\mathcal{D}},\ k=1,\dots,m$$

where $\tilde{\mathcal{D}} = (\mathbb{R}^{n-1}\backslash\overline{\tilde{\Omega}}) \times \mathbb{R}$. We set

$$\mathfrak{B}(\lambda) = \big(\mathcal{L}^\circ(0,\partial_y), 1, D_\nu, \dots, D_\nu^{m-1}\big).$$

Inequality (10.5.12) implies an analogous inequality for the operator $\mathcal{L}_0(0,\partial_y)$. Inserting a function of the form $e^{\lambda y_n}\,v(y')$ into the latter inequality, we obtain Gårding's inequality also for the operator $\mathcal{L}^\circ(0,\partial_{y'},\lambda)$, $\operatorname{Re}\lambda = 0$. Hence each of the mappings

$$\mathfrak{B}(\lambda)\ :\ W_2^m(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \to (W_2^m(\mathbb{R}^{n-1}\backslash\tilde{\Omega}))^* \times \prod_{k=1}^m W_2^{m-k+1/2}(\partial\tilde{\Omega}),$$

where $\operatorname{Re}\lambda = 0,\ \lambda \neq 0$, and

$$\mathfrak{B}(0)\ :\ V_{2,0}^m(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \to (V_{2,0}^m(\mathbb{R}^{n-1}\backslash\tilde{\Omega}))^* \times \prod_{k=1}^m W_2^{m-k+1/2}(\partial\tilde{\Omega})$$

is an isomorphism. From this we conclude that the mappings

$$\mathfrak{B}(\lambda)\ :\ W_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \to W_{2,\beta}^{l-2m}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \times \prod_{k=1}^m W_2^{l-k+1/2}(\partial\tilde{\Omega})$$

for $\operatorname{Re}\lambda = 0,\ \lambda \neq 0$, and

$$\mathfrak{B}(0)\ :\ V_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \to V_{2,\beta}^{l-2m}(\mathbb{R}^{n-1}\backslash\tilde{\Omega}) \times \prod_{k=1}^m W_2^{l-k+1/2}(\partial\tilde{\Omega})$$

are isomorphisms for all l, β satisfying (10.5.16). This allows us to apply Theorem 10.3.2, according to which the operator of the second limit problem

$$\mathcal{A}_2\ :\ V_{2,\beta}^l(\tilde{\mathcal{D}}) \to V_{2,\beta}^{l-2m}(\tilde{\mathcal{D}}) \times \prod_{k=1}^m V_{2,\beta}^{l-k+1/2}(\partial\tilde{\mathcal{D}})$$

realizes an isomorphism if l and β satisfy (10.5.16).

According to (10.5.12), there exists a unique variational solution $u \in \overset{\circ}{W}{}_2^m(\mathcal{G})$ of problem (10.5.10), (10.5.11) for $g_k = 0$, $f \in (\overset{\circ}{W}{}_2^m(\mathcal{G}))^*$. Using Hardy's inequality, we obtain that this solution belongs to the space $V_{2,-m,0}^0(\mathcal{G})$. By means of an estimate of the form (10.5.9), it holds $u \in V_{2,l-m,0}^l(\mathcal{G})$ if $f \in V_{2,l-m,0}^{l-2m}(\mathcal{G})$, $l \geq 2m$.

Applying Theorem 10.5.2, we arrive at the following theorem.

THEOREM 10.5.3. *Let $2m > n-1$, n even or $2m < n-1$. Furthermore, let $l \geq 2m$, $\gamma \in (\gamma_-,\gamma_+)$, where γ_- and γ_+ are a negative and a positive number, respectively, such that $-\gamma_+ < \operatorname{Re}\lambda < -\gamma_-$ is the widest strip in the λ-plane free of the spectrum of the pencil $\mathfrak{A}(\lambda)$ defined on the space $V_{2,m}^{2m}(\Omega)$. Moreover, we assume that*

$$\beta \in \begin{cases} (l-m-1/2,\, l-m+1/2) & \text{if } 2m > n-1,\ n \text{ even},\\ (l-(n-1)/2\,,\, l-2m+(n-1)/2) & \text{if } 2m < n-1.\end{cases}$$

Then the following statements hold:

1) *The operator*

$$\mathcal{A} \,:\, V^l_{2,\beta,\gamma}(\mathcal{G}) \to V^{l-2m}_{2,\beta,\gamma}(\mathcal{G}) \times \prod_{k=1}^{m} V^{l-k+1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$$

of the Dirichlet problem (10.5.10), (10.5.11) realizes an isomorphism.

2) *If $u \in W^m_2(\mathcal{G})$ is a generalized solution of problem (10.5.10), (10.5.11) and $f \in V^{l-2m}_{2,\beta,\gamma}(\mathcal{G})$, $g_k \in V^{l-k+1/2}_{2,\beta,\gamma}(\partial\mathcal{G})$, then $u \in V^l_{2,\beta,\gamma}(\mathcal{G})$.*

REMARK 10.5.2. Everything said before about the Dirichlet problem for $2m < n-1$ is also true for general boundary value problems (10.1.2), (10.1.3) with $J = 0$ (i.e., there do not appear unknowns $\underline{u}^{(s)}$ on the boundary of $\mathcal{G}$) which can be written in the variational form. We suppose that the orders of the differential operators $B_{s,k}$ are less than $2m$ and there exists a sesquilinear form $a(\cdot,\cdot)$ on $W^m_2(\mathcal{G}) \times W^m_2(\mathcal{G})$ such that the Green formula

$$a(u,v) = \int_{\mathcal{G}} Lu \cdot \overline{v}\, dx + \sum_{(s,k)\in I_1} \int_{\Gamma_s} B_{s;k}u \cdot \overline{T_{s;k}v}\, d\sigma + \sum_{(s,k)\in I_2} \int_{\Gamma_s} S_{s;k}u \cdot \overline{B_{s;k}v}\, d\sigma$$

is valid for all functions u, $v \in W^{2m}_2(\mathcal{G})$ with compact support. Here the set I_1 contains all pairs (s,k) such that $m \le \operatorname{ord} B_{s;k} \le 2m-1$, while the set I_2 contains all pairs (s,k) such that $\operatorname{ord} B_{s;k} < m$. As for the Dirichlet problem, we assume that Gårding's inequality

$$|\operatorname{Re} a(u,u)| \ge c\, \|u\|^2_{W^m_2(\mathcal{G})}$$

is satisfied for all $u \in W^m_2(\mathcal{G})$ such that $B_{s;k}u = 0$ on Γ_s for $(s;k) \in I_2$, i.e., the sesquilinear form $a(\cdot,\cdot)$ is V-elliptic (see Definition 4.3.1). Furthermore, we assume that $\beta \in \big(l-(n-1)/2\,,\, l-2m+(n-1)/2$. Then assertions 1) and 2) of Theorem 10.5.3 (in the case $2m < n-1$) are valid for the boundary value problem (10.1.2), (10.1.3).

10.5.3. Boundary value problems in the exterior of a "cusp", "paraboloid", or "infinite funnel". Let $\mathfrak{G}$ be a domain in $\mathbb{R}^n$ with a compact closure $\overline{\mathfrak{G}}$ and boundary $\partial\mathfrak{G}$. We suppose that the surface $\partial\mathfrak{G}$ is everywhere smooth (of class C^l), except at the point 0 which coincides with the origin.

For an arbitrary point $x = (x', x_n) \in \mathbb{R}^n$ we set $\rho = |x|$, $\omega = x/|x|$, $r = |x'|$. The coordinates $x' = (x_1, \ldots, x_{n-1})$ are also local coordinates in a small neighbourhood of the north pole $N = (0, \ldots, 0, 1)$ on the sphere S^{n-1}. We denote by $\tilde{\Omega}$ a domain in $\mathbb{R}^{n-1}$ with a smooth boundary and assume that

$$\partial B_\rho \backslash \overline{\mathfrak{G}} = \{x \in \mathbb{R}^n \,:\, |x| = \rho,\; x'/\rho \in \varphi(\log\rho^{-1})\,\tilde{\Omega}\},$$

where $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$ and φ is a function satisfying condition (10.1.1). By $\mathcal{V}^l_{2,\beta,\gamma}(\mathfrak{G})$ we denote the weighted Sobolev space with the norm

$$\|u\|_{\mathcal{V}^l_{2,\beta,\gamma}(\mathfrak{G})} = \Big(\sum_{|\alpha|\le l} \int_{\mathfrak{G}} \rho^{2(\gamma-l+|\alpha|)}\, \theta^{2(\beta-l+|\alpha|)}\, |D^\alpha_x u|^2\, dx \Big)^{1/2}.$$

Here θ is the angle between the vector $0x$ and the x_n-axis. For $l \ge 1$ we denote the corresponding trace space by $\mathcal{V}^{l-1/2}_{2,\beta,\gamma}(\partial\mathfrak{G})$.

Let $\mathcal{G}$ be the image of the domain $\mathfrak{G}$ under the mapping $x \to (\omega, t)$, where $t = \log|x|^{-1}$ and $\omega = x/|x|$. This domain satisfies the conditions of the previous subsections.

The coordinate transformation $x \to (\omega, t)$ takes the space $\mathcal{V}^l_{2,\beta,\gamma}(\mathfrak{G})$ into the space $V^l_{2,\beta,-\gamma-n/2}(\mathcal{G})$. Hence from Theorems 10.5.1 - 10.5.3 for the boundary value problems in the domain $\mathcal{G}$ we obtain the same results for problems in the domain $\mathfrak{G}$. We give an example.

Let the boundary value problem

$$L(x,\partial_x)\,u = f \quad \text{in } \mathfrak{G}, \tag{10.5.17}$$

$$B_k(x,\partial_x)\,u = g_k \quad \text{on } \partial\mathfrak{G},\ k = 1,\dots,m, \tag{10.5.18}$$

be given. We assume that the coefficients of the operator L are smooth in $\mathbb{R}^n$, while those of the operators B_k are smooth outside the origin in a neighbourhood of $\partial\mathfrak{G}$. Moreover, we suppose that the operators B_k have the representation

$$B_k(x,\partial_x) = r^{-\mu_k} \sum_{|\alpha|+j\le\mu_k} b_{k;\alpha,j}(x'/\varphi(\log\rho^{-1}))\,(r\partial_{x'})^\alpha\,(r\partial_\rho)^j$$

in a neighbourhood of the point 0, where $\mu_k < 2m$ and $b_{k;\alpha,j}$ are smooth functions on $\partial\tilde{\Omega}$, and that the operator $\mathcal{A}$ of problem (10.5.17), (10.5.18) is V-elliptic (see Remark 10.5.2). This implies the unique solvability of problem (10.5.17), (10.5.18) in the space $W^m_2(\mathfrak{G})$ for $g_k = 0$. By Hardy's inequality, the last space coincides with $\mathcal{V}^m_{2,0,0}(\mathfrak{G})$ if $2m < n-1$. Therefore, the condition of V-ellipticity of the operator $\mathcal{A} = (L, B_k)$ turns into the condition of V-ellipticity of the operator $e^{t(m-n/2)}(\tilde{L},\tilde{B}_k)e^{-t(m-n/2)}$, where $\tilde{L}$, $\tilde{B}_k$ are the operators obtained from $\rho^{2m}L$, $\rho^{\mu_k}B_k$ by the coordinate change $x \to (\omega,t)$, $t = \log\rho^{-1}$.

Now we can apply the result given in Remark 10.5.2 (taking into account Remark 10.5.1). In our case the numbers γ_-, γ_+ occurring in Theorem 10.5.3 can be computed explicitly.

Let $\tilde{L}(\infty)$ be the operator obtained from $\tilde{L}$ replacing the coefficients by their limits for $t \to \infty$. It can be directly verified that the inverse substitution $(\omega,t) \to x$ takes the operator $\tilde{L}(\infty)$ into the operator $\rho^{2m}L^\circ(0,\partial_x)$, where L° is the principal part of L. This circumstance allows us to find all the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ generated by the operator $\tilde{L}(\infty)$ and defined on the space $V^{2m}_{2,\beta}(S^{n-1})$, $\beta \in (2m-(n-1)/2, (n-1)/2)$. For this it is sufficient to find all nontrivial solutions of the equation $L^\circ(0,\partial_x)w = 0$ outside the x_n-axis having the form $\rho^\lambda\Phi_\lambda(\omega)$. Since the function Φ_λ satisfies a homogeneous elliptic equation with coefficients from $C^\infty(S^{n-1})$ on the sphere S^{n-1} outside the north pole N, it follows that it is either smooth on S^{n-1} or has a power-like singularity at the point N of order $2m - n - k$, $k = -1,0,1,\dots$. The latter is not possible since $\Phi_\lambda \in V^{2m}_{2,\beta}(S^{n-1})$ and $\beta < (n-1)/2$. Hence we may assume that $L^\circ(0,\partial_x)w = 0$ for $|x| > 0$. The solutions of this equation are either homogeneous polynomials of degrees $\lambda = 0,1,\dots$ or linear combinations of the derivatives of the fundamental solution of the operator $L^\circ(0,\partial_x)$. In this case, $\lambda = 2m-n, 2m-n-1,\dots$. Due to the fact that $\beta > 2m-(n-1)/2$, the restrictions of all these solutions to the sphere S^{n-1} belong to $V^{2m}_{2,\beta}(S^{n-1})$.

Thus, the eigenvalues of the pencil $\mathfrak{A}(\lambda)$ are $2m-n, 2m-n-1,\dots$ and $0,1,2,\dots$. Consequently, the eigenvalues of the pencil generated by the operator $e^{t(m-n/2)}\tilde{L}e^{-t(m-n/2)}$ are $\pm(n/2-m), \pm(n/2-m+1),\dots$. Therefore, the largest strip in the λ-plane which is free of eigenvalues of the pencil generated by the operator $e^{t(m-n/2)}\tilde{L}e^{-t(m-n/2)}$ and contains the imaginary axis has the form $|\mathrm{Re}\,\lambda| < n/2 - m$.

Applying the result formulated in Theorem 10.5.3 (see also Remark 10.5.2) and returning to the operator $\mathcal{A}$ in the domain $\mathfrak{G}$, we obtain the following theorem.

THEOREM 10.5.4. *Let* $2m < n-1$, $l \geq 2m$, $\beta \in (l-(n-1)/2, l-2m+(n-1)/2)$ *and* $\gamma \in (l-n/2, l-2m+n/2)$. *Then the following assertions are true.*

1) *The operator of problem (10.5.17), (10.5.18) realizes an isomorphism*

$$\mathcal{V}^l_{2,\beta,\gamma}(\mathfrak{G}) \to \mathcal{V}^{l-2m}_{2,\beta,\gamma}(\mathfrak{G}) \times \prod_{k=1}^{m} \mathcal{V}^{l-\mu_k-1/2}_{2,\beta,\gamma}(\partial\mathfrak{G}),$$

2) *If* $u \in W^m_2(\mathfrak{G})$ *is a generalized solution of problem (10.5.17), (10.5.18), where* $f \in \mathcal{V}^{l-2m}_{2,\beta,\gamma}(\mathfrak{G})$, $g_k \in \mathcal{V}^{l-\mu_k-1/2}_{2,\beta,\gamma}(\partial\mathfrak{G})$, *then* $u \in \mathcal{V}^l_{2,\beta,\gamma}(\mathfrak{G})$.

An analogous result holds for a domain $\mathfrak{G}$ with infinity as singular point, when for large ρ

$$\partial B_\rho \backslash \mathfrak{G} = \{x \in \mathbb{R}^n \,:\, |x| = \rho,\ x' \in \rho\varphi(\log\rho)\tilde{\Omega}\}.$$

Then the transformation $x \to (\omega, \log|x|)$ maps the initial domain into the complement of the contracting tube with respect to the cylinder $S^{n-1} \times \mathbb{R}$ and we can repeat the considerations in the proof of Theorem 10.5.4.

10.6. Notes

The Dirchlet problem for second order elliptic equations in domains with singular boundary points including vertices of inside cusps was studied by V. G. Maz'ya and G. M. Verzhbinskiĭ [**159, 160**]. These two authors described, in particular, the principal term in the asymptotics of the solutions near the singular points.

Let $\mathcal{G}$ be a threedimensional domain which coincides in a neighbourhood of the origin with the rotational domain

$$\{x \,:\, 0 < r < 1,\ 0 < \theta < \pi - \varepsilon(r)\},$$

where $\varepsilon(r) \to 0$ as $r \to 0$. Here $r = |x|$ and θ denotes the angle between the x_3-axis and the vector $0x$. It is proved in [**160**] that, under certain conditions on $\varepsilon(\cdot)$, there exists a harmonic function u in $\mathcal{G}$ vanishing on $\partial\mathcal{G} \cap \{x : |x| < 1\}$ which admits the asymptotic representation

$$u(r,\theta) = r^{-1} \exp\Big(\frac{1}{2}\int_r^1 \frac{d\rho}{\rho\,|\log\varepsilon(\rho)|}\Big)\Big(1 - \frac{\log\cos\frac{\theta}{2}}{\log\sin\frac{\varepsilon(r)}{2}} + o(1)\Big).$$

The solvability of general elliptic boundary value problems in domains with inside cusps and the regularity of the solutions were investigated in the paper [**137**] of V. G. Maz'ya, S. A. Nazarov, and B. A. Plamenevskiĭ. The results of Chapter 10 are contained (for boundary value problems without unknowns on the boundary) in this paper.

Furthermore, we refer to the papers of A. B. Movchan, S. A. Nazarov [**170, 171**], where problems of linear elasticity in domains with cuspidal points are studied.

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List of Symbols

$\check{u}$ Laplace transform of u, 156
∂_x^α partial derivative, 158
$\tilde{\mathcal{W}}_{2,\beta}^{l,k}(\mathcal{C})$ weighted Sobolev space, 162
$(\cdot,\cdot)_{\mathcal{C}}$ scalar product in $L_2(\mathcal{C})$, 166
$(\cdot,\cdot)_{\partial\mathcal{C}}$ scalar product in $L_2(\partial\mathcal{C})$ and $L_2(\partial\mathcal{C})^k$, 166

Chapter 6

$\mathcal{K}$ cone in $\mathbb{R}^n$, 191
$V_{2,\beta}^l(\mathcal{K})$ weighted Sobolev space, 191
$V_{2,\beta}^{l-1/2}(\partial\mathcal{K})$ trace space, 191
$\mathcal{M}_{r\to\lambda}$ Mellin transformation, 194
$\tilde{u}$ Mellin transform of u, 194
$\tilde{V}_{2,\beta}^{l,k}(\mathcal{K})$ weighted Sobolev space, 195
$\mathcal{G}$ domain in $\mathbb{R}^n$, 212
$\mathcal{S} = \{x^{(1)},\dots,x^{(d)}\}$ set of conical points, 212
$\mathcal{U}_\tau$ neighbourhood of $x^{(\tau)}$, 212
$\mathcal{K}_\tau$ cone with vertex $x^{(\tau)}$, 212
Ω_τ domain on the sphere, 212
$V_{2,\beta}^l(\mathcal{G})$, $\tilde{V}_{2,\beta}^{l,k}(\mathcal{G})$ weighted Sobolev spaces, 212
$V_{2,\beta}^{l-1/2}(\partial\mathcal{G})$ trace space, 212
$P^{(\tau)}$ leading part of P at $x^{(\tau)}$, 214
$D_{2,\beta}^{l,k}(\mathcal{G})$ space of functionals, 224
$E_{2,\beta}^l(\mathcal{K})$ weighted Sobolev space, 248
$E_{2,\beta}^{l-1/2}(\partial\mathcal{K})$ trace space, 248
δ Laplace-Beltrami operator, 265

Chapter 7

$\overset{\circ}{u}$ average of u, 268
$p_l(u)$ Taylor polynomial of degree l, corresponding to u, 269
$W_{2,\beta}^l(\mathcal{G})$ weighted Sobolev space, 270
Π_l polynomials of degree $\le l$, 270
$W_{2,\beta}^{l-1/2}(\partial\mathcal{G})$ trace space, 273
Ψ_l, Υ_l sets of polynomials, 274
K integral operator in $W_2^{1/2}(\mathbb{R}_+)$, 288
$\sim$ equivalence relation in $W_2^{1/2}(\mathbb{R}_+)$, 290
$\Pi_l^{(0)}$ homogeneous polynomials of degree l, 294
$\Psi_l^{(0)}$, $\Upsilon_l^{(0)}$ sets of homogeneous polynomials, 294

Chapter 9

$\mathcal{G}$ domain in $\mathbb{R}^n$, 337, 347, 350
$\mathcal{C}_+$ half-cylinder, 337
$\mathcal{W}_{2,\beta,\gamma}^l(\mathcal{G})$ weighted Sobolev space, 339, 348, 350
$\mathcal{W}_{2,\beta,\gamma}^{l-1/2}(\mathcal{G})$ trace space, 340, 348, 350
$V_{2,\beta,\gamma}^l(\mathcal{G})$ weighted Sobolev space, 353

Chapter 10

$\mathcal{G}$ unbounded domain in $\mathbb{R}^n$, 359
Ω, $\tilde{\Omega}$ domains in $\mathbb{R}^{n-1}$, 359
$\mathcal{C} = \Omega \times \mathbb{R}$ infinite cylinder, 359
$\mathcal{D}$ infinite tube, 359
$\mathcal{O}_k^\mu$ class of differential operators, 359
$\mathcal{C}^\circ = (\Omega\backslash\{0\}) \times \mathbb{R}$, 361
$\tilde{\mathcal{D}}$ exterior of a cylinder, 361
$V_{2,\beta,\gamma}^l(\mathcal{C}^\circ)$ weighted Sobolev space, 363
$V_{2,\beta,\gamma}^{l-1/2}(\mathcal{C}^\circ)$ trace space, 363
$V_{2,\beta}^l(\Omega)$ weighted Sobolev space, 363
$V_{2,\beta}^l(\mathbb{R}^{n-1})$, $E_{2,\beta}^l(\mathbb{R}^{n-1})$ weighted Sobolev spaces, 364
μ_-, μ_+ certain real numbers, 365
$V_{2,\beta}^l(\tilde{\mathcal{D}})$ weighted Sobolev space, 370
$W_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$, $V_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde{\Omega})$, weighted Sobolev spaces, 370
$\tilde{\Omega}_\lambda$ domain in $\mathbb{R}^{n-1}$, 372
$E_{2,\beta}^l(\mathbb{R}^{n-1}\backslash\tilde{\Omega}_\lambda)$ weighted Sobolev space, 372
$V_{2,\beta}^{l-1/2}(\partial\tilde{\Omega}_\lambda)$, $E_{2,\beta}^{l-1/2}(\partial\tilde{\Omega}_\lambda)$ trace spaces, 372
$V_{2,\beta,\gamma}^l(\mathcal{C}\backslash\mathcal{D})$ weighted Sobolev space, 381
$V_{2,\beta,\gamma}^{l-1/2}(\partial\mathcal{C})$, $V_{2,\beta,\gamma}^{l-1/2}(\partial\mathcal{D})$ trace spaces, 381
$V_{2,\beta,\gamma}^l(\mathcal{G})$ weighted Sobolev space, 388
$V_{2,\beta,\gamma}^{l-1/2}(\Gamma_s)$ trace space, 388
$\mathfrak{G}$ domain with an inside cusp, 394
$\mathcal{V}_{2,\beta,\gamma}^l(\mathfrak{G})$ weighted Sobolev space, 394
$\mathcal{V}_{2,\beta,\gamma}^{l-1/2}(\partial\mathfrak{G})$ trace space, 394